Classification of spaces of continuous functions on ordinals

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Abstract. We conclude the classification of spaces of continuous functions on ordinals carried out by Górak [Górak R., *Function spaces on ordinals*, Comment. Math. Univ. Carolin. **46** (2005), no. 1, 93–103]. This gives a complete topological classification of the spaces $C_p([0,\alpha])$ of all continuous real-valued functions on compact segments of ordinals endowed with the topology of pointwise convergence. Moreover, this topological classification of the spaces $C_p([0,\alpha])$ completely coincides with their uniform classification.

Keywords: space of continuous functions; pointwise topology; homeomorphism of function spaces; uniform homeomorphism; ordinal number

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1. Introduction

Our terminology basically follows [4]. In particular, we understand cardinals as initial ordinals, compare [4, page 6]. A segment of the ordinals $[0,\alpha]$ is endowed with a standard order topology. The symbol $C_p([0,\alpha])$ denotes the set of all continuous real-valued functions defined on $[0,\alpha]$ and endowed with the topology of pointwise convergence.

A complete linear topological classification of Banach spaces $C([0,\alpha])$ was carried out in [7] and independently in [8] (for the initial part of this classification, see also [3] and [9]). Similar complete linear topological classification for $C_p([0,\alpha])$ can be found in [6], [2].

The topological classification of the spaces $C_p([0,\alpha])$ is carried out in the R. Górak's paper [5], in which the question whether the spaces $C_p([0,\alpha])$ and $C_p([0,\beta])$ are homeomorphic is solved for all ordinals α and β with except for the case $\alpha = k^+ \cdot k$, $\beta = k^+ \cdot k^+$, where k is the initial ordinal, and k^+ is the smallest initial ordinal greater than k. We note that an ordinal of the form k^+ is always regular ordinal. In this paper we prove the following theorem.

Theorem 1. *Let τ be an arbitrary initial regular ordinal, σ and λ be initial ordinals satisfying the inequality $\omega \le \sigma < \lambda \le \tau$. Then the space $C_p([0,\tau\cdot\sigma])$ is not homeomorphic to the space $C_p([0,\tau\cdot\lambda])$.*

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If we combine this result with the results of [5], we get a complete topological classification of the spaces $C_p([0,\alpha])$ (which coincides with the uniform classification). We can write it in the form of the following theorem.

Theorem 2. *Let α and β be ordinals and $\alpha \le \beta$.*

(a) *If $|\alpha| \ne |\beta|$, then $C_p([0,\alpha])$ and $C_p([0,\beta])$ are not homeomorphic.*
(b) *If τ is an initial ordinal, $|\alpha| = |\beta| = \tau$ and either $\tau = \omega$ or τ is a singular ordinal or $\beta \ge \alpha \ge \tau^2$, then the spaces $C_p([0,\alpha])$ and $C_p([0,\beta])$ are (uniformly) homeomorphic.*
(c) *If τ is a regular uncountable ordinal and $\alpha, \beta \in [\tau, \tau^2]$, then the space $C_p([0,\alpha])$ is (uniformly) homeomorphic to the space $C_p([0,\beta])$ if and only if $\tau \cdot \sigma \le \alpha \le \beta < \tau \cdot \sigma^+$, where σ is the initial ordinal, $\sigma < \tau$, and σ^+ is the smallest initial ordinal, exceeding σ.*

2. Proof of Theorem 1

We need some notation and auxiliary statements. For an arbitrary ordinal α and the initial ordinal $\lambda \le \alpha$ we set

$$A_{\lambda,\alpha} = \{t \in [0,\alpha] \colon \chi(t) = |\lambda|\},$$

where $\chi(t)$ is the character of the point $t \in [0,\alpha]$. In particular, $A_{\omega,\alpha}$ is the set of all limit points of $t \in [0,\alpha]$, having a countable base of neighborhoods.

Let α be a limit ordinal. The smallest order type of sets $A \subset [0,\alpha]$ cofinal in $[0,\alpha)$, is called *cofinality* of the ordinal α and denoted by $\mathrm{cf}(\alpha)$.

It is easy to see that $|\mathrm{cf}(\alpha)| = \chi(\alpha)$ for the limit ordinal α. The initial ordinal α is called *regular* if $\mathrm{cf}(\alpha) = \alpha$. Otherwise, the initial ordinal is called *singular*.

The symbol $D(x)$ denotes the set of points of discontinuity of the function x.

The proof of the following two lemmas is standard (see Example 3.1.27 in [4]).

Lemma 1. *Let α be an arbitrary ordinal and let τ be an initial ordinal such that $\omega < \tau \le \alpha$, $t_0 \in A_{\tau,\alpha}$ and a function $x\colon [0,\alpha] \to \mathbb{R}$ is continuous at all points of the set $A_{\omega,\alpha}$. Then there is an ordinal $\gamma < t_0$ such that $x|_{(\gamma,t_0)} = const$.*

Lemma 2. *If a function $x\colon [0,\alpha] \to \mathbb{R}$ is continuous at all points of the set $A_{\omega,\alpha}$, then the set $D(x)$ is at most countable.*

For the function $x \in \mathbb{R}^{[0,\alpha]}$ and the initial ordinal $\lambda \le \alpha$ the symbol $G_\lambda(x)$ denotes the family

$$G_\lambda(x) = \left\{ \bigcap_{s \in S} V_s \colon V_s \text{ is standard neighborhoods of } x \text{ in } \mathbb{R}^{[0,\alpha]} \text{ and } |S| = |\lambda| \right\}.$$

The elements of the family $G_\lambda(x)$ will be called λ-*neighborhoods of the function* x.

For a regular ordinal $\tau \geq \omega_1$ and an initial ordinal $\sigma \leq \tau$ we put

$$M_{\tau\sigma} = \{x \in \mathbb{R}^{[0,\tau\cdot\sigma]} \colon x \text{ is continuous at those points } t \in [0,\tau\cdot\sigma] \text{ for which } \mathrm{cf}(t) < \tau\}.$$

It is clear that $C([0,\tau\cdot\sigma]) \subset M_{\tau\sigma}$.

Lemma 3. *Let $\tau \geq \omega_1$ be an initial regular ordinal and let σ be an initial ordinal such that $\sigma \leq \tau$. Then*

$$M_{\tau\sigma} = \{x \in \mathbb{R}^{[0,\tau\cdot\sigma]} \colon V \cap C_p([0,\tau\cdot\sigma]) \neq \emptyset \text{ for every } V \in G_\lambda(x) \text{ and each } \lambda < \tau\}.$$

Proof: We denote by $L_{\tau\sigma}$ the right-hand side of the equality and assume that $x \notin M_{\tau\sigma}$, that is, x is discontinuous at some point t_0 for which $\mathrm{cf}(t_0) < \tau$. Since $|\mathrm{cf}(t_0)| = \chi(t_0)$, there exists a base $\{U_j(t_0)\}_{j\in J}$ of neighborhoods of the point t_0 such that $|J| < \tau$. Since x is discontinuous at t_0, there exists a number $\varepsilon_0 > 0$ such that for each $j \in J$ there is a point $t_j \in U_j(t_0)$ such that $|x(t_j) - x(t_0)| \geq \varepsilon_0$. Let $V = \bigcap\{V(x,t_j,t_0,1/n) \colon j \in J,\ n \in \mathbb{N}\}$, where $V(x,t_j,t_0,1/n)$ is the standard neighborhood of the function x in the space $\mathbb{R}^{[0,\tau\cdot\sigma]}$. If $y \in V$, then $y(t_j) = x(t_j)$ and $y(t_0) = x(t_0)$. Hence, the function y is discontinuous at the point t_0 and then $y \notin C_p([0,\tau\cdot\sigma])$. Thus, $V \cap C_p([0,\tau\cdot\sigma]) = \emptyset$, that is, $x \notin L_{\tau\sigma}$.

Now let $x \in M_{\tau\sigma}$, i.e. the function x can be discontinuous only at the points of the set $A_{\tau,\tau\cdot\sigma}$. It is easy to see that the set $A_{\tau,\tau\cdot\sigma}$ has the form

$$A_{\tau,\tau\cdot\sigma} = \{\tau\cdot(\xi+1) \colon 0 \leq \xi < \sigma\}, \text{ or}$$
$$A_{\tau,\tau\cdot\sigma} = \{\tau\cdot(\xi+1) \colon 0 \leq \xi < \tau\} \cup \{\tau\cdot\tau\} \quad \text{if } \sigma = \tau.$$

By Lemma 2, the set $D(x)$ is at most countable and therefore

$$A_{\tau,\tau\cdot\sigma} \cap D(x) = \{\tau\cdot(\xi_n+1) \colon \xi_n < \sigma,\ n \in \mathbb{N}\}, \text{ or}$$
$$A_{\tau,\tau\cdot\sigma} \cap D(x) = \{\tau\cdot(\xi_n+1) \colon \xi_n < \tau,\ n \in \mathbb{N}\} \cup \{\tau\cdot\tau\} \quad \text{if } \sigma = \tau.$$

Let $\lambda < \tau$ and $V(x) = \bigcap\{U(x,\eta,1/n) \colon \eta \in S,\ n \in \mathbb{N}\}$ be a λ-neighbourhood of the point x. Then $|S| < |\tau|$.

Since the countable set $A_{\tau,\tau\cdot\sigma} \cap D(x)$ is not cofinal in the regular ordinal $\tau \geq \omega_1$, for each $n \in \mathbb{N}$ there is an ordinal γ_n such that $\tau\xi_n < \gamma_n < \tau(\xi_n+1)$ and $(\gamma_n, \tau(\xi_n+1)) \cap S = \emptyset$. In the case $\sigma = \tau$ there is also an ordinal $\gamma_0 < \tau^2$, such that $(\gamma_0, \tau^2) \cap S = \emptyset$ and $(\gamma_0, \tau^2) \cap \{\tau(\xi_n+1)\}_{n=1}^\infty = \emptyset$.

Consider the function

$$\tilde{x}(t) = \begin{cases} x(\tau(\xi_n+1)) & \text{if } t \in (\gamma_n, \tau(\xi_n+1)); \\ x(\tau^2) & \text{if } t \in (\gamma_0, \tau^2); \\ x(t) & \text{otherwise.} \end{cases}$$

It is not difficult to see that the function $\tilde{x}$ is continuous at all points $t \in [0,\tau\cdot\sigma]$, and since $\tilde{x}|_S = x|_S$, $\tilde{x} \in V(x)$, that is, $V(x) \cap C_p([0,\tau\cdot\sigma]) \neq \emptyset$ and therefore $x \in L_{\tau\sigma}$. $\square$

If X is a Tychonoff space, then the symbol νX denotes the Hewitt completion of the space X. The proof of the following lemma can be found in [4, page 218].

Lemma 4. *If $\varphi\colon X \to Y$ is a homeomorphism of Tychonoff spaces, then there exists a homeomorphism $\widetilde{\varphi}\colon \nu X \to \nu Y$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for each $x \in X$.*

Lemma 5. *Let α be an arbitrary ordinal. Then*

$$\nu(C_p([0,\alpha])) = \{x \in \mathbb{R}^{[0,\alpha]} \colon x \text{ is continuous at all points of the set } A_{\omega,\alpha}\}.$$

Proof: It is known (see [10, page 382]) that for an arbitrary Tychonoff space X the space $\nu(C_p(X))$ coincides with the set of all strictly $\aleph_0$-continuous functions from X to $\mathbb{R}$. In this case, the function $f \in \mathbb{R}^X$ is called strictly $\aleph_0$-continuous (see [1]) if for any countable set $A \subset X$ there is a continuous function $g \in \mathbb{R}^X$ such that $f|_A = g|_A$.

Since for each countable set $A \subset [0,\alpha]$, its closure $\bar{A}$ is also countable, by the Tietze-Urysohn theorem we obtain that the set of all strictly $\aleph_0$-continuous functions in $[0,\alpha]$ in $\mathbb{R}$ coincides with the set of all those functions that are continuous on each countable subset $A \subset [0,\alpha]$. It is easy to see that these are precisely all those functions that are continuous at all points of the set $A_{\omega,\alpha}$. $\square$

Corollary 6. *If $\tau \geq \omega_1$ is the initial regular ordinal and $\sigma \leq \tau$ is the initial ordinal, then $M_{\tau\sigma} \subset \nu(C_p([0,\tau\cdot\sigma]))$.*

For the initial ordinal σ we denote by Γ_σ the discrete space of cardinality $|\sigma|$ and consider the space

$$c_0(\Gamma_\sigma) = \left\{x \in \mathbb{R}^{\Gamma_\sigma} \colon \{t \in \Gamma_\sigma \colon |x(t)| \geq \varepsilon\} \text{ is finite for any } \varepsilon > 0\right\}.$$

Lemma 7. *Let $\tau \geq \omega_1$ be an initial regular ordinal, $\sigma \leq \tau$ be an initial ordinal. Then there exists a homeomorphic embedding $f\colon c_0(\Gamma_\sigma) \to M_{\tau\sigma}$ such that $f(0) = 0$ and $f(x) \in M_{\tau\sigma} \setminus C_p([0,\tau\cdot\sigma])$ if $x \neq 0$.*

Proof: We enumerate the points of the set Γ_σ by the ordinals $\xi \in [0,\sigma)$. Then $\Gamma_\sigma = \{t_\xi\}_{\xi\in[0,\sigma)}$. For each characteristic function $\chi_{\{t_\xi\}} \in c_0(\Gamma_\sigma)$ we put $f(\chi_{\{t_\xi\}}) = \chi_{\{\tau(\xi+1)\}}$. It is obvious that $\chi_{\{\tau(\xi+1)\}} \in M_{\tau\sigma} \setminus C_p([0,\tau\cdot\sigma])$. It remains to extend the map f in the standard way to the space $c_0(\Gamma_\sigma)$. $\square$

Lemma 8. *Let $\tau \geq \omega_1$ be an initial regular ordinal, σ, λ be an initial ordinals and $\omega \leq \lambda < \sigma \leq \tau$. If $f\colon c_0(\Gamma_\sigma) \to M_{\tau\lambda}$ is an injective mapping such that $f(0) = 0$ and $f(x) \in M_{\tau\lambda} \setminus C_p([0,\tau\cdot\lambda])$ for $x \neq 0$, then the map f is not continuous.*

Proof: Suppose that there exists a continuous map $f\colon c_0(\Gamma_\sigma) \to M_{\tau\lambda}$ with the above-mentioned properties. As in Lemma 7, let $\Gamma_\sigma = \{t_\xi\}_{\xi\in[1,\sigma)}$. Since the space $c_0(\Gamma_\sigma)$ is considered in the topology of pointwise convergence, any sequence of the form $\chi_{\{t_{\xi_n}\}}$ converges to zero in this space. Consequently, at each point $\gamma \in [0,\tau\cdot\lambda]$ only a countable number of functions $f(\chi_{\{t_\xi\}})$ is nonzero. Since by the condition $f(\chi_{\{t_\xi\}}) \in M_{\tau\lambda} \setminus C_p([0,\tau\cdot\lambda])$, each function $f(\chi_{\{t_\xi\}})$ is discontinuous at some point of the set $A_{\tau,\tau\lambda} \subset [0,\tau\cdot\lambda]$.

Let us take

$$B_\gamma = \{f(\chi_{\{t_\xi\}})\colon f(\chi_{\{t_\xi\}}) \text{ is discontinuous at a point } \tau(\gamma+1) \in A_{\tau,\tau\lambda}\}.$$

Since $\bigcup_{\gamma<\lambda} B_\gamma = f\left(\{\chi_{\{t_\xi\}}\colon \xi < \sigma\}\right)$ and $|\lambda| = |A_{\tau,\tau\lambda}| < |\sigma|$, there is a point $\gamma_0 < \lambda$, such that $|B_{\gamma_0}| = |\sigma|$. Since at the point $\tau(\gamma_0+1)$ only a countable number of functions from B_{γ_0} are nonzero, without loss of generality we can assume that all functions from B_{γ_0} at the point $\tau(\gamma_0+1)$ are equal to zero. By Lemma 1, for each function $f(\chi_{\{t_\xi\}}) \in B_{\gamma_0}$ there exists an ordinal $\gamma_\xi < \tau(\gamma_0+1)$ such that $f(\chi_{\{t_\xi\}})|_{[\gamma_\xi,\tau(\gamma_\xi+1))} = \text{const} = C_\xi$. Since $|B_{\gamma_0}| = |\sigma| > \omega$, in B_{γ_0} there is an uncountable family of functions for which $|C_\xi| \geq \varepsilon_0$. Consider the sequence $\{f(\chi_{\{t_{\xi_n}\}})\}_{n=1}^\infty$ of such functions and put $\gamma_0 = \sup\{\gamma_{\xi_n} : n = 1, 2, \ldots\}$. Since $\mathrm{cf}(\tau(\gamma_0+1)) > \omega$, $\gamma_0 < \tau(\gamma_0+1)$ and therefore $|f(\chi_{\{t_{\xi_n}\}})(t)| \geq \varepsilon_0$ for each $t \in (\gamma_0, \tau(\gamma_0+1))$. But this contradicts the fact that the sequence $\{f(\chi_{\{t_{\xi_n}\}})\}_{n=1}^\infty$ converges pointwise to zero. □

Proof of Theorem 1: Suppose that there exists a homeomorphism $\varphi\colon C_p([0, \tau\cdot\sigma]) \to C_p([0, \tau\cdot\lambda])$. We can assume that $\varphi(0) = 0$. By Lemma 4, there exists a homeomorphism $\widetilde{\varphi}\colon \nu(C_p([0,\tau\cdot\sigma])) \to \nu(C_p([0,\tau\cdot\lambda]))$ such that $\widetilde{\varphi}(C_p([0, \tau\cdot\sigma])) = C_p([0,\tau\cdot\lambda])$. By Corollary 6, $M_{\tau\sigma} \subset \nu(C_p([0,\tau\cdot\sigma]))$, and by Lemma 3 $\widetilde{\varphi}(M_{\tau\sigma}) = M_{\tau\lambda}$. By Lemma 7 the mapping $\widetilde{\varphi}\cdot f\colon c_0(\Gamma_\sigma) \to M_{\tau\lambda}$ is continuous, $(\widetilde{\varphi}\cdot f)(0) = 0$ and $(\widetilde{\varphi}\cdot f)(M_{\tau\sigma}) \subset M_{\tau\lambda} \setminus C_p([0,\tau\cdot\lambda])$ for $x \neq 0$. In this case, the map $\widetilde{\varphi}|_{c_0(\Gamma_\sigma)}$ is a homeomorphism of the space $c_0(\Gamma_\sigma) \subset M_{\tau\sigma}$ onto the subspace $M_{\tau\lambda}$ such that $\widetilde{\varphi}(0) = 0$ and $\widetilde{\varphi}(x) \subset M_{\tau\lambda} \setminus C_p([0,\tau\cdot\lambda])$ for $x \neq 0$. But this is impossible by Lemma 8. □

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