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Making holes in the cone, suspension and hyperspaces of some continua

José G. Anaya, Enrique Castañeda-Alvarado,
Alejandro Fuentes-Montes de Oca, Fernando Orozco-Zitli

Abstract. A connected topological space Z is *unicoherent* provided that if $Z = A \cup B$ where A and B are closed connected subsets of Z, then $A \cap B$ is connected. Let Z be a unicoherent space, we say that $z \in Z$ makes a hole in Z if $Z - \{z\}$ is not unicoherent. In this work the elements that make a hole to the cone and the suspension of a metric space are characterized. We apply this to give the classification of the elements of hyperspaces of some continua that make them hole.

Keywords: continuum; hyperspace; hyperspace suspension; property (b); unicoherence; cone; suspension

Classification: 54B15, 54B20, 54F55

1. Introduction

The unicoherence is a very important topological property. It arised when different authors, among which we can mention L. E. J. Brouwer (1910), F. Hausdorff (1914), Z. Janiszewsky (1913), S. Mazurkiewicz (1922) and R. L. Moore (1924), studied the topological properties of the plane and sphere. As we know K. Kuratowski was the first who defined this notion (see [19] and [20]). In [6], K. Borsuk introduced the use of continuous functions of a space to the circumference, technique that has been useful for proving that a space is unicoherent.

Intuitively we can say that a space is unicoherent if it does not have "holes". We may think that a connected space has a "hole", if we can cover it with two closed connected subsets such that their intersection is not connected. With this idea we can see that S^1, $S^1 \times I$ and $S^1 \times S^1$ are examples of spaces with a "hole". We can ask what happens to a unicoherent space if we remove one of its elements. If the space loses the unicoherent property, then we have that the element "makes a hole". This is basically the idea that we will try to use throughout this paper.

Throughout this paper X will denote a *continuum* (a nondegenerate compact connected metric space) with metric d. A *subcontinuum* is a continuum which is

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a subset of a space. Consider the following hyperspaces of X for $n \in \mathbb{N}$:

$$\begin{aligned} 2^X &= \{A \subset X \colon A \text{ is closed and nonempty}\}, \\ C_n(X) &= \{A \in 2^X \colon A \text{ has at most } n \text{ components}\}, \\ F_n(X) &= \{A \subset X \colon A \neq \emptyset,\ A \text{ has at most } n \text{ points}\}. \end{aligned}$$

All the hyperspaces are considered with the Hausdorff metric (see [17, Theorem 2.2, page 11]). If $n = 1$, then we write $C(X)$ for $C_1(X)$ and it is called the *hyperspace of subcontinua* of X. The hyperspace $F_n(X)$ is called the *nth-symmetric product* of X. The space $C_n(X)$ is called the *nth-fold hyperspace* of X. The hyperspace $HS(X)$, introduced by S. B. Nadler Jr. in [27], is defined as the quotient space $C_1(X)/F_1(X)$ and is called *hyperspace suspension* of X. The natural function associated with the decomposition space is denoted by $q_X \colon C(X) \to HS(X)$. Let $F_X = q_X(F_1(X))$ be the point in the hyperspace suspension where $F_1(X)$ is identified.

To illustrate the importance of unicoherence and its usefulness to distinguishing topological spaces, we mention the following case: A. Illanes in [14, Lemmas 2.1 and 2.2, pages 348–349] shows that $C_2([0,1]) - \{A\}$ is unicoherent for all $A \in C_2([0,1])$, while $C_2(S^1) - \{S^1\}$ is not unicoherent, where S^1 is the unit circle. As a result, A. Illanes obtained that $C_2([0,1])$ and $C_2(S^1)$ are not homeomorphic; this is in contrast to the fact that $C([0,1])$ and $C(S^1)$ are homeomorphic.

We divide this paper in five sections. In Section 2 we give the definitions and results that we need in the development of the work. In Section 3 we prove some results about what elements make a hole in the cone and suspension of a metric space. In Section 4 we present some examples of continua, in these examples we apply the results of the previous section to show which elements make a hole in its cone and its suspension. In Section 5 we show what elements make a hole in hyperspaces of some continua.

2. Definitions and preliminaries

We will represent the set of positive integers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, the interval $[0,1]$ by I, the n-cell by I^n for $n \in \mathbb{N}$, the *Hilbert cube* $I \times I \times I \times \cdots$ by I^∞ and the n-sphere by S^n for $n \in \mathbb{N}$. The *cardinality* of a set A is denoted by $|A|$. Let Z be a topological space and $A \subset Z$, the *closure* of A in Z is denoted by $\mathrm{cl}_Z(A)$, the *boundary* of A by $\mathrm{bd}_Z(A)$ and the *interior* of A by $\mathrm{int}_Z(A)$. When there is no confusion we omit the subscript Z. When two topological spaces Y and Z are homeomorphic we write $Y \approx Z$. A *manifold of dimension n*, or more concisely an *n-manifold*, is a Hausdorff space M in which each point has an open neighborhood homeomorphic to $\mathbb{R}^n$. A compact n-manifold is called *closed*. Generalizing the definition of an n-manifold, an *n-manifold with boundary* is a Hausdorff space M in which each point has an open neighborhood homeomorphic either to $\mathbb{R}^n$ or to the half-space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon x_n \geq 0\}$.

Definition 2.1. Let Z be a topological space. An *arc* is any topological space homeomorphic to I. An arc pq, with *end points* p and q, is called a *free arc in* Z if $pq - \{p, q\}$ is open. A *simple closed curve* is any topological space homeomorphic to S^1. A simple closed curve S is called *free* in Z if $S \neq Z$ and there exists $p \in S$ such that $S - \{p\}$ is open.

Definition 2.2. A topological space Z is *locally connected* if for every $z \in Z$ and any neighborhood U of z there exists an open connected set C such that $z \in C \subset U$.

Definition 2.3. Let Y be a subspace of a topological space Z. A continuous function $r\colon Z \to Y$ is called a *retraction* of Z in Y, if $r(y) = y$ for all $y \in Y$. We say that Y is a *deformation retract* of Z, if there exists a retraction $r\colon Z \to Y$ and a continuous function $H\colon Z \times I \to Z$ such that $H(z, 0) = z$ and $H(z, 1) = r(z)$ for each $z \in Z$.

Definition 2.4. A topological space Z is said to be *contractible*, if there exists $z \in Z$, such that $\{z\}$ is a deformation retract of Z.

The *cone over* Z, denoted by $\mathrm{Cone}(Z)$, is the decomposition space $(Z \times I)/(Z \times \{1\})$ where $Z \times I$ has the product topology. The *vertex* of $\mathrm{Cone}(Z)$ is the point $Z \times \{1\}$, v always denotes the vertex of a cone. One element in $\mathrm{Cone}(Z) - \{v\}$ will be denoted by $[z, t]_c$ with $z \in Z$ and $t \in [0, 1)$. The *base of* $\mathrm{Cone}(Z)$ is the set $B(Z) = \{[z, 0]_c \colon z \in Z\}$. The decomposition space $\mathrm{Cone}(Z)/B(Z)$ is called the *suspension over* Z, and is denoted by $\mathrm{Sus}(Z)$. The point of identification is denoted by B_Z. Throughout this paper for every $[z, t]_c \in \mathrm{Cone}(Z) - \{B(Z), v\}$ its class in $\mathrm{Sus}(Z)$ is denoted by $[z, t]_s$.

Definition 2.5. Let Z be a topological space. We say that Z has *property* (b) if for each continuous function $f\colon Z \to S^1$, there exists a continuous function $h\colon Z \to \mathbb{R}$ such that $f = \exp \circ h$, where exp is the function of $\mathbb{R}$ on S^1 defined by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$. The function h is called *lifting of* f.

Proposition 2.6 ([1, Proposition 8, page 2001])**.** *Let Z be a topological space such that $Z = A \cup B$, where A and B are closed connected subsets. If A and B have property* (b) *and $A \cap B$ is connected, then Z has property* (b).

Theorem 2.7 ([2, Theorem 2.4, page 3])**.** *Let Z be a normal connected space. We assume that Z has property* (b). *Let $p \in Z$ and W be a neighborhood of p in Z such that $W - \{p\}$ has property* (b). *If there is a neighborhood U of p such that* $\mathrm{bd}(U)$ *is connected and* $\mathrm{cl}(U) \subset W$, *then $Z - \{p\}$ has the property* (b).

The following result was proved by S. Mardešić in [23].

Lemma 2.8 ([23, Lemma 5, page 39])**.** *Let Z be a topological space and let $f\colon Z \to S^1$ be a continuous function. Then f has a lifting if and only if f is homotopic to a constant function.*

Proposition 2.9. *Let Z be a topological space and let Y be a deformation retract of Z. Then Z has property* (b) *if and only if Y has property (b).*

Proof: Since Y is a deformation retract of Z, there exist a retraction $r\colon Z \to Y$ and a continuous function $H\colon Z\times I \to Z$ such that $H(z,0)=z$ and $H(z,1)=r(z)$ for each $z\in Z$.

Let $f\colon Y \to S^1$ be a continuous function. If Z has property (b), then there exists a lifting $h\colon Z \to \mathbb{R}$ of $f\circ r$. We consider $h|_Y\colon Y\to\mathbb{R}$. For $y\in Y$ we have $\exp\circ h|_Y(y) = f\circ r(y) = f(y)$. Thus f has a lifting. Therefore Y has property (b).

Now we consider a continuous function $f\colon Z\to S^1$. We affirm that f is homotopic to a constant function. If Y has property (b), then by Lemma 2.8, $f|_Y$ is homotopic to a constant function. Therefore, there exist $s_0\in S^1$ and $F\colon Y\times I\to S^1$ such that $F(y,0)=f|_Y(y)$ and $F(y,1)=s_0$ for each $y\in Y$. We define $F_1\colon Z\times I\to S^1$ by

$$F_1(z,t)=\begin{cases} f(H(z,2t)) & \text{if } t\in\left[0,\frac{1}{2}\right],\\ F(r(z),2t-1) & \text{if } t\in\left[\frac{1}{2},1\right].\end{cases}$$

As $f(H(z,1))=f(r(z))$ and $F(r(z),0)=f(r(z))$ for all $z\in Z$ we have that the function F_1 is continuous. Also, for each $z\in Z$ we have $F_1(z,0)=f(H(z,0))=f(z)$ and $F_1(z,1)=F(r(z),1)=s_0$. Thus f is homotopic to a constant function, and by Lemma 2.8 f has a lifting. Therefore Z has property (b). □

Corollary 2.10. *Each contractible topological space has property* (b).

Proof: If Z is contractible, then there exists $z\in Z$ such that $\{z\}$ is a deformation retract of Z. As $\{z\}$ has property (b), by Proposition 2.9, Z has property (b). □

Theorem 2.11 ([31, Theorem 7.5, page 228])**.** *If each of two continua Z and Y has property* (b), *so also has their cartesian product $Z\times Y$.*

Definition 2.12. A connected topological space Z is *unicoherent* provided that if $Z=A\cup B$, where A and B are closed connected subsets of Z, then $A\cap B$ is connected.

Theorem 2.13 ([10, Théorème 2 and 3, pages 69 and 70] or [31, Theorem 7.3, page 227])**.** *Let Z be a normal topological space. If Z has property* (b), *then Z is unicoherent.*

The converse of Theorem 2.13 is not true.

Example 2.14. Let $(SP)_1 = S^1\cup\{[1+1/t]\exp(\mathrm{i}t)\colon t\geq 1\}$. This continuum is the compactification of the ray $R=[0,\infty)$ with remainder S^1. Observe that there are only four types of subcontinua in $(SP)_1$: points, arcs, S^1 and topological copies of $(SP)_1$. It is easy to see that $(SP)_1$ is unicoherent. But the function $f\colon (SP)_1\to S^1$ such that $f|_{S^1}=i_{S^1}$ (identity function) has not a lifting.

There is an equivalence for these two properties.

Theorem 2.15 ([10, Theorem 3, page 70])**.** *Let Z be a normal T_1 locally connected topological space. Then Z is unicoherent if and only if Z has property* (b).

Definition 2.16. Let Y and Z be topological spaces. A function $f\colon Y \to Z$ is said to be *monotone* provided that f is continuous and $f^{-1}(z)$ is connected for every $z \in Z$.

Proposition 2.17. *Let Y be a connected space. If $X \times Y$ is unicoherent, then X is unicoherent.*

Proof: Suppose that $X \times Y$ is unicoherent space. Consider the projection $\pi_X\colon X \times Y \to X$, given by $\pi_X(x, y) = x$ for every $(x, y) \in X \times Y$. We have that π_X is monotone. By Theorem [13, Theorem 3.6, page 40] we have that $\pi_X(X \times Y) = X$ is unicoherent. □

It is not known if the converse of the previous theorem is true, in fact this question belongs to a more general open problem which is known in some cases. K. Kuratowski (1930) proposed the problem whether the product of two unicoherent Peano continua is also a unicoherent Peano continuum. K. Bursuk in [6] answered this question affirmatively. S. Eilenberg in [10] proved a better result that the product of two metric, locally connected unicoherent spaces is unicoherent. T. Ganea in [12, Theorem 1.3, page 35] proved a generalization of this, namely, that the product of an arbitrary family of locally connected unicoherent spaces is unicoherent. On the other hand, the continuum of Example 2.14 satisfies that its product with itself is not unicoherent (see [13, Example 5.5, pages 48 and 49]). A particular case of this problem that is of our interest is:

Question 2.18. *If X is a unicoherent space, then is $X \times I$ a unicoherent space?*

3. Making holes in cones and suspensions

We begin with the definition of our interest.

Definition 3.1. Let Z be a unicoherent space and $z \in Z$, we say that z *makes a hole* in Z if $Z - \{z\}$ is not unicoherent.

With this definition we will prove some results that indicate us what elements of the cone of a metric space (the suspension of a metric space) make a hole.

Proposition 3.2. *Let Z be a metric space. The following are normal spaces:*

a) $Z \times I$,
b) $\mathrm{Cone}(Z)$,
c) $\mathrm{Sus}(Z)$,
d) $\mathrm{Cone}(Z) - \{[z, t]_c\}$ *for* $z \in Z$ *and* $t \in [0, 1)$, *and*
e) $\mathrm{Sus}(Z) - \{[z, t]_s\}$ *for* $z \in Z$ *and* $t \in (0, 1)$.

Proof: For a), since Z is a metric space, so is $Z \times I$. Thus $Z \times I$ is a normal space. For b), let $f\colon Z \times I \to \mathrm{Cone}(Z)$ be the quotient map and let A and B be disjoint closed subsets of $\mathrm{Cone}(Z)$. Without loss of generality, suppose that $v \in A$. Then $f^{-1}(A) = (Z \times \{1\}) \cup \{f^{-1}(a)\colon a \in A - \{v\}\}$ and $f^{-1}(B)$ are disjoint closed subsets of $Z \times I$. By a) and since $f|_{Z\times[0,1)}$ is a homeomorphism it is possible to find U and V disjoint open subsets in $Z \times I$ such that $A \subset f(U)$, $B \subset f(V)$ and $f(U), f(V)$ are disjoint open subset of $\mathrm{Cone}(Z)$. Therefore $\mathrm{Cone}(Z)$ is a normal space. For c), d) and e) the proof is similar, only for d) and e) we use that $Z \times I$ is a completely normal space. □

Remark 3.3. Let Z be a metric space. By Proposition 3.2 we have that $\mathrm{Cone}(Z)$ is a normal space. If v is the vertex of the $\mathrm{Cone}(Z)$, then $\{v\}$ is a deformation retract of $\mathrm{Cone}(Z)$, thus $\mathrm{Cone}(Z)$ is contractible. By Corollary 2.10 we have that $\mathrm{Cone}(Z)$ has property (b), and by Theorem 2.13 is unicoherent.

In Proposition 3.2 the hypothesis that Z is a metric space is necessary, since there is an example such that Z is a normal space and $Z \times I$ is not a normal space (see [29]).

Proposition 3.4. *Let Z be a metric space. Then for each $z \in Z$, $[z, 0]_c$ does not make a hole in* $\mathrm{Cone}(Z)$.

Proof: We have that $\{v\}$ is a deformation retract of $\mathrm{Cone}(Z) - \{[z, 0]_c\}$. Thus $\mathrm{Cone}(Z) - \{[z, 0]_c\}$ is contractible and by Corollary 2.10 has property (b). By Proposition 3.2 $\mathrm{Cone}(Z) - \{[z, 0]_c\}$ is a normal space and by Theorem 2.13 is unicoherent. Thus $[z, 0]_c$ does not make a hole in $\mathrm{Cone}(Z)$. □

Remark 3.5. If Z is homeomorphic to I^2, then by Proposition 3.4 and Theorem 2.15 we have that $Z - \{z\}$ has property (b) for all $z \in \partial(Z)$, where $\partial(Z)$ is the boundary of Z as manifold.

Proposition 3.6. *Let Z be a metric space and suppose that pq is a free arc in Z such that $p, q \notin \mathrm{int}(pq)$. If $r \in pq$, then for all $t_0 \in (0, 1)$ we have that $[r, t_0]_c$ makes a hole in* $\mathrm{Cone}(Z)$.

Proof: Let $r_0 \in pq$ such that $r_0 \neq r$. We consider rr_0 an arc from r to r_0 contained in pq. Let $\mathcal{C}_1 = \{[s, t]_c \colon s \in rr_0,\ t \in [0, t_0]\} - \{[r, t_0]_c\}$ and $\mathcal{C}_2 = \mathrm{cl}(\mathrm{Cone}(Z) - \mathcal{C}_1) - \{[r, t_0]_c\}$. Then $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathrm{Cone}(Z) - \{[r, t_0]_c\}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{[r, t]_c \colon t \in [0, t_0)\} \cup \{[r_0, t]_c \colon t \in [0, t_0]\} \cup \{[s, t_0]_c \colon s \in rr_0,\ s \neq r\}$ is not connected. Thus $[r, t_0]_c$ makes a hole in $\mathrm{Cone}(Z)$. □

Proposition 3.7. *Let Z be a metric space and suppose that pq is a free arc in Z. If $p \in \mathrm{int}(pq)$, then for all $t_0 \in (0, 1)$ we have that $[p, t_0]_c$ does not make a hole in* $\mathrm{Cone}(Z)$.

Proof: By Remark 3.3 $\mathrm{Cone}(Z)$ has property (b). We have that $\mathrm{Cone}(pq)$ is a connected neighborhood of $[p, t_0]_c$ in $\mathrm{Cone}(Z)$ and $\mathrm{bd}(\mathrm{Cone}(pq))$ is connected. By Remark 3.5 $\mathrm{Cone}(pq) - \{[p, t_0]_c\}$ has property (b). Therefore, by Theorem 2.7 $\mathrm{Cone}(Z) - \{[p, t_0]_c\}$ has property (b) and by Theorem 2.13 is unicoherent. □

Proposition 3.8. *Let Z be a metric space and let $z_0 \in Z$ such that $Z - \{z_0\}$ is connected. If Z has property* (b), *then for all $t_0 \in (0, 1)$ we have that $[z_0, t_0]_c$ does not make a hole in* $\mathrm{Cone}(Z)$.

Proof: Let z_0 be the element of the hypothesis and let $t_0 \in (0,1)$. Let $\mathcal{C}_1 = \{[z,t]_c \colon z \in Z,\ t \in [0,t_0]\} - \{[z_0,t_0]_c\}$ and $\mathcal{C}_2 = \mathrm{cl}\{\mathrm{Cone}(Z) - \mathcal{C}_1\} - \{[z_0,t_0]_c\}$, which they are closed connected subsets of $\mathrm{Cone}(Z) - \{[z_0,t_0]_c\}$. Furthermore, $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathrm{Cone}(Z) - \{[z_0,t_0]_c\}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = Z \times \{t_0\} - \{[z_0,t_0]_c\}$. Since Z has property (b), by Proposition 2.9 the space $\mathcal{C}_1$ has property (b) and by Corollary 2.10 the space $\mathcal{C}_2$ has property (b). As $\mathcal{C}_1 \cap \mathcal{C}_2$ is connected, by Proposition 2.6 the space $\mathrm{Cone}(Z) - \{[z_0,t_0]_c\}$ has property (b). By Proposition 3.2 the space $\mathrm{Cone}(Z) - \{[z_0,t_0]_c\}$ is a normal space and by Theorem 2.13 it is unicoherent. □

The following proposition is a direct consequence of Proposition 2.17.

Proposition 3.9. *Let Z be a topological space. If Z is not unicoherent, then v makes a hole in* $\mathrm{Cone}(Z)$.

Proposition 3.10. *For every connected metric space Z, B_Z does not make a hole in* $\mathrm{Sus}(Z)$.

Proof: Notice that $\{v\}$ is a deformation retract of $\mathrm{Cone}(Z) - B(Z)$. Thus $\mathrm{Cone}(Z) - B(Z)$ is contractible and, by Corollary 2.10, has property (b). By Proposition 3.2 the space $\mathrm{Cone}(Z) - B(Z)$ is a normal space, and by Theorem 2.13, is unicoherent. The proposition follows easily because $\mathrm{Cone}(Z) - B(Z) \approx \mathrm{Sus}(Z) - B_Z$. □

Proposition 3.11. *For every connected metric space Z, v does not make a hole in* $\mathrm{Sus}(Z)$.

Proof: Since $\{B_Z\}$ is a deformation retract of $\mathrm{Sus}(Z) - \{v\}$. The proposition follows easily because $\mathrm{Cone}(Z) - B(Z) \approx \mathrm{Sus}(Z) - \{v\}$. □

Theorem 3.12. *Let Z be a connected metric space and let $z \in Z$. Then $Z - \{z\}$ is connected if and only if $[z,t_0]_s$ does not make a hole in* $\mathrm{Sus}(Z)$ *for each $t_0 \in (0,1)$.*

Proof: Since $\{v\}$ is a deformation retract of $\mathcal{C}_1 = (Z \times [t_0,1]/Z \times \{1\}) - \{[z,t_0]_s\}$ and $\{B_Z\}$ is a deformation retract of $\mathcal{C}_2 = (Z \times [0,t_0]/B(Z)) - \{[z,t_0]_s\}$. By Corollary 2.10 both $\mathcal{C}_1$ and $\mathcal{C}_2$ have property (b). As $(Z \times \{t_0\}) - \{[z,t_0]_s\} \approx Z - \{z\}$, then $\mathcal{C}_1 \cap \mathcal{C}_2$ is connected. By Proposition 2.6 the space $\mathcal{C}_1 \cup \mathcal{C}_2$ has property (b). By Proposition 3.2 the space $\mathrm{Sus}(Z) - \{[z,t_0]_s\}$ is a normal space and by Theorem 2.13 it is unicoherent. On the other hand, let $z \in Z$ and $t_0 \in (0,1)$ such that $[z,t_0]_s$ does not make a hole in $\mathrm{Sus}(Z)$. Let $\mathcal{C}_1 = (Z \times [t_0,1]/Z \times \{1\}) - \{[z,t_0]_s\}$ and $\mathcal{C}_2 = (Z \times [0,t_0]/B(Z)) - \{[z,t_0]_s\}$. Both are closed, connected subsets of $\mathrm{Sus}(Z) - \{[z,t_0]_s\}$ and $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathrm{Sus}(Z) - \{[z,t_0]_s\}$. Thus, by the unicoherence of $\mathrm{Sus}(Z) - \{[z,t_0]_s\}$ we have that $\mathcal{C}_1 \cap \mathcal{C}_2 \approx Z - \{z\}$ is connected. □

Let Z be a connected space and let $z \in Z$. If $Z - \{z\}$ is not connected, then z is a *cut point* of Z.

Corollary 3.13. *Let Z be a connected metric space. Then Z does not have cut points if and only if no element of* $\mathrm{Sus}(Z)$ *makes a hole.*

The following result is a direct consequence of [13, Theorem 3.6, page 40].

Proposition 3.14. *Let Z be a connected metric space. Let $A = v$ or $A = [p,t]_c$ with $p \in Z$ and $t \in (0,1)$. If $\mathrm{Cone}(Z) - \{A\}$ is unicoherent, then $\mathrm{Sus}(Z) - \{A_s\}$ is unicoherent, where A_s is the class of A in $\mathrm{Sus}(Z)$.*

4. Examples

4.1 Graphs. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint, or intersect only in one or both of their end points. By a *segment* of a graph G we shall always mean one of those arcs. The end points of the segments of G are called *vertices* of G. Let $p \in G$, we say that p is of *order* n in G, written $\mathrm{ord}(p, G) = n$, provided that for every open set U such that $p \in U$, there exists open set V such that $p \in V \subset U$ and $|\mathrm{bd}(V)| = n$, see [28, Theorem 9.12, page 146]. If $\mathrm{ord}(p, G) = 1$, then p is called an *end point* of G. If $\mathrm{ord}(p, G) \geq 3$, then p is called a *ramification point* of G. By a *simple n-od*, $n \geq 3$, we mean a graph G with only one ramification point, exactly n end points and without *circles* (simple closed curves). The *complete graph K_m* is the graph with exactly m vertices such that any two vertices are joined by a segment of the graph.

Remark 4.1. A graph G is not unicoherent if and only if G contains circles.

We have the classification of the elements of $\mathrm{Cone}(G)$ that make a hole.

Theorem 4.2. *Let G be a graph and let $p \in G$.*

a) *The graph G has circles if and only if v makes a hole in $\mathrm{Cone}(G)$.*
b) *If $\mathrm{ord}(p, G) = 1$ and $t \in (0,1)$, then $[p,t]_c$ does not make a hole in $\mathrm{Cone}(G)$.*
c) *If $\mathrm{ord}(p, G) > 1$ and $t \in (0,1)$, then $[p,t]_c$ makes a hole in $\mathrm{Cone}(G)$, and*
d) *the element $[p,0]_c$ does not make a hole in $\mathrm{Cone}(G)$.*

Proof: a) If G contains circles, then by Remark 4.1 the space G is not unicoherent. By Proposition 2.17 the space $G \times [0,1)$ is not unicoherent. As $G \times [0,1) \approx \mathrm{Cone}(G) - \{v\}$ the necessity follows. If G does not contain circles, then by Remark 4.1 is the graph G unicoherent. As G is locally connected, then by Theorem 2.15 G has property (b). By Theorem 2.11 the space $G \times [0,1)$ has property (b) and by Theorem 2.13 it is unicoherent. The sufficiency follows.

b) Let $p \in G$ such that $\mathrm{ord}(p, G) = 1$. Let r be the vertex of a segment that contains the end point p. We have that $p \in \mathrm{int}(pr)$, then by Proposition 3.7 the element $[p,t]_c$ does not make hole in $\mathrm{Cone}(G)$.

c) Let $p \in G$ such that $\mathrm{ord}(p, G) > 1$. It is easy to see that there exists $r \in G$ such that pr is a free arc in G and $p, r \notin \mathrm{int}(pr)$. By Proposition 3.6 the element $[p,t]_c$ makes a hole in $\mathrm{Cone}(G)$.

d) If $p \in G$, then by Proposition 3.4 the element $[p,0]_c$ does not make a hole in $\mathrm{Cone}(G)$. $\square$

Corollary 4.3. *The set $A \in \mathrm{Cone}(G)$ makes a hole if and only if $A = v$ and G have circles, or $A = [p,t]_c$ with $\mathrm{ord}(p,G) > 1$ and $t \in (0,1)$.*

For the suspension of a graph we have the following result.

Theorem 4.4. *The set $A \in \mathrm{Sus}(G)$ makes a hole if and only if $A = [p,t]_s$ such that $G - \{p\}$ is not connected and $t \in (0,1)$.*

PROOF: If $A \in \mathrm{Sus}(G)$ makes a hole, then by Proposition 3.10 the space$A \neq B_Z$ and by Proposition 3.11 the space $A \neq v$. Thus $A = [p,t]_s$ with $t \in (0,1)$ and by Theorem 3.12 the space $G - \{p\}$ is not connected. The sufficiency follows from Theorem 3.12. □

4.2 n-cells and the Hilbert cube. In [14, Lemma 2.1, page 348] it is shown that $I^4 - \{z\}$ is unicoherent for each $z \in I^4$. We will give another proof of this statement using what was developed in this paper. The following result is easy to see.

Proposition 4.5. *The cell $I^n \approx \mathrm{Cone}(I^{n-1}) \approx \mathrm{Sus}(I^{n-1})$ for each $n > 1$.*

Lemma 4.6. *The space $I^n - \{A\}$ is unicoherent for all $n \geq 3$ and for all $A \in I^n$.*

PROOF: By Proposition 4.5 we can work with $\mathrm{Cone}(I^{n-1})$. By Theorems 2.11 and 2.13 the space $I^{n-1} \times [0,1)$ is unicoherent. So, if $A = v$, then $\mathrm{Cone}(I^{n-1}) - \{A\}$ is unicoherent. If $A = [p,0]_c$, then by Proposition 3.4 the space A does not make hole in $\mathrm{Cone}(I^{n-1})$. If $A = [p,t]_c$ with $p \in I^{n-1}$ and $t \in (0,1)$, since $n \geq 3$ then by Proposition 3.8 the space A does not make hole in $\mathrm{Cone}(I^{n-1})$. □

Corollary 4.7. *The space $\mathrm{Cone}(I^n) - \{A\}$ is unicoherent for all $A \in \mathrm{Cone}(I^n)$ and $n \geq 2$.*

Corollary 4.8. *The space $\mathrm{Sus}(I^n) - \{A\}$ is unicoherent for all $A \in \mathrm{Sus}(I^n)$ and $n \geq 2$.*

A topological space Y is said to be *homogeneous* provided that for any $p, q \in Y$, there is a homeomorphism h of Y onto Y such that $h(p) = q$.

Lemma 4.9. *The space $I^\infty - \{A\}$ is unicoherent for all $A \in I^\infty$.*

PROOF: Notice that $I^\infty - \{(0,0,0,\dots)\}$ is unicoherent. Since I^∞ is homogeneous, see [18], then $I^\infty - \{A\}$ is unicoherent for all $A \in I^\infty$. □

4.3 Manifolds.

Theorem 4.10. *Let M be a connected closed n-manifold with $n > 1$.*

a) *If M is unicoherent, then $\mathrm{Cone}(M) - \{A\}$ is unicoherent for all $A \in \mathrm{Cone}(M)$.*
b) *If M is not unicoherent, then the only element that makes a hole in $\mathrm{Cone}(M)$ is v.*

Proof: a) We have that M is locally connected, and by [12, Theorem 1.3, page 35] $M \times [0,1)$ is unicoherent. Thus $\mathrm{Cone}(M) - \{v\}$ is unicoherent. Let $A = [x,t]_c$ with $x \in M$ and $t \in [0,1)$. If $t = 0$, then by Proposition 3.4 the set A does not make a hole in $\mathrm{Cone}(M)$. Now for $t \neq 0$, let $I_0 = [a,b]$ such that $0 < a < t < b < 1$. Let V be an open neighborhood of x homeomorphic to $\mathbb{R}^n$. Then $W = \mathrm{cl}(V) \times I_0$ is homeomorphic to a closed neighborhood of A in $\mathrm{Cone}(M)$ and $W \approx I^{n+1}$. Since $n > 1$, by Lemma 4.6 the space $W - \{A\}$ is unicoherent and by Theorem 2.15 has property (b). As $\mathrm{bd}(W)$ is connected, by Theorem 2.7 the space $\mathrm{Cone}(M) - \{A\}$ has property (b). Therefore A does not make a hole in $\mathrm{Cone}(M)$.

b) By Proposition 3.9 the element v makes a hole in $\mathrm{Cone}(M)$. For $A = [x,t]_c$ with $x \in M$ and $t \in [0,1)$ the proof is the same as in a). □

Remark 4.11. The last theorem is true, if we suppose that M is an n-manifold with boundary. In the proof we consider V to be an open neighborhood of A homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Thus $W \approx I^{n+1}$.

Corollary 4.12. *Let $n > 1$. If M is a connected closed n-manifold with or without boundary, then $\mathrm{Sus}(M) - \{A\}$ is unicoherent for all $A \in \mathrm{Sus}(M)$.*

4.4 Elsa continua. A compactification of $R = [0,\infty)$ with an arc J as the remainder is called an *Elsa continuum*, this family of continua was defined by S. B. Nadler Jr. in [25]. There are uncountably many topologically different Elsa continua (see [24, page 184]), a particular example of an Elsa continuum is the familiar $\sin(1/x)$-*continuum* which is $J \cup R = \mathrm{cl}\{(x, \sin(1/x)) \in \mathbb{R}^2 \colon x \in (0,1]\}$, where J is the rectilinear arc joining the points $(0,-1)$ and $(0,1)$ in $\mathbb{R}^2$, which we denote by S_0.

Remark 4.13. If X is an Elsa continuum, then X has property (b) (see [28, 12.66, page 269]).

Now we enunciate a result (see [25, Lemma 3.1, page 330]), that will serve to distinguish the continua that we work with.

Lemma 4.14. *If X is an Elsa continuum, then X can be embedded in the plane in such a way that the remainder is $J = \{(0,y) \in \mathbb{R}^2 \colon y \in [-1,1]\}$ and the rest of the continuum is the graph of a continuous function $f_X \colon (0,1] \to [-1,1]$, denoted by G_{f_X}.*

Remark 4.15. Notice that every Elsa continuum X can also be embedded in the plane in such a way that the remainder is $J = \{(0,y) \in \mathbb{R}^2 \colon y \in [-1,1]\}$ and the rest of the continuum is the graph G_{g_X} of a continuous function $g_X \colon [-1,0) \to [-1,1]$. So, we can write an Elsa continuum X in two different ways $X = J \cup G_{f_X}$ or $X = J \cup G_{g_X}$.

Theorem 4.16. *Let $X = J \cup G_{f_X}$ be an Elsa continuum, where J and f_X are as in Lemma 4.14. Let $p \in X$.*

a) *The element* v *does not make a hole in* $\mathrm{Cone}(X)$.
b) *If* $p = f_X(1)$, *then* $[p,t]_c$ *does not make a hole in* $\mathrm{Cone}(X)$ *for every* $t \in (0,1)$.
c) *If* $p \in G_{f_X}$ *and* $p \neq f_X(1)$, *then* $[p,t]_c$ *makes a hole in* $\mathrm{Cone}(X)$ *for every* $t \in (0,1)$.
d) *If* $p \in J$, *then* $[p,t]_c$ *does not make a hole in* $\mathrm{Cone}(X)$ *for every* $t \in (0,1)$, *and*
e) *the element* $[p,0]_c$ *does not make a hole in* $\mathrm{Cone}(X)$.

Proof: a) By Remark 4.13 the space X has property (b), thus by Theorem 2.11 the space $X \times [0,1)$ has property (b). Therefore v does not make a hole in $\mathrm{Cone}(X)$.

b) Let $p = f_X(1)$. It is easy to see that p is an end point of an arc contained in G_{f_X}. By Proposition 3.7 the space $[p,t]_c$ does not make a hole in $\mathrm{Cone}(X)$ for every $t \in (0,1)$,

c) Let $p \in G_{f_X}$ such that $p \neq f_X(1)$. It is easy to see that p is contained in a free arc r_0r_1 such that $r_0, r_1 \notin \mathrm{int}(r_0r_1)$. By Proposition 3.6 we have that $[p,t]_c$ makes a hole in $\mathrm{Cone}(X)$ for every $t \in (0,1)$.

d) Let $[p,t_0]_c \in \mathrm{Cone}(X)$, where $p \in J$ and $t_0 \in (0,1)$. We consider $\mathcal{C}_1 = \{[x,t]_c \colon x \in X,\ t \in [0,t_0]\} - \{[p,t_0]_c\}$ and $\mathcal{C}_2 = \mathrm{cl}\{\mathrm{Cone}(X) - \mathcal{C}_1\} - \{[p,t_0]_c\}$, which are clearly closed connected subsets of $\mathrm{Cone}(X) - \{[p,t_0]_c\}$. Furthermore, $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathrm{Cone}(X) - \{[p,t_0]_c\}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = (X \times \{t_0\} - \{[p,t_0]_c\})$. We note that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a connected subset. By Remark 4.13 the space X has property (b), thus by Proposition 2.9 the space $\mathcal{C}_1$ has property (b). By Corollary 2.10 the space $\mathcal{C}_2$ has property (b). By Proposition 2.6 we can conclude that $\mathcal{C}_1 \cup \mathcal{C}_2$ has property (b), and by Theorem 2.13 is unicoherent. Therefore $[p,t]_c$ does not make a hole in $\mathrm{Cone}(X)$ with $t \in (0,1)$.

e) If $p \in X$, then by Proposition 3.4 the element $[p,0]_c$ does not make a hole in $\mathrm{Cone}(X)$. □

Corollary 4.17. *The set* $A \in \mathrm{Cone}(X)$ *makes a hole if and only if* $A = [p,t]_c$ *such that* $p \in G_{f_X}$ *and* $p \neq f_X(1)$ *with* $t \in (0,1)$.

For the suspension of an Elsa continuum we have the following result.

Theorem 4.18. *The set* $A \in \mathrm{Sus}(X)$ *makes a hole if and only if* $A = [p,t]_s$ *such that* $p \in G_{f_X}$ *and* $p \neq f_X(1)$ *with* $t \in (0,1)$.

Proof: For the necessity, if $A \in \mathrm{Sus}(X)$ makes a hole, then by Proposition 3.10 the space $A \neq B_Z$ and by Proposition 3.11 the space $A \neq v$. Thus $A = [p,t]_s$, with $p \in X$ and $t \in (0,1)$. By Proposition 3.14 we have that $[p,t]_c$ makes a hole in $\mathrm{Cone}(X)$. Finally by Corollary 4.17 we have that $p \in G_{f_X}$ and $p \neq f_X(1)$. The sufficiency follows by Theorem 4.16. □

4.5 Elsa circle and double Elsa circle.

Definition 4.19. Let $X = J \cup G_{f_X}$ be an Elsa continuum, where J and f_X are as in the Lemma 4.14. Let A be an arc which joins the points $(0,-1)$ and $(1, f_X(1))$, such that $X \cap A = \{(0,-1),(1,f_X(1))\}$. We call the continuum $\mathrm{EC} = X \cup A$ *Elsa circle.*

For the rest of this paper, EC will always denote an Elsa circle with X and A as in the previous definition. These continua are described in the proof of [25, Theorem 3.11, page 334], however not called by a special name. It is the paper [3] where the authors give them the name.

Definition 4.20. Let $X_1 = J \cup G_{f_{X_1}}$ and $X_2 = J \cup G_{g_{X_2}}$ be Elsa continua, where J, f_{X_1} and g_{X_2} are as in Remark 4.15. Let A be an arc which joins the points $(-1, g_{X_2}(-1))$ and $(1, f_{X_1}(1))$, such that

$$(X_1 \cup X_2) \cap A = \{(-1, g_{X_2}(-1)), (1, f_{X_1}(1))\}.$$

We call the continuum $\mathrm{DEC} = X_1 \cup X_2 \cup A$ the *double Elsa circle.*

For the rest of this paper, DEC will always denote a double Elsa circle with X_1, X_2 and A as in the previous definition. This family of continua are described in the proof of [25, Theorem 3.11, page 334]. Since there are uncountably many topologically different Elsa continua, we conclude that there are uncountably many topologically different Elsa circles and double Elsa circles.

The following remark is easy to see.

Remark 4.21. Any EC (DEC) is not unicoherent, and therefore does not have the property (b).

The next result shows the classification of the elements of Cone(EC) that makes a hole.

Theorem 4.22. *For every EC we have the following:*

a) v *makes a hole in* Cone(EC),
b) *if* $p \in J - \{(0,-1)\}$, *then* $[p,t_0]_c$ *does not make a hole in* Cone(EC) *for all* $t_0 \in (0,1)$,
c) *if* $p \in \mathrm{EC} - \{J - \{(0,-1)\}\}$, *then* $[p,t_0]_c$ *makes a hole in* Cone(EC) *for all* $t_0 \in (0,1)$, *and*
d) *if* $p \in \mathrm{EC}$, *then* $[p,0]_c$ *does not make a hole in* Cone(EC).

Proof: a) By Remark 4.21 the space EC is not unicoherent, by Proposition 2.17 the space $\mathrm{Cone(EC)} - \{v\}$ is not unicoherent. Thus v makes a hole in Cone(EC).

b) Let $[p,t_0]_c \in \mathrm{Cone(EC)}$, where $p \in J - \{(0,-1)\}$ and $t_0 \in (0,1)$. We consider $\mathcal{C}_1 = \{[x,t]_c \colon x \in X,\ t \in [0,t_0]\} - \{[p,t_0]_c\}$ and $\mathcal{C}_2 = \mathrm{cl}\{\mathrm{Cone(EC)} - \mathcal{C}_1\} - \{[p,t_0]_c\}$, which are clearly closed connected subsets of $\mathrm{Cone(EC)} - \{[p,t_0]_c\}$. Furthermore, $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathrm{Cone(EC)} - \{[p,t_0]_c\}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = (X \times \{t_0\} - \{[p,t_0]_c\}) \cup (\{(0,-1)\} \times [0,t_0]) \cup (\{(1,f_X(1))\} \times [0,t_0])$. We note that $\mathcal{C}_1 \cap \mathcal{C}_2$ is a connected subset. We have that $\mathcal{C}_1$ has property (b) (see Example 4.13) and by Corollary 2.10 the set $\mathcal{C}_2$ has property (b). By Proposition 2.6 we can conclude that $\mathcal{C}_1 \cup \mathcal{C}_2$ has property (b), and by Theorem 2.13 is unicoherent.

c) Let $[p,t_0]_c \in \mathrm{Cone(EC)}$, where $p \in \mathrm{EC} - \{J - \{(0,-1)\}\}$ and $t_0 \in (0,1)$. We have that $p \in G_{f_X} \cup A$. Let $p_0 \in G_{f_X} \cup A$ such that $p_0 \neq p$. We have that p_0p is a free arc in EC, thus by Proposition 3.6 the space $\mathrm{Cone}(\mathrm{EC}) - \{[p, t_0]_c\}$ is not unicoherent.

d) If $p \in \mathrm{EC}$, then by Proposition 3.4 the element $[p, 0]_c$ does not make a hole in Cone(EC). □

Corollary 4.23. *The space $A \in \mathrm{Cone}(\mathrm{EC})$ makes a hole if and only if $A = v$ or $A = [p, t_0]_c$ with $p \in \mathrm{EC} - \{J - \{(0, -1)\}\}$ and $t_0 \in (0, 1)$.*

By Corollary 3.13 the following result is not difficult to see.

Theorem 4.24. *If $A \in \mathrm{Sus}(\mathrm{EC})$, then $\mathrm{Sus}(\mathrm{EC}) - \{A\}$ is unicoherent.*

The following result shows the elements of Cone(DEC) that makes a hole.

Theorem 4.25. *For every DEC we have the following:*

a) *v makes a hole in Cone(DEC),*
b) *if $p \in J$, then $[p, t_0]_c$ does not make a hole in Cone(DEC) for all $t_0 \in (0, 1)$,*
c) *if $p \in \mathrm{DEC} - J$, then $[p, t_0]_c$ makes a hole in Cone(DEC) for all $t_0 \in (0, 1)$, and*
d) *if $p \in \mathrm{DEC}$, then $[p, 0]_c$ does not make a hole in Cone(DEC).*

Proof: The proof of a), c) and d) is similar to the proof of Theorem 4.22. We only prove b). Let $\mathcal{C}_1 = \{X_2 \times [0, t_0]\} - \{[p, t_0]_c\}$ and $\mathcal{C}_2 = \mathrm{cl}\{\mathrm{Cone}(\mathrm{DEC}) - \mathcal{C}_1\} - \{[p, t_0]_c\}$, notice that $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathrm{Cone}(\mathrm{DEC}) - \{[p, t_0]_c\}$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \{(J \times [0, t_0]) \cup (X_2 \times \{t_0\}) \cup (\{(-1, g_{X_2}(-1))\} \times [0, t_0])\} - \{[p, t_0]_c\}$. □

Corollary 4.26. *The set $A \in \mathrm{Cone}(\mathrm{DEC})$ makes a hole if and only if $A = v$ or $A = [p, t_0]_c$ with $p \in \mathrm{DEC} - J$ and $t_0 \in (0, 1)$.*

Similarly to Theorem 4.24, we have the following result.

Theorem 4.27. *If $A \in \mathrm{Sus}(\mathrm{DEC})$, then $\mathrm{Sus}(\mathrm{DEC}) - \{A\}$ is unicoherent.*

5. Making holes in hyperspaces

In [26] S. B. Nadler Jr. proved that the hyperspaces 2^X and $C(X)$ are unicoherent for any X, later in [27] he proved that the hyperspace suspension has property (b), and by Theorem 2.13 we have that $HS(X)$ is also unicoherent. In [22] S. Macías proved that the hyperspace $C_n(X)$ has property (b) and is unicoherent for all $n \geq 1$. In [21] S. Macías proved that the hyperspace $F_n(X)$ is unicoherent for all $n \geq 3$.

So it becomes interesting to determine the elements that make a hole in these hyperspaces. In this direction, see [1], [2] and [4], for what is known about this problem. This section is dedicated to giving an application in hyperspaces of what we have developed.

We know that if X is a locally connected, then 2^X is homeomorphic to the Hilbert cube (see [17, Theorem 11.3, page 89]). And if X is locally connected without free arcs, also $C_n(X)$ is homeomorphic to the Hilbert cube (see [22, Theorem 7.1, page 250]). So, by Lemma 4.9 we have the following two theorems.

Theorem 5.1. *If X is locally connected, then A does not make a hole in 2^X for all $A \in 2^X$.*

Theorem 5.2. *If X is locally connected without free arcs, then A does not make a hole in $C_n(X)$ for all $A \in C_n(X)$ and for each $n \in \mathbb{N}$.*

5.1 Hyperspaces of a graph. We know that a graph is locally connected space and by Theorem 5.1 none element of 2^X makes a hole.

For $C_n(X)$ we have the following partial result.

Theorem 5.3. *Let G be a graph, $R(G)$ the set of ramification points of G and $n \geq 2$. If $A \in C_n(G) - C_{n-1}(G)$ and $A \cap R(G) = \emptyset$, then A does not make a hole in $C_n(G)$.*

Proof: As G is locally connected, we have $C_n(G)$ is locally connected (see [22, Theorem 3.2, page 240]). By [15, Lemma 3.2, page 182] A has a neighborhood $\mathcal{U}$ such that $\mathcal{U} \approx I^{2n}$ in $C_n(G)$. By Lemma 4.6 $\mathcal{U} - \{A\}$ is unicoherent and by Theorem 2.15 it has property (b). As $\mathrm{bd}(\mathcal{U})$ is connected, by Theorem 2.7 $C_n(G) - \{A\}$ has property (b). Therefore A does not make a hole in $C_n(G)$. □

With a similar argument, but using [9, Lemma 4.2, page 1441] (which is also true for $n = 3$) we can prove the following.

Theorem 5.4. *Let G be a graph, $R(G)$ the set of ramification points of G and $n \geq 3$. If $A \in F_n(G) - F_{n-1}(G)$ and $A \cap R(G) = \emptyset$, then A does not make a hole in $F_n(G)$.*

5.2 Hyperspaces of cones. Let $A \subset X$ and $\delta > 0$. We consider the set $N(\delta, A) = \{x \in X\colon$ there exists $a \in A$ such that $d(a, x) < \delta\}$. Let $A, B \in 2^X$. An *order arc* from A to B is a continuous function $\alpha\colon [0,1] \to 2^X$ such that $\alpha(0) = A$, $\alpha(1) = B$, and if $0 \leq t < s \leq 1$, then $\alpha(t) \subset \alpha(s)$ and $\alpha(t) \neq \alpha(s)$.

Theorem 5.5. *Let Y be a compact metric space. We consider $X = \mathrm{Cone}(Y)$ and the hyperspace $\mathcal{H}(X) \in \{2^X, C_n(X), F_n(X)\}$ for $n \in \mathbb{N}$. Then $\mathcal{H}(X) \approx \mathrm{Cone}(Z)$, where Z is a continuum.*

Proof: The proof in the case $\mathcal{H}(X) = F_n(X)$ was made by R. M. Schori in [30, Cororally of Theorem 5, page 82]. We suppose that $\mathcal{H}(X) \in \{2^X, C_n(X)\}$. Let $Z = \{A \in \mathcal{H}(X)\colon A \cap B(Y) \neq \emptyset\}$. We prove that Z is a continuum.

Let $\{A_n\}$ be a sequence in Z such that $A_n \to A$. Suppose that $A \cap B(Y) = \emptyset$, let $\delta = (1/2)\min\{s \in [0,1]\colon [y,s]_c \in A\}$. Notice that $\delta > 0$ and $N(\delta, A) \cap B(Y) = \emptyset$. As $A_n \to A$ there exists $N \in \mathbb{N}$ such that $A_n \subset N(\delta, A)$ for every $n \geq N$. Thus, $A_n \cap B(Y) = \emptyset$ for every $n \geq N$, this is a contradiction. So Z is closed.

As for the connectedness of Z, if $A \in 2^X$, then by [26, Theorem 1.8, page 59] there exists an order arc from A to X. It is easy to see that if α is an order arc in 2^X and $\alpha(0) \in C_n(X)$, then $\alpha(t) \in C_n(X)$ for all $t \in [0,1]$. Therefore Z is connected.

Let $F\colon Z \times I \to \mathcal{H}(X)$, defined by $F(A,t) = A^t$ where

$$A^t = \{[y, (1-t)s + t]_c\colon [y,s]_c \in A\}.$$

Notice that $F(A,0)=A$ and $F(A,1)=v$ for all $A\in Z$, v is the vertex of X.
Claim 1. $F|_{Z\times[0,1)}$ is one-to-one.
Let $A_1,A_2\in Z$ and $t_1,t_2\in[0,1)$. Suppose that $F(A_1,t_1)=F(A_2,t_2)$, then $A_1^{t_1}=A_2^{t_2}$. Let $s_0=\min\{s\in[0,1)\colon [y,(1-t_1)s+t_1]_c\in A_1^{t_1}\}=\min\{s\in[0,1)\colon [y,(1-t_2)s+t_2]_c\in A_2^{t_2}\}$. As $A_1,A_2\in Z$, then $s_0=t_1$ and $s_0=t_2$. Therefore $t_1=t_2$. For A^t with $t\in[0,1)$, we define the set $(A^t)^{-t}$ by $\{[y,(s-t)/(1-t)]_c\colon [y,s]_c\in A\}$. Then $(A_1^{t_1})^{-t_1}=A_1$ and $(A_2^{t_2})^{-t_2}=A_2$. Thus $A_1=A_2$.
Claim 2. F is onto.
Let $B\in\mathcal{H}(X)$ and $s_0=\min\{s\in I\colon [y,s]_c\in B\}$. If $B=v$, then $s_0=1$. In this case, we take $y\in Y$, then $F(\{y\},1)=v$. If $B\neq v$, then $s_0<1$, thus for all $[y,s_0]_c\in B$ we have that $[y,0]_c\in B^{-s_0}\cap B(Y)$ and $F(B^{-s_0},s_0)=B$. So F is onto.

Observe that $F(A,t)=v$ if and only if $t=1$. By Transgression lemma (see [28, Exercise 3.22, page 45]) F induces a homeomorphism $\mathcal{F}\colon \mathrm{Cone}(Z)\to\mathcal{H}(X)$. □

Corollary 5.6. *Let Y be a compact metric space. We consider $X=\mathrm{Cone}(Y)$ and the hyperspace $\mathcal{H}(X)\in\{2^X,C_n(X),F_n(X)\}$ for $n\in\mathbb{N}$. If $A\in Z=\{A\in\mathcal{H}(X)\colon A\cap B(Y)\neq\emptyset\}$, then A does not make a hole in $\mathcal{H}(X)$. In particular X does not make a hole in $\mathcal{H}(X)$.*

Proof: We have that $[A,0]_c\in B(Z)$ and the result follows from Proposition 3.4. □

5.2.1 The 2nd-symmetric product of a simple $n-od$. Let X be a simple $n-od$ with $n\geq 2$. Here we consider $X\approx I$ when $n=2$. Suppose that $X=\bigcup_{i=1}^n\overline{oe_i}$ where $E(X)=\{e_1,\dots,e_n\}\subset\mathbb{R}^n$ is the standard basis of $\mathbb{R}^n$, o is the origin of $\mathbb{R}^n$ and $\overline{oe_i}$ denotes the convex segment in $\mathbb{R}^n$ that joins o and e_i. Note that $X=\mathrm{Cone}(\{e_1,\dots,e_n\})$. Let $Z=\{A\in F_2(X)\colon e_i\in A$ for some $i\in\{1,\dots,n\}\}$. In [8, Lemma 1, page 68], the author proved there exists $\widetilde{F}\colon \mathrm{Cone}(Z)\to F_2(X)$ a homeomorphism such that $\widetilde{F}(v)=\{o\}$. This homeomorphism is induced by $F\colon Z\times I\to F_2(X)$ defined as $F(A,t)=(1-t)A$, where $(1-t)A=\{(1-t)a\colon a\in A\}$. In [8, Lemma 2, page 70], the author proved that Z is the union of K_n and n pairwise disjoint arcs each of them intersecting K_n in exactly one of its vertices.

In this context, as Z is homeomorphic to a finite graph and $F_2(X)\approx\mathrm{Cone}(Z)$, a direct application of Corollary 4.3 gives the following theorem, cf. Theorem 2.7 and Theorem 3.10 in [5, pages 86 and 91].

Theorem 5.7. *Let X be a simple $n-od$. Then $A\in F_2(X)$ makes a hole if and only if either $A\cap E(X)=\emptyset$ and A has two points or $A=\{o\}$.*

5.2.2 Hyperspaces of the interval. If $Y=\{y\}$, then $I\approx\mathrm{Cone}(Y)$. By Theorem 5.5 the hyperspaces 2^X, $C_n(X)$ and $F_n(X)$ for all $n\in\mathbb{N}$, are cones over a continuum. First consider the hyperspace 2^I, then by Theorem 5.1 no element of 2^I makes hole it. For the case of $C(I)$ we have that $A\in C(I)$ makes a hole if and only if A is a free arc pq such that $p,q\notin\mathrm{int}(A)$ (see [2, Theorem 7.1, page 13]), cf. Proposition 5.15 in this paper. In [14, Lemma 2.2, page 349] the author proved that $C_2(I) \approx I^4$. Thus, by [14, Lemma 2.1, page 348] or Lemma 4.6 the set A does not make a hole in $C_2(I)$ for all $A \in C_2(I)$. Additionally, for $n \geq 3$, by Theorem 5.3, if $A \in C_n(I) - C_{n-1}(I)$, then A does not make a hole in $C_n(I)$. Let $\mathcal{H}(I) \in \{2^I, F_n(I), C_n(I)\}$ for $n \in \mathbb{N}$ and let $A \in \mathcal{H}(I)$. By Corollary 5.6 if $0 \in A$ or $1 \in A$, then A does not make a hole in $\mathcal{H}(I)$. So, we have the following.

Conjecture 5.8. *A does not make a hole in $C_n(I)$ for all $A \in C_n(I)$ and $n \geq 2$.*

On the other hand, for $F_2(I)$ by Theorem 5.7 we have that $A \in F_2(I)$ makes a hole in $F_2(I)$ if and only if $|A| = 2$ and $A \cap \{0,1\} = \emptyset$. In [7, Theorem 6, page 880] the authors prove that $F_3(I) \approx I^3$, by Lemma 4.6 the space A does not make a hole in $F_3(I)$ for all $A \in F_3(I)$. Moreover, for $n \geq 4$, by Theorem 5.4, if $A \in F_n(I) - F_{n-1}(I)$, then A does not make a hole in $F_n(I)$. So, we have the following.

Conjecture 5.9. *The space A does not make a hole in $F_n(I)$ for all $A \in F_n(I)$ and $n \geq 4$.*

5.2.3 The 2nd-fold hyperspace of S^1. Let T be a solid torus. Since T is a 3-manifold with boundary and is not unicoherent, and by Remark 4.11 the unique element of $\mathrm{Cone}(T)$ such that makes a hole is v. In [16] the author proved that $C_2(S^1) \approx \mathrm{Cone}(T)$ and in [14] the author proved that $C_2(S^1) - \{S^1\}$ is not unicoherent (in the following theorem we give an alternative proof of this), thus the unique element of $C_2(S^1)$ such that makes a hole is S^1.

Theorem 5.10. *The space $C_2(S^1) - \{S^1\}$ is not unicoherent.*

Proof: Define $S = \{\mathrm{e}^{\mathrm{i}t} \colon t \in \mathbb{R}\}$. We shall prove that $C_2(S) - \{S\}$ is not unicoherent. Let

$$\mathcal{H} = \{A_1 \cup A_2 \in C_2(S) - \{S\} \colon \mathrm{Im}(z_1 z_2) \geq 0 \text{ where } z_i \text{ is the midpoint of } A_i \text{ for } i = 1,2\},$$

and

$$\mathcal{K} = \{A_1 \cup A_2 \in C_2(S) - \{S\} \colon \mathrm{Im}(z_1 z_2) \leq 0 \text{ where } z_i \text{ is the midpoint of } A_i \text{ for } i = 1,2\}$$

where $\mathrm{Im}(z_1 z_2)$ is the imaginary part of the complex number $z_1 z_2$. It is easy to check that $\mathcal{H}$ and $\mathcal{K}$ are closed in $C_2(S) - \{S\}$. It is clear that $\mathcal{H} \neq \emptyset \neq \mathcal{K}$ and $\mathcal{H} \cup \mathcal{K} = C_2(S) - \{S\}$.

We will prove that $\mathcal{H}$ is connected. Let $P = A_1 \cup A_2 \in \mathcal{H}$ and suppose that for $j = 1,2$ the extreme points of A_j are $\mathrm{e}^{\mathrm{i}t_j^0}$ and $\mathrm{e}^{\mathrm{i}t_j^1}$, respectively. In this situation the midpoint of A_j is $z_j = \mathrm{e}^{\mathrm{i}t_j}$ where $t_j = (t_j^0 + t_j^1)/2$. Since $P \in \mathcal{H}$ then $\mathrm{Im}(\mathrm{e}^{\mathrm{i}(t_1+t_2)}) \geq 0$. For $u \in [0,1]$ and $j = 1,2$ we define $a(u) = (1-u)t_j^0 + ut_j$, $b(u) = (1-u)t_j^1 + ut_j$ and the arc A_j^u in S has extreme points $\mathrm{e}^{\mathrm{i}a(u)}$ and $\mathrm{e}^{\mathrm{i}b(u)}$. In order to see that $\mathcal{H}$ is connected it is enough to show that there exists a connected subset $\mathcal{L}$ of $\mathcal{H}$ such that P and $\{1\}$ are elements of $\mathcal{L}$. Suppose for example that

$0 \le t_1 \le t_2 \le 2\pi$, then $0 \le t_1 + t_2 \le 4\pi$. Since $\mathrm{Im}(\mathrm{e}^{\mathrm{i}(t_1+t_2)}) \ge 0$ we have that $0 \le t_1 + t_2 \le \pi$ or $2\pi \le t_1 + t_2 \le 3\pi$. We consider the following two cases:
Case I. If $0 \le t_1 + t_2 \le \pi$, then define the set

$$\begin{aligned}\mathcal{L} &= \{A_1^u \cup A_2^u \in C_2(S) - \{S\} \colon u \in [0,1]\} \\ &\quad \cup \Big\{\{\mathrm{e}^{\mathrm{i}(t_1+r)}, \mathrm{e}^{\mathrm{i}(t_2-r)}\} \colon r \in \Big[0, \frac{t_1 - t_2}{2}\Big]\Big\} \cup \Big\{\{\mathrm{e}^{\mathrm{i}r}\} \colon r \in \Big[0, \frac{t_1 + t_2}{2}\Big]\Big\}.\end{aligned}$$

Notice that $\mathcal{L}$ is an arc joining P and $\{1\}$, then $\mathcal{L}$ is arcwise connected. To see that $\mathcal{L} \subset \mathcal{H}$ take a point $Q \in \mathcal{L}$, and then we have three possibilities.

a) If $Q = \{\mathrm{e}^{\mathrm{i}(t_1+r)}, \mathrm{e}^{\mathrm{i}(t_2-r)}\}$, then we have $\mathrm{Im}(\mathrm{e}^{\mathrm{i}(t_1+r)}\mathrm{e}^{\mathrm{i}(t_2-r)}) = \mathrm{Im}(\mathrm{e}^{\mathrm{i}(t_1+t_2)}) \ge 0$.
b) If $Q = \{\mathrm{e}^{\mathrm{i}r}\}$ with $r \in [0, (u+r)/2]$, then $0 \le r \le \pi/2$. Thus $\mathrm{Im}(\mathrm{e}^{\mathrm{i}r}\mathrm{e}^{\mathrm{i}r}) = \mathrm{Im}(\mathrm{e}^{2\mathrm{i}r}) \ge 0$.
c) Suppose that $Q = A_1^u \cup A_2^u$, in this case notice that the midpoints of A_1^u and A_2^u are z_1 and z_2, respectively. Then $\mathrm{Im}(z_1 z_2) \ge 0$.

Case II. If $2\pi \le t_1 + t_2 \le 3\pi$, then we consider $l = ((t_1 + t_2)/2 - \pi)$. Thus $0 \le l \le \pi/2$. Notice that $0 \le 2\pi - t_2 \le (t_1 - t_2)/2 + \pi$. Define the set

$$\begin{aligned}\mathcal{L} &= \{A_1^u \cup A_2^u \in C_2(S) - \{S\} \colon u \in [0,1]\} \\ &\quad \cup \Big\{\{\mathrm{e}^{\mathrm{i}(t_1+r)}, \mathrm{e}^{\mathrm{i}(t_2-r)}\} \colon r \in \Big[0, (t_1 - t_2)/2 + \pi\Big]\Big\} \cup \{\{\mathrm{e}^{\mathrm{i}r}\} \colon r \in [0, l]\}.\end{aligned}$$

Notice that $\mathcal{L}$ is an arc joining P and $\{1\}$, then $\mathcal{L}$ is arcwise connected. The proof of $\mathcal{L} \subset \mathcal{H}$ is similar to the one in Case I. Therefore, in both cases $\mathcal{L}$ is arcwise connected, it lies in $\mathcal{H}$. Thus $\mathcal{H}$ is connected.

Now, define $h \colon S \to S$ by $h(z) = \mathrm{e}^{t+\pi/2}$ if $z = \mathrm{e}^{\mathrm{i}t} \in S$. It is easy to show that h is continuous. Define $H \colon C_2(S) \to C_2(S)$ by $H(A) = h(A)$, then H is continuous and $H(\mathcal{H}) = \mathcal{K}$. Thus $\mathcal{K}$ is connected.

Finally, let

$$\begin{aligned}\mathcal{A} = \{A_1 \cup A_2 \in C_2(S) - \{S\} \colon &z_1 z_2 \in \mathbb{R} \text{ and } z_1 z_2 \ge 0 \\ &\text{where } z_i \text{ is the midpoint of } A_i \text{ for } i = 1, 2\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{B} = \{A_1 \cup A_2 \in C_2(S) - \{S\} \colon &z_1 z_2 \in \mathbb{R} \text{ and } z_1 z_2 \le 0 \\ &\text{where } z_i \text{ is the midpoint of } A_i \text{ for } i = 1, 2\}.\end{aligned}$$

It is easy to see that $\mathcal{H} \cap \mathcal{K} = \mathcal{A} \cup \mathcal{B}$ is not connected. Therefore, this completes the proof, showing that $C_2(S) - \{S\}$ is not unicoherent. □

5.3 C-H continua. The following definition was introduced by S. B. Nadler Jr. in [25].

Definition 5.11. If X is such that $\mathrm{Cone}(X) \approx C(X)$, then we will say that X is a *C-H continuum*.

The set X is *decomposable* provided that X can be written as the union of two proper subcontinua. The set X is said to be *hereditarily decomposable* provided that each nondegenerate subcontinuum of X is decomposable. In [25] S. B. Nadler Jr. shows that there are exactly eight C-H continua hereditarily decomposable. These continua are:

1) the interval I;
2) the unit circle S^1;
3) the $\sin(1/x)$-*continuum* S_0;
4) the *Warsaw circle* $S_2^1 = S_0 \cup A$, where A is an arc from $(0,-1)$ to $(1,\sin(1))$, such that $S_0 \cap A = \{(0,-1),(1,\sin(1))\}$;
5) the *double Warsaw circle* $S_3^1 = S_0 \cup \{(x,\sin(1/x))\colon x \in [-1,0)\} \cup A$, where A is an arc from $(-1,\sin(-1))$ to $(1,\sin(1))$, such that
$$\left(S_0 \cup \left\{\left(x,\sin\left(\frac{1}{x}\right)\right)\colon x\in[-1,0)\right\}\right) \cap A = \{(-1,\sin(-1)),(1,\sin(1))\};$$
6) $(SP)_1$, see Example 2.14;
7) $(SP)_2 = (SP)_1 \cup \{(1-1/t)\exp(it)\colon t \geq 1\}$ with the points $2\exp(i)$ and $(0,0)$ identified;
8) $(SP)_3 = (SP)_1 \cup \{(1+1/t)\exp(it)\colon t \leq -1\}$ with the points $2\exp(i)$ and $(0,0)$ identified.

Theorem 5.12 ([11, Theorem 6.5, page 117])**.** *If X is a C-H continuum dimensionally finite, then there exists $h\colon C(X) \to \mathrm{Cone}(X)$ a homeomorphism such that $h(F_1(X)) = B(X)$.*

A couple of results that follow directly from Theorem 5.12 are:

Corollary 5.13 ([11, Theorem 6.6, page 118])**.** *If X is a C-H continuum dimensionally finite, then $HS(X) \approx \mathrm{Sus}(X)$.*

Corollary 5.14. *Let X be a C-H continuum dimensionally finite. If $h\colon HS(X) \to \mathrm{Sus}(X)$ is the homeomorphism ensuring by the previous corollary, then $h(F_X) = B_X$.*

The set X is said to have the *cone = hyperspace property* if there is a homeomorphism $h\colon \mathrm{Cone}(X) \to C(X)$ such that $h(B(X)) = F_1(X)$ and $h(v) = X$. Even more, we can have that if $[p,0]_c \in \mathrm{Cone}(X)$, then $h([p,0]_c) = \{p\}$.

By [17, Exercise 40.3, page 262] the only graphs such that have the cone = hyperspace property are I and S^1.

The following proposition is a consequence of [2, Theorem 7.1, page 13], and we give an alternative proof.

Proposition 5.15. *The space $A \in C(I)$ makes a hole if and only if $A = pq$ a free arc such that $p, q \notin \mathrm{int}(A)$.*

Proof: Let $h\colon \mathrm{Cone}(I) \to C(I)$ be a homeomorphism such that $h(B(I)) = F_1(I)$, $h(v) = I$ and $h([p,0]_c) = \{p\}$ for $p \in I$. As for the necessity, let $A \in C(I)$ such that A makes a hole in $C(X)$. Then $h^{-1}(A)$ makes a hole in $\mathrm{Cone}(I)$. By

Corollary 4.3 we have that $h^{-1}(A) = [p,t]_c$ with $\mathrm{ord}(p,I) > 1$ and $t \in (0,1)$. Thus $h^{-1}(A) \notin B(I)$, $h^{-1}(A) \neq v$ and $h^{-1}(A) \notin \{[0,t]_c \colon t \in (0,1)\} \cup \{[1,t]_c \colon t \in (0,1)\}$. By [17, Example 5.1, page 33] we have that A is a free arc. The sufficiency follows by [1, Theorem 1, page 2001]. □

Proposition 5.16. *The set $A \in C(S^1)$ makes a hole if and only if $A = S^1$ or $A = pq$ is a free* arc.

Proof: Let $h \colon \mathrm{Cone}(S^1) \to C(S^1)$ be a homeomorphism such that we have $h(B(S^1)) = F_1(S^1)$, $h(v) = S^1$ and $h([p,0]_c) = \{p\}$ for $p \in S^1$. As for the necessity, let $A \in C(S^1)$ such that A makes a hole in $C(S^1)$. Then $h^{-1}(A)$ makes a hole in $\mathrm{Cone}(S^1)$. By Corollary 4.3 we have that $h^{-1}(A) = v$ or $h^{-1}(A) = [p,t]_c$ with $\mathrm{ord}(p,I) > 1$ and $t \in (0,1)$. Thus $A = S^1$ and $h^{-1}(A) \notin B(S^1)$. As for the sufficiency, if A is a free arc, then by [1, Theorem 1, page 2001] A makes a hole in $C(S^1)$. If $A = S^1$, then since $C(S^1)$ is homeomorphic to the unit disk and A belongs to the manifold interior of $C(S^1)$, it is easy to see that S^1 makes a hole in $C(S^1)$. □

For S_0 we have the following result (see [4, Theorem 4.4, page 138]).

Proposition 5.17. *The space $A \in C(S_0)$ makes a hole if and only if $A = pq$ a free* arc *such that $p, q \notin \mathrm{int}(A)$.*

By Theorem 5.12 and Corollary 5.13 we have the following lemma.

Lemma 5.18. *For $i = 2,3$, the space* $\mathrm{Cone}(S_i^1) \approx C(S_i^1)$ *and* $\mathrm{Sus}(S_i^1) \approx HS(S_i^1)$.

But it is not all we know, for $i = 2,3$ there exists $h_i \colon \mathrm{Cone}(S_i^1) \to C(S_i^1)$ a homeomorphism such that $h_i(B(S_i^1)) = F_1(S_i^1)$ (see Theorem 5.12), $h_i(v) = J$ and $h_i(B(J)) = F_1(J)$ (see the proof of [25, Theorem 3.11, page 334]). Thus, $(p,t) \in \mathrm{Cone}(J) - \{v\}$ if and only if $h_i(p,t) \in C(J) - \{J\}$. Therefore, we can determine for $i = 2,3$, the elements that make a hole in $C(S_i^1)$ and $HS(S_i^1)$.

Theorem 5.19. *Let $i \in \{2,3\}$. A subcontinuum Y of S_i^1 makes a hole in $C(S_i^1)$ if and only if $Y = J$ or $Y \not\subseteq J$.*

Proof: Suppose that Y makes a hole in $C(S_i^1)$. Let $h_i \colon \mathrm{Cone}(S_i^1) \to C(S_i^1)$ be the homeomorphism described in the previous paragraph. We have that $h_i^{-1}(Y)$ makes a hole in $\mathrm{Cone}(S_i^1)$. For $i = 2$, by Corollary 4.23 we have that $h_i^{-1}(Y) = v$ or $h_i^{-1}(Y) = [p,t_0]_c$, with $p \in S_2^1 - \{J - \{(0,-1)\}\}$ and $t_0 \in (0,1)$. For $i = 3$, by Corollary 4.26 we have that $h_i^{-1}(Y) = v$ or $h_i^{-1}(Y) = [p,t_0]_c$, with $p \in S_3^1 - J$ and $t_0 \in (0,1)$. This implies that $Y = J$ or $Y \not\subseteq J$. The converse implication follows directly by Theorems 4.22 and 4.25. □

The corresponding theorem for $HS(S_i^1)$, where $i = 2,3$, follows from Theorems 4.24 and 4.27.

Theorem 5.20. *For $i = 2,3$, no element of $HS(S_i^1)$ makes a hole.*

Remark 5.21. We make a comparison of the circle S^1, the Warsaw circle S_2^1 and the double Warsaw circle S_3^1.

- These three continua are not unicoherent.
- The three continua are hereditarily decomposable dimensionally finite C-H continua. Thus $\mathrm{Cone}(S^1) \approx C(S^1)$, $\mathrm{Sus}(S^1) \approx HS(S^1)$, $\mathrm{Cone}(S^1_i) \approx C(S^1_i)$ and $\mathrm{Sus}(S^1_i) \approx HS(S^1_i)$ for $i = 2, 3$.
- We have that any $A \in HS(S^1)$ does not make a hole in $HS(S^1)$, and any $A \in HS(S^1_i)$ does not make a hole in $HS(S^1_i)$ for $i = 2, 3$.

From the last statement of the previous remark, the following question arises:

Question 5.22. *For which families of continua does not any element of hyperspace suspension make a hole?*

Unfortunately for $(SP)_1$, $(SP)_2$ and $(SP)_3$ we do not have an answer to this question. Basically the problem lies in the fact that these continua do not have property (b). That is why Question 2.18 is of particular importance to us. What we know is:

Proposition 5.23. *The following are true:*

a) *v makes a hole in* $\mathrm{Cone}((SP)_i)$ *for* $i = 2, 3$;
b) *if* $p \in (SP)_i - S^1$ *with* $i = 2, 3$, *then* $[p, t_0]_c$ *makes a hole in* $\mathrm{Cone}((SP)_i)$ *for all* $t_0 \in (0, 1)$;
c) *if* $p \in (SP)_1 - S^1$ *and* $p \neq 2\exp(i)$, *then* $[p, t_0]_c$ *makes a hole in* $\mathrm{Cone}((SP)_1)$ *for all* $t_0 \in (0, 1)$;
d) *if* $p \in (SP)_1 - S^1$ *and* $p = 2\exp(i)$, *then* $[p, t_0]_c$ *does not make a hole in* $\mathrm{Cone}((SP)_1)$ *for all* $t_0 \in (0, 1)$; *and*
e) *if* $p \in (SP)_i$ *with* $i = 1, 2, 3$, *then* $[p, 0]_c$ *does not make a hole in* Cone(EC).

Proposition 5.24. *The following are true:*

a) *v does not make a hole in* $\mathrm{Sus}((SP)_i)$ *for* $i = 1, 2, 3$;
b) *if* $p \in (SP)_i$ *with* $i = 2, 3$, *then* $[p, t_0]_s$ *does not make a hole in* $\mathrm{Sus}((SP)_i)$ *for all* $t_0 \in (0, 1)$;
c) *if* $p \in (SP)_1 - S^1$ *and* $p \neq 2\exp(i)$, *then* $[p, t_0]_s$ *makes a hole in* $\mathrm{Sus}((SP)_1)$ *for all* $t_0 \in (0, 1)$;
d) *if* $p \in S^1$ *or* $p = 2\exp(i)$, *then* $[p, t_0]_s$ *does not make a hole in* $\mathrm{Sus}((SP)_1)$ *for all* $t_0 \in (0, 1)$; *and*
e) $B_{(SP)_i}$ *does not make a hole in* $\mathrm{Sus}((SP)_i)$ *for* $i = 1, 2, 3$.

Corollary 5.25. *If* $A \in \mathrm{Sus}(SP)_i$ *for* $i = 2, 3$, *then* $\mathrm{Sus}(SP)_i - \{A\}$ *is unicoherent.*

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J. G. Anaya, E. Castañeda-Alvarado, A. Fuentes-Montes de Oca, F. Orozco-Zitli:
Universidad Autónoma del Estado de México, Facultad de Ciencias, Instituto Literario No. 100, Col. Centro, C. P. 50000, Toluca, Estado de México, México

E-mail: jgao@uaemex.mx
eca@uaemex.mx
afuentesm@uaemex.mx
foz@uaemex.mx

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