The small Ree group ${}^2G_2(3^{2n+1})$ and related graph

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Abstract. Let G be a finite group. The main supergraph $\mathcal{S}(G)$ is a graph with vertex set G in which two vertices x and y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. In this paper, we will show that $G \cong {}^2G_2(3^{2n+1})$ if and only if $\mathcal{S}(G) \cong \mathcal{S}({}^2G_2(3^{2n+1}))$. As a main consequence of our result we conclude that Thompson's problem is true for the small Ree group ${}^2G_2(3^{2n+1})$.

Keywords: main supergraph; simple Ree group; Thompson's problem

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1. Introduction

Let G be a finite group and $x \in G$. The order of x is denoted by $o(x)$. The set of all element orders of G is denoted by $\pi_e(G)$ and the set of all prime factors of $|G|$ is denoted by $\pi(G)$. It is clear that the set $\pi_e(G)$ is closed and partially ordered by divisibility, and hence it is uniquely determined by $\mu(G)$, the subset of its maximal elements. Let $i \in \pi_e(G)$. Set $m_i = m_i(G) = |\{g \in G\colon o(g) = i\}|$, and nse$(G)$= $\{m_k(G)\colon k \in \pi_e(G)\}$ be the set of the numbers of elements with the same order.

We define the graph $\mathcal{S}(G)$ with vertex set G such that two vertices x and y are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. This graph is called *main supergraph* of *power graph* G and was introduced in [8]. The power graph $\mathcal{P}(G)$ of a group G is the graph with group elements as vertex set and two elements are adjacent if one is a power of the other. The main properties of this graph were investigated by P. J. Cameron in [3] and I. Chakrabarty et al. in [4]. The *proper main supergraph* $\mathcal{S}^*(G)$ is the graph constructed from $\mathcal{S}(G)$ by removing the identity element of G. We write $x \sim y$ when two vertices x and y are adjacent.

We say that groups G_1 and G_2 are of the same *order type* if and only if $m_t(G_1) = m_t(G_2)$ for all t. By the definition of the main supergraph, it is clear that if G_1 and G_2 are groups with the same order type, then $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. The converse of this result is not generally correct. To prove this, we consider $G_1 = C_4 \times C_4$ and $G_2 = C_2 \times C_2 \times C_4$. Since G_1 and G_2 are 2-groups, we have $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. But $m_4(G_1) = 12 > 8 = m_4(G_2)$ and $m_2(G_1) = 3 < 7 = m_2(G_2)$.

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In 1987, J. G. Thompson, see [9, Problem 12.37], posed the following problem: **Thompson's problem.** Suppose that G_1 and G_2 are two groups of the same order type. If G_1 is solvable, is it true that G_2 is also necessarily solvable?

Obviously, if G_1 and G_2 are the same order type, then $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$ and $|G_1| = |G_2|$. Therefore, if a group G has been uniquely determined by its order and $\mathrm{nse}(G)$, then Thompson's problem is true for G. In [6], the authors proved that Thompson's problem is true for the small Ree group ${}^2G_2(q)$, where $q \pm \sqrt{3q} + 1$ is a prime number ($q = 3^{2n+1}$ and n is a natural number) by its nse and order.

Clearly, for two groups G_1 and G_2 that are the same order type, we have $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. Therefore, if a group G has been uniquely determined by $\mathcal{S}(G)$, then Thompson's problem is true for G. If G is the alternating group of degrees p, $p+1$ or $p+2$ or the symmetric group of degree p, where p is prime, then it is proved that these groups are uniquely determined by their main supergraph, see [1]. Also, in [2], it is proved that the groups $\mathrm{PSL}_2(p)$, $\mathrm{PGL}_2(p)$, where p is prime, and all of the sporadic simple groups are uniquely determined by their main supergraph. In this paper, we remove the assumption $q \pm \sqrt{3q}+1$ is a prime number in [6] and as the main result, conclude that Thompson's problem is true for ${}^2G_2(q)$. In fact, we prove the following theorem.

Theorem 1.1. *Let G be a finite group. If $\mathcal{S}(G) \cong \mathcal{S}({}^2G_2(3^{2n+1}))$, where n is a natural number, then $G \cong {}^2G_2(3^{2n+1})$.*

As noted above, as an immediate consequence of Theorem 1.1, we have that

Corollary 1.2. *If G is a finite group with the same type as ${}^2G_2(3^{2n+1})$, then G is isomorphic to ${}^2G_2(3^{2n+1})$.*

We construct the *prime graph* of G, which is denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if G has an element of order pq, $p \neq q$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, \ldots, \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we always suppose $2 \in \pi_1$.

Throughout this paper we denote by $\varphi(n)$, where n is a natural number, Euler's totient function. Let r be a prime number and $\mathrm{Syl}_r(G)$ be the set of Sylow r-subgroups of group G. We denote by P_r a Sylow r-subgroup of G and $n_r(G)$ is the number of Sylow r-subgroups of G, that is, $n_r(G) = |\mathrm{Syl}_r(G)|$. The other notations and terminologies in this paper are standard, and the reader is referred to [14] if necessary.

2. Preliminary results

We first quote some lemmas that are used in deducing the theorem of this paper.

Lemma 2.1 ([7])**.** *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \colon g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.2 ([11])**.** *Let R be the small Ree group ${}^2G_2(3^{2n+1})$, where n is a natural number. Then $\pi_e(G)$ exactly consists of divisors of 6, 9, $q-1$, $(q+1)/2$ and $q \pm \sqrt{3q}+1$.*

Lemma 2.3. *Let H be a finite simple group. Then $5 \nmid |H|$ holds if and only if H is isomorphic to one of the following simple groups:*

(a) Z_p, $p \neq 5$;
(b) $\mathrm{PSL}_n(q)$, $n = 2, 3$, *where* $q = p^f$ *(f is odd),* $p \neq 5$ *and* $p \neq 5k \pm 1$ *for some* $k > 0$;
(c) $G_2(q)$, *where* $q = p^f$ *(f is odd),* $p \neq 5$ *and* $p \neq 5k \pm 1$ *for some* $k > 0$;
(d) $\mathrm{PSU}_3(q)$, *where* $q = p^f$ *(f is odd),* $p \neq 5$ *and* $p \neq 5k \pm 1$ *for some* $k > 0$;
(e) ${}^3D_4(q)$, *where* $q = p^f$ *(f is odd),* $p \neq 5$ *and* $p \neq 5k \pm 1$ *for some* $k > 0$;
(f) ${}^2G_2(3^{2n+1})$, *where n is a natural number.*

Proof: See [15, Lemma 2.5] or [10]. □

Definition 2.1. A finite group G is a *Frobenius group* if it has a proper nontrivial subgroup H such that $H \cap H^g = 1$ for all $g \in G - H$. The subgroup H with these properties is called a *Frobenius complement* of G. The *Frobenius kernel* of G, with respect to H, is defined by $K = (G - \bigcup_{g\in G} H^g) \cup \{1\}$. A group G is a 2-*Frobenius group* if there exists a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively.

We quote some known results about Frobenius group and 2-Frobenius group which are useful in the sequel.

Lemma 2.4 ([5])**.** *Let G be a 2-Frobenius group of even order, i.e., G is a finite group and has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then:*

(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ *and* $\pi_2 = \pi(K/H)$;
(b) G/K *and* K/H *are cyclic,* $|G/K| \mid (|K/H|-1)$, $(|G/K|, |K/H|) = 1$ *and* $G/K \leq \mathrm{Aut}(K/H)$.

Lemma 2.5 ([5])**.** *Suppose that G is a Frobenius group of even order and H, K are the Frobenius kernel and the Frobenius complement of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$ and $\pi(K)$.*

Lemma 2.6 ([13, Theorem A])**.** *If G is a finite group such that $t(G) \geq 2$, then G has one of the following structures:*

(a) *G is a Frobenius group or a 2-Frobenius group;*
(b) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1$ and K/H is a non-abelian simple group. In particular, H is nilpotent, $G/K \leq \mathrm{Out}(K/H)$ and the odd order components of G are the odd order components of K/H.*

3. Proof of Theorem 1.1

In this section, $q = 3^{2n+1}$, where n is a natural number. Now, we prove the theorem stated in the introduction.

PROOF: By the definition of the main supergraph and our assumption, we have $|G| = |{}^2G_2(q)|$ (note that $|{}^2G_2(q)| = q^3(q^3+1)(q-1)$, by [14, page 137]). Also, by $\mathcal{S}({}^2G_2(q)) \cong \mathcal{S}(G)$ and the definition of the proper main supergraph, we have $\mathcal{S}^*({}^2G_2(q)) \cong \mathcal{S}^*(G)$.

We will show that $q-\sqrt{3q}+1$, $q+\sqrt{3q}+1$, $q-1$ (or $q+1$) are all mutually coprime. Let $r \mid (q-\sqrt{3q}+1)$ and $r \mid (q-1)$, where r is a prime number. Since $r \mid (q-\sqrt{3q}+1)$, we have $r \mid (q-\sqrt{3q}+1)(q+\sqrt{3q}+1) = q^2-q+1 = q^2-(q-1)$. On the other hand, $r \mid (q-1)$. It follows that $r \mid q^2$, which is a contradiction. Similarly, if $r \mid (q+\sqrt{3q}+1)$ and $r \mid (q-1)$, then we get a contradiction.

Now, let $r \mid (q-\sqrt{3q}+1)$ and $r \mid (q+1)$, where r is a prime number. Since $r \mid (q+\sqrt{3q}+1)$, we have $r \mid (q+\sqrt{3q}+1)(q-\sqrt{3q}+1) = q^2-q+1 = q^2+2-(q+1)$. Therefore, $r \mid (q^2+2)$. On the other hand, $r \mid (q+1)$. It follows that $r \mid (q^2+q)$. Since $r \mid (q^2+2)$ and $r \mid (q^2+q)$, we have $r \mid (q-2)$. Hence, $r \mid 3$. Because $q = 3^{2n+1}$ and $r \mid (q+1)$, we get a contradiction. Similarly, if $r \mid (q+\sqrt{3q}+1)$ and $r \mid (q-1)$, then we get a contradiction.

By Lemma 2.2, $\mu({}^2G_2(q)) = \{6,\ 9,\ q-1,\ (q+1)/2,\ q \pm \sqrt{3q}+1\}$. Thus ${}^2G_2(q)$ has not any element of order rp, where $r \in \pi(q^3(q^2-1)(q+\sqrt{3q}+1))$ and $p \in \pi(q-\sqrt{3q}+1)$. Also it has not any element of order rp, where $r \in \pi(q^3(q^2-1)(q-\sqrt{3q}+1))$ and $p \in \pi(q+\sqrt{3q}+1)$. It follows that $\mathcal{S}^*(G)$ is a disconnected graph with three connected components. We denote them by T_+, T_- and T_0 such that the vertices of T_+ are elements $x \in G$ with $o(x) \mid (q+\sqrt{3q}+1)$, the vertices of T_- are elements $x \in G$ with $o(x) \mid (q-\sqrt{3q}+1)$ and the vertices of T_0 are elements $x \in G$ with $o(x) \mid q^3(q^2-1)$.

Let x be an arbitrary vertex of T_+ such that $o(x) = r$, where r is a prime and let y be an arbitrary vertex of T_- such that $o(y) = s$, where s is a prime. If $rs \in \pi_e(G)$, then there exists $z \in G$ such that $o(z) = rs$. By definition of $\mathcal{S}^*(G)$, we have $x \sim z$ and $y \sim z$. Thus T_+ and T_- are connected in $\mathcal{S}^*(G)$, a contradiction. It follows that $rs \notin \pi_e(G)$. Therefore, r and s are not joined by an edge in prime graph G. Similarly, we can prove it for T_+ and T_0 and also for T_- and T_0. Thus $\Gamma(G)$ has at least three connected components.

Since $t(G) \geq 3$, Lemmas 2.4 (a) and 2.5 show that G is neither a Frobenius group nor a 2-Frobenius group. By Lemma 2.6, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a non-abelian simple group, $|G/K|$ divides $|\mathrm{Out}(K/H)|$.

By Lemma 2.3, $|G| = |{}^2G_2(q)|$ is coprime to 5. Again by Lemma 2.3, since $|K/H| \mid |G|$, K/H is isomorphic to one the following groups: $\mathrm{PSL}_2(f)$, $\mathrm{PSL}_3(f)$, $\mathrm{PSU}_3(f)$, $G_2(f)$, ${}^3D_4(f)$, where $f \equiv \pm 2 \pmod 5$ (f is a power of prime p) and ${}^2G_2(f)$, where $f = 3^{2m+1} \geq 27$.

We will show that the order of a Sylow 2-subgroup of G is 8. As noted at the beginning of the proof, $|G| = |{}^2G_2(q)| = q^3(q^3+1)(q-1) = q^3(q^2-q+1)(q^2-1)$, where $q = 3^{2n+1}$. Clearly, $2 \nmid q^3(q^2-q+1)$. Since $q^2-1 = (q-1)(q+1) = (3^{2n+1}-1)(3^{2n+1}+1) = 8(3^{2n}+3^{2n-1}+\cdots+1)(3^{2n}-3^{2n-1}+3^{2n-2}-\cdots+1)$ and $2 \nmid (3^{2n}+3^{2n-1}+\cdots+1)(3^{2n}-3^{2n-1}+3^{2n-2}-\cdots+1)$, we have $|P_2| = 8$.

By [14, Sections 4.3.3, 4.6.2], $|G_2(f)| = f^6(f+1)^2(f-1)^2(f^2-f+1)(f^2+f+1)$ and $|{}^3D_4(f)| = f^{12}(f^8+f^4+1)(f^6-1)(f^2-1)$. Clearly, the order of a Sylow 2-subgroup of $G_2(f)$ or ${}^3D_4(f)$ is greater than 8. Therefore, we can rule out the cases $G_2(f)$ and ${}^3D_4(f)$.

If K/H is isomorphic to $\mathrm{PSL}_3(f)$ or $\mathrm{PSU}_3(f)$, then the order of K/H is divisible by $(f \pm 1)(f^2-1)$. When f is odd, this is always divisible by 16 and so f must be even. Thus K/H is isomorphic to one of the groups: $\mathrm{PSL}_2(f)$ with $f \equiv \pm 2 \pmod 5$, $\mathrm{PSL}_3(2^u)$ with $u \geq 2$, $\mathrm{PSU}_3(2^u)$ with $u \geq 2$ and ${}^2G_2(f)$. Since 16 divides the order of $\mathrm{PSL}_3(2^u)$, $\mathrm{PSU}_3(2^u)$, K/H is isomorphic to $\mathrm{PSL}_2(f)$ or ${}^2G_2(f)$.

Let K/H be isomorphic to $\mathrm{PSL}_2(f)$ and let $f = p^m$, where p is a prime number and m a natural number. By the above discussion, $q \pm \sqrt{3q}+1$ are odd order components of K/H.

If $p = 2$, then $f+1$ and $f-1$ are the odd order components of $\mathrm{PSL}_2(f)$, so $q+\sqrt{3q}+1 = f+1$ and $q-\sqrt{3q}+1 = f-1$, which is impossible.

If $p \neq 2$, then the odd order components of $\mathrm{PSL}_2(f)$ are f and $(f \pm 1)/2$. Thus $q+\sqrt{3q}+1 = f$ and $q-\sqrt{3q}+1 = (f+1)/2$, or $q+\sqrt{3q}+1 = f$ and $q-\sqrt{3q}+1 = (f-1)/2$.

If the latter case holds, $q - 3\sqrt{3q}+2 = 0$. This equation has no solutions in positive integer. Then the former case occur in which we have that $q - 3\sqrt{3q} = 0$. It follows that $q = 27$ and $f = 37$. Therefore, $K/H = \mathrm{PSL}_2(37)$. In this case $\mathcal{S}^*({}^2G_2(27))$ has three components such that two components are complete graphs (T_+ and T_-). We show that the vertices of T_+ or T_- are elements of order $37 = 27+\sqrt{3\cdot 27}+1$. We know that order of T_+ or T_- is $m_{37} = 1633531536$.

First, let x and y be two vertices of T_+ or T_- such that $o(x) = r$ and $o(y) = s$, where $r \neq s$ and $r,\ s \in \pi(G)$. Since T_+ and T_- are complete, we have $x \sim y$, a contradiction. Let r be a prime and the vertices of T_+ or T_- be all of $x \in G$ such that $o(x) = r, r^2, \ldots,$ or r^k (note that $\exp(P_r) = r^k$). Then with considering $m = |P_r|$ in Lemma 2.1, $|P_r| \mid (1+m_r+m_{r^2}+\cdots+m_{r^k}) = 1+m_{37} = 1633531537$. It follows that $r = 37$. Hence, the vertices of T_+ or T_- are $x \in G$ such that $o(x) = 37^k$, where $k \geq 1$ is an integer.

Since $|G| = 2^3\cdot 3^9\cdot 7\cdot 13\cdot 19\cdot 37$, we have $37^2 \notin \pi_e(G)$. Therefore, the vertices of T_+ or T_- are all of elements of order 37 in G. Therefore, G has not any element of order $37r$, where $r \in \pi(G)$.

By Lemma 2.6 (b), $|G/K|$ divides $|\mathrm{Out}(K/H)| = |\mathrm{Out}(PSL_2(37))| = 2$. Since $|K/H| = |\mathrm{PSL}_2(37)| = 2^2\cdot 3^2\cdot 19\cdot 37$, $|G| = |{}^2G_2(27)| = 2^3\cdot 3^9\cdot 7\cdot 13\cdot 19\cdot 37$ and $|G| = |G/K|\cdot|K/H|\cdot|H|$, we have $|H| = 3^7\cdot 7\cdot 13$ or $2\cdot 3^7\cdot 7\cdot 13$. Thus $|H| \mid 2\cdot 3^7\cdot 7\cdot 13$. Since $H \trianglelefteq G$, we have $n_{13}(H) = n_{13}(G) = m_{13}(G)/12$. Since G has not any element of order $37\cdot 13$, P_{37} acts fixed point freely on the elements of order 13. Thus $37 = |P_{37}| \mid m_{13}(G) = m_{13}(H)$. By Sylow's theorem $n_{13}(H) \mid |H|$. This implies that $2^2\cdot 3^9\cdot 7\cdot 19\cdot 37 \leq m_{13}(G) = m_{13}(H) < 2\cdot 3^7\cdot 7\cdot 13$, which is a contradiction.

By the above discussion, K/H is isomorphic to ${}^2G_2(f)$, where $f = 3^{2m+1}$ and m is a natural number. Hence, $t(K/H) = 3$ and $f \pm \sqrt{3f}+1$ are odd order components of K/H (see [12], Table Id). On the other hand, $q \pm \sqrt{3q}+1$ are also the odd order components of K/H. This implies that $q \pm \sqrt{3q}+1 = f \pm \sqrt{3f}+1$. Consequently, $f = q$. Therefore, $K/H \cong {}^2G_2(q)$. Since $|G| = |K/H| = |{}^2G_2(q)|$, we deduce that $G \cong {}^2G_2(q)$. $\square$

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