On pseudocompactness and related notions in ZF

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Abstract. We study in ZF and in the class of T_1 spaces the web of implications/non-implications between the notions of pseudocompactness, light compactness, countable compactness and some of their ZFC equivalents.

Keywords: axiom of choice; countably compact; lightly compact topological space; pseudocompact topological space

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1. Notation and terminology

Let $\mathbf{X} = (X, T)$ be a topological space and $\mathcal{U}$ be a family of subsets of X. An element $x \in X$ is called a *cluster point* of $\mathcal{U}$ if and only if every neighborhood of x meets nontrivially infinitely many members of $\mathcal{U}$. The set $\mathcal{U}$ is said to be *locally finite* (or *point finite*, respectively) if each point of X has a neighborhood intersecting a finite number of elements of $\mathcal{U}$ (or each point of X belongs to finitely many members of $\mathcal{U}$, respectively). An *open refinement* of an open cover $\mathcal{U}$ of $\mathbf{X}$ is a new open cover $\mathcal{V}$ of $\mathbf{X}$ such that each set in $\mathcal{V}$ is contained in some member of $\mathcal{U}$.

The space $\mathbf{X}$ is said to be *metacompact* if and only if every open cover of $\mathbf{X}$ has a point finite open refinement.

The space $\mathbf{X}$ is said to be *compact* (or *countably compact*, respectively) if and only if every open cover $\mathcal{U}$ of $\mathbf{X}$ (or countable open cover $\mathcal{U}$ of $\mathbf{X}$, respectively) has a finite subcover $\mathcal{V}$.

The space $\mathbf{X}$ is said to be *pseudocompact* if and only if every continuous real-valued function on $\mathbf{X}$ is bounded. Pseudocompact spaces were introduced and investigated by E. Hewitt in [2].

The space $\mathbf{X}$ is said to be *lightly compact* (or *countably lightly compact*, respectively) if and only if $\mathbf{X}$ has no infinite (or no countably infinite, respectively) locally finite family of open subsets.

Light compactness has been introduced in [5]. Lightly compact spaces are also called *feebly compact*, see, e.g., [7].

Countable light compactness is condition (B_3) in [1] and is equivalent to light compactness in ZFC, i.e., the Zermelo–Fraenkel set theory ZF together with axiom of choice (AC).

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The space $\mathbf{X}$ is said to be *ineptly compact* (or *countably ineptly compact*, respectively) if and only if $\mathbf{X}$ has no infinite (or no countably infinite, respectively) locally finite family of closed sets. Inept compactness has been introduced in [4]. It is known to be stronger than countable compactness in ZF, but equivalent to the latter property in ZFC.

Let X be an infinite set. We say that X is *Dedekind infinite* (or *weakly Dedekind infinite*, respectively) if and only if X ($\mathcal{P}(X)$, respectively) has a countably infinite subset. Otherwise, X is called *Dedekind finite* (or *weakly Dedekind finite*, respectively).

Below we list the weak forms of the axiom of choice we shall use in this paper.

- DC (the axiom of dependent choice): For any nonempty set X and any binary relation R on X such that for every $x \in X$ there is a $y \in X$ with xRy, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of X such that $x_n R x_{n+1}$ for all $n \in \mathbb{N}$.
- CAC (the countable axiom of choice): For every countable family $\mathcal{A}$ of nonempty sets there exists a function f such that for all $x \in \mathcal{A}$, $f(x) \in x$.
- CMC (the countable axiom of multiple choice): For every family $\mathcal{A} = \{A_i \colon i \in \omega\}$ of pairwise disjoint nonempty sets there exists a family $\mathcal{B} = \{B_i \colon i \in \omega\}$ of nonempty finite sets such that for all $i \in \omega$, $B_i \subseteq A_i$.

The set $\mathcal{B}$ in the statement of CMC is called *multiple choice set* of $\mathcal{A}$. CMC is equivalent (see [3]) to the assertion: For every family $\mathcal{A} = \{A_i \colon i \in \omega\}$ of pairwise disjoint nonempty sets there exists a subfamily $\mathcal{B} = \{A_{k_i} \colon i \in \omega\}$ of $\mathcal{A}$ with a multiple choice set $\mathcal{C}$ which is called *partial multiple choice set* of $\mathcal{A}$.

- IDI: Every infinite set is Dedekind infinite.
- IDI($\mathbb{R}$): IDI restricted to subsets of the real line $\mathbb{R}$.
- IWDI: Every infinite set is weakly Dedekind infinite.
- NT: Every normal space satisfies the Tietze extension theorem.

For ZF models satisfying any single weak choice axiom, or its negation, from the above list we refer the reader to [3].

2. Introduction and some known results

The intended context for reasoning in this paper will be ZF unless otherwise noted. In order to stress that a result is proved in ZF (or ZF + WFC, respectively) we shall write in the beginning of the statements of the theorems (ZF) (or (ZF + WFC), respectively), where WFC will stand for some weak form of the axiom of choice listed in the first section.

It is well known, see e.g. [4] and references therein, or it is easy to see that on any topological space $\mathbf{X} = (X, T)$ each of the following properties implies "$\mathbf{X}$ is pseudocompact" in ZFC.

(A_1): Every pairwise disjoint locally finite family of open sets of $\mathbf{X}$ is finite.
(A_2): Every locally finite open cover of $\mathbf{X}$ is finite.

(B_1): Every countable open covering $\mathcal{U}$ of $\mathbf{X}$ has a finite subcollection whose closures cover X.
(B_2): Every countable family $\mathcal{U}$ of nonempty open subsets of $\mathbf{X}$ has a cluster point in $\mathbf{X}$.
(B_3): Every pairwise disjoint family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of nonempty open subsets of $\mathbf{X}$ has a cluster point in $\mathbf{X}$.
(B_4): Every countable open filterbase has a point of adherence.
(B_5): Every countable, locally finite, disjoint collection of open sets of $\mathbf{X}$ is finite.
(B_6): If $\mathcal{U}$ is a countable open cover of $\mathbf{X}$ and A is an infinite subset of X, then the closure of some member of $\mathcal{U}$ contains infinitely many points of A.
(C_1): Every locally finite family of subsets of $\mathbf{X}$ is finite.
(C_2): Every pairwise disjoint, locally finite family of subsets of $\mathbf{X}$ is finite.
(C_3): Every pairwise disjoint, locally finite family of closed subsets of $\mathbf{X}$ is finite.
(C_4): Every countable locally finite family of subsets of $\mathbf{X}$ is finite.
(C_5): Every countable pairwise disjoint, locally finite family of subsets of $\mathbf{X}$ is finite.
(C_6): Every countable pairwise disjoint, locally finite family of closed subsets of $\mathbf{X}$ is finite.

The question which pops up at this point is whether the statement "$\mathbf{X}$ is pseudocompact" implies back, in ZF, any statement of the above list.

Regarding the ZF implications/non-implications which hold amongst the members of the list, the following results are known.

Theorem 1 ([4], (ZF))**.** *On a topological space $\mathbf{X} = (X, T)$ the following hold.*

(i) *Properties* (A_1) *and* (A_2) *are equivalent to "$\mathbf{X}$ is lightly compact".*
(ii) *Properties* (B_1)–(B_5) *are equivalent.*
(iii) (B_1) *implies* (B_6).
(iv) *Properties* (C_1) *and* (C_2) *are equivalent to the statement "$\mathbf{X}$ is ineptly compact".*
(v) (C_1) *implies* (C_3).
(vi) *If $\mathbf{X}$ is ineptly compact then it is lightly compact and countably compact.*

Theorem 2 ([4])**.** (i) *The statement: "Every topological space satisfying* (B_6) *satisfies* (B_1) *(or is pseudocompact, respectively)" implies* IDI($\mathbb{R}$).

(ii) *Each of the statements: "Every pseudocompact, completely regular, T_4 space is lightly compact (or ineptly compact, respectively)";*
"every countably compact T_4 space is lightly compact (or ineptly compact, respectively)";
"every T_4 space satisfying condition (B_5) *(or* (B_6)*, respectively) is lightly compact";*
"every countably compact space is ineptly compact";
"every countably compact space is lightly compact";

"every countably compact space satisfies condition (C_1) *(or* (C_3)*, respectively)" implies* IWDI*. In particular, none of the above-mentioned statements is a theorem of* ZF*.*

(iii) *The statement: "Every* T_1 *topological space satisfying* (C_3) *satisfies* (C_1)*" implies* CMC*.*

Theorem 3 ([4], (ZF + CAC))**.** (i) *A topological space satisfies condition* (C_1) *if and only if it satisfies property* (C_3)*.*

(ii) (ZF + IDI) *A topological space* $\mathbf{X} = (X, T)$ *is ineptly compact if and only if it is countably compact.*

Theorem 4 ([4], (ZF + DC))**.** *Let* $\mathbf{X} = (X, T)$ *be a* T_4 *topological space. The following are equivalent:*

(i) *the space* $\mathbf{X}$ *is ineptly compact;*
(ii) *the space* $\mathbf{X}$ *is lightly compact;*
(iii) *the space* $\mathbf{X}$ *is pseudocompact;*
(iv) *the space* $\mathbf{X}$ *is countably compact.*

Theorem 5 ([8], [6], (ZFC))**.** *A Tychonoff space* $\mathbf{X}$ *is compact if and only if it is pseudocompact and metacompact.*

The following web of ZF implications/non-implications, whose interpretation is self-evident, pictures the results stated in Theorems 1 and 2.

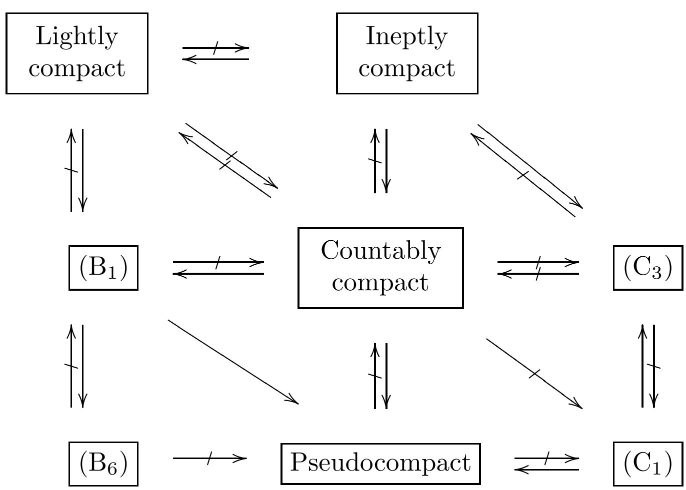

Diagram 1.

Countable light (countable inept, respectively) compactness, and properties (C_4), (C_5) and (C_6) are introduced here. We show in the forthcoming Proposition 6 that properties (C_4), (C_5) and countable inept compactness are equivalent to countable compactness, and each of the conditions (C_3), (C_4) implies (C_6). Our main aim in this paper is to add to Diagram 1 condition (C_6) and study in ZF all the implications/non-implications which hold between the properties in the augmented diagram.

3. Main results

In this section all topological spaces will assume to satisfy at least the T_1 separation axiom.

Proposition 6. *Let $\mathbf{X} = (X, T)$ be a topological space. Then, the following hold:*

(i) (ZF) *The space $\mathbf{X}$ is countably lightly compact if and only if $\mathbf{X}$ satisfies (B_i), $i = 1, \ldots, 5$.*
(ii) (ZF) *Properties (C_4) and (C_5) are equivalent to "$\mathbf{X}$ is countably ineptly compact".*
(iii) (ZF) *The space $\mathbf{X}$ is countably ineptly compact if and only if it is countably compact.*
(iv) (ZF) *If $\mathbf{X}$ is countably ineptly compact then it is countably lightly compact and satisfies (C_6).*
(v) (ZF) *If $\mathbf{X}$ satisfies condition (C_3) then $\mathbf{X}$ satisfies (C_6).*
(vi) CMC *implies "every topological space satisfying condition (C_6) is countably compact" and "every countably ineptly compact space is ineptly compact".*
(vii) CMC *if and only if "every topological space satisfying (C_6) satisfies (C_1)". In particular,* CMC *implies that properties (C_1)–(C_6) are equivalent.*

Proof: (i) It is straightforward to see that $\mathbf{X}$ is countably lightly compact if and only if $\mathbf{X}$ satisfies (B_2). The conclusion of part (i) now follows from Theorem 1.

(ii) If $\mathbf{X}$ is countably ineptly compact then $\mathbf{X}$ satisfies (C_4), and $(C_4) \to (C_5)$ are straightforward.

Assume that $\mathbf{X}$ satisfies (C_5) and show that $\mathbf{X}$ is countably ineptly compact. Fix a countably infinite, locally finite family $\mathcal{U}$ of closed subsets of $\mathbf{X}$. Without loss of generality we may assume that $\mathcal{U}$ is closed under finite intersections. Let "$\sim$" be the equivalence relation on $Y = \bigcup \mathcal{U}$ given by:

$$x \sim y \text{ if and only if for every } U \in \mathcal{U}, \quad x \in U \leftrightarrow y \in U. \tag{1}$$

Let $P = Y/\sim$ be the quotient set of "$\sim$". Clearly, P is pairwise disjoint. For every $x \in Y$ let U_x denote the intersection of all members of $\mathcal{U}$ including x and $[x]$ denote the "$\sim$" equivalence class of x. We claim that for every $x, y \in Y$, $[x] \subseteq U_x$, and $[x] \neq [y]$ if and only if $U_x \neq U_y$. Indeed, $[x] \subseteq U_x$ is an immediate consequence of (1). For the second assertion we note that:

$$[x] \neq [y] \text{ if and only if there is } U \in \mathcal{U} \text{ such that}$$
$$(x \in U \wedge y \notin U) \vee (x \notin U \wedge y \in U) \text{ if and only if } U_x \neq U_y.$$

By our assumption, for every $x \in Y$, $U_x \in \mathcal{U}$. Hence $\{U_x : x \in Y\}$, and consequently P, is countable. Therefore, by our hypothesis, P is finite. Since $\mathcal{U}$ is locally finite and every member of $\mathcal{U}$ includes a member of P it follows easily that $\mathcal{U}$ is finite.

(iii) ($\leftarrow$) Assume the contrary and fix a locally finite family $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ of closed subsets of $\mathbf{X}$. For every $n \in \mathbb{N}$, let

$$F_n = \bigcup\{G_i : i \geq n\}.$$

Clearly, $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ is a descending family of closed subsets of $\mathbf{X}$. Hence, by our hypothesis, $F = \bigcap \mathcal{F} \neq \emptyset$. It is easy to see that for every $x \in F$ and every neighborhood V of x, $V \cap F_n \neq \emptyset$ for all $n \in \mathbb{N}$. Hence, V meets infinitely many members of $\mathcal{G}$ meaning that $\mathcal{G}$ is not locally finite. This leads us to a contradiction. Hence, $\mathbf{X}$ is countably ineptly compact as required.

($\rightarrow$) Assume the contrary and fix a descending family $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ of closed subsets of $\mathbf{X}$ with $\bigcap \mathcal{G} = \emptyset$. Clearly, $\mathcal{G}$ is locally finite. Hence, by our hypothesis, $\mathcal{G}$ is finite contradicting our assumption.

(iv) If $\mathbf{X}$ is countably ineptly compact then $\mathbf{X}$ satisfies (C_4) which in turn implies (C_6). To see that $\mathbf{X}$ is countably lightly compact, fix a locally finite family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of open subsets of $\mathbf{X}$. Clearly, $\mathcal{G} = \{G_n = \overline{U}_n : n \in \mathbb{N}\}$ is locally finite. Therefore, by our hypothesis, $\mathcal{G}$ is finite. Since for every $G \in \mathcal{G}$, $|\{U \in \mathcal{U} : \overline{U} = G\}| < \aleph_0$ ($\mathcal{U}$ is locally finite), it follows that $\mathcal{U}$ is finite as a finite union of finite sets.

(v) This is straightforward.

(vi) Fix a topological space $\mathbf{X} = (X, T)$ satisfying condition (C_6). We show that $\mathbf{X}$ is countably compact. To see this, we assume the contrary and fix $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ a strictly descending family of closed subsets of $\mathbf{X}$ with empty intersection. Fix by CMC, for every $n \in \mathbb{N}$, a finite subset $K_n \subseteq G_n \backslash G_{n+1}$. Since finite subsets of T_1 spaces are closed, it follows that $\mathcal{K} = \{K_n : n \in \mathbb{N}\}$ is a pairwise disjoint family of closed subsets of $\mathbf{X}$ without cluster points (any cluster point x of $\mathcal{K}$ is in $\bigcap \mathcal{G}$). Thus, $\mathcal{K}$ is locally finite, and by our hypothesis finite. Contradiction!

The second assertion can be proved similarly and we leave it as an easy exercise for the reader.

(vii) ($\leftarrow$) This follows from Theorem 2 (iii) and the fact that every space satisfying (C_3) satisfies (C_6) also.

($\rightarrow$) This follows from part (vi) and Theorem 1 (iv). □

Example 7 (ZF)**.** A lightly compact, pseudocompact topological space satisfying condition (B_6) but not (C_6), hence not (C_3) also.

Let T be the co-countable topology on $\mathbb{R}$ ($O \in T$ if and only if $O = \emptyset$ or $|\mathbb{R}\backslash O| \leq \aleph_0$). Since every two nonempty open sets of $\mathbb{R}$ meet nontrivially, it follows that $(\mathbb{R}, T)$ is lightly compact, countably lightly compact and pseudocompact. Hence, it satisfies condition (B_6) also. Furthermore, $\{\{n\}: n \in \mathbb{N}\}$ is an infinite pairwise disjoint locally finite family of closed subsets of $(\mathbb{R}, T)$. Thus, $(\mathbb{R}, T)$ is not countably compact and does not satisfy properties (C_6) and (C_3).

Example 8 (ZF)**.** A pseudocompact topological space satisfying the negation of (B_6).

Let $X = \{(n, m): n, m \in \mathbb{N}\}$ be endowed with the topology T in which basic neighborhoods of points $(n, m) \in X$ are all cofinite subsets of

$$A_{n,m} = \{(n, i): i \in \mathbb{N}\} \cup \{(i, m): i \in \mathbb{N}\}$$

including (n, m). Clearly, for every $(n, m) \in X$, $A_{n,m}$ is a clopen (simultaneously closed and open) set of $\mathbf{X}$. Hence, $\mathcal{U} = \{A_{n,m}: n, m \in \mathbb{N}\}$ is a countable open cover of $\mathbf{X}$. Since for all $n, m \in \mathbb{N}$, $A_{n,m} \cap \{(n, n): n \in \mathbb{N}\}$ is finite, it follows that $\mathbf{X}$ does not satisfy condition (B_6).

The space $\mathbf{X}$ is pseudocompact. To see this, fix a continuous function $f: \mathbf{X} \to \mathbb{R}$. Clearly, for every $n, m \in \mathbb{N}$ the restriction of f to each of the subspaces $\mathbf{Y}_n, Y_n = \{(n, i): i \in \mathbb{N}\}$ and $\mathbf{Z}_m, Z_m = \{(i, m): i \in \mathbb{N}\}$ is constant (the subspace topology on $\mathbf{Y}_n$, $\mathbf{Z}_n$ coincides with the cofinite one). Since $Y_n \cap Z_m \neq \emptyset$, it follows that f is constant on $A_{n,m}$. Similarly, the fact that for all $n, m, u, v \in \mathbb{N}$, $A_{n,m} \cap A_{u,v} \neq \emptyset$, implies f is constant on $A_{n,m} \cup A_{u,v}$. Therefore f is constant on $X = \bigcup \mathcal{U}$ and $\mathbf{X}$ is pseudocompact as required.

Example 9 (ZF)**.** A pseudocompact topological space satisfying condition (B_6) but not conditions (C_3) and (B_1).

Fix a pairwise disjoint family $\mathcal{A} = \{A_i: i \in \mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ whose union is not a cofinite subset of $\mathbb{N}$ and let $B = \{s_i: i \in \mathbb{N}\}$ be an infinite subset of $\mathbb{N}$ disjoint from $\bigcup \mathcal{A}$. For every $i \in \mathbb{N}$ let $X_i = A_i \cup \{s_i\}$. Let T be the topology on $X = \bigcup\{X_i: i \in \mathbb{N}\}$ in which basic open neighborhoods of points $x \in A_n$, $n \in \mathbb{N}$, are all cofinite subsets of A_n including x, and for all $n \in \mathbb{N}$ neighborhoods of s_n are all cofinite subsets of $U_n = \bigcup\{X_i: i \leq n\}$ including s_n. It is easy to verify that $\mathcal{A}$ is a pairwise disjoint locally finite family of open sets of $\mathbf{X}$ (if $x \in X$ then $x \in X_n$ for some $n \in \mathbb{N}$. Hence, x has a neighborhood meeting at most n members of $\mathcal{A}$), and $\{\{s_i\}: i \in \mathbb{N}\}$ is a pairwise disjoint locally finite family of closed subsets of $\mathbf{X}$. Thus, $\mathbf{X}$ is not countably lightly compact and does not satisfy condition (C_6).

We show next that $\mathbf{X}$ is pseudocompact. To this end, fix a continuous function $f: \mathbf{X} \to \mathbb{R}$. Clearly, the restriction of f to B is constant (every two open sets U, V of $\mathbf{X}$ meeting nontrivially B have a nonempty intersection). Since for every $i \in \mathbb{N}$ the restriction of f to $\mathbf{X}_i$ is constant it follows that f is constant and $\mathbf{X}$ is pseudocompact as required.

Finally, we show that $\mathbf{X}$ satisfies condition (B_6). To this end, fix an infinite subset $A = \{a_i : i \in \mathbb{N}\}$ of X, $X \subseteq \mathbb{N}$, and let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be an open cover of $\mathbf{X}$. Since $\mathcal{U}$ is a cover of X it follows that $s_1 \in U_n$ for some $n \in \mathbb{N}$. Since $\overline{U_n} = X$ it follows that $A \subseteq \overline{U_n}$ and $\mathbf{X}$ satisfies (B_6) as required.

Theorem 10. (i) CMC *if and only if every topological space satisfying property* (C_3) *is countably lightly compact.*

In particular, it is relatively consistent with ZF *the existence of a non-countably lightly compact (or non-countably compact, respectively) topological space satisfying the* (C_3) *condition.*

(ii) (ZF) *Every topological space satisfying* (C_6) *(or* (C_3)*, respectively) is pseudocompact and satisfies property* (B_6).

Proof: (i) ($\rightarrow$) This follows from Proposition 6 (vii), and Theorem 1 (iv) and (vi).

($\leftarrow$) Assume the contrary and fix a pairwise disjoint family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ of nonempty sets without a partial multiple choice set. For every $n \in \mathbb{N}$ let $X_n = \bigcup\{Y_i : i \leq n\}$, where $Y_n = A_n \cup B_n$ and $B_n = A_n \times \{n\}$ is a disjoint copy of A_n.

Define a topology T on $X = \bigcup\{X_n : n \in \mathbb{N}\}$ by requiring:

(1) Basic neighborhoods of points $x \in A_n$, $n \in \mathbb{N}$, are all subsets S of A_n such that $x \in S$ and $|A_n \backslash S| < \aleph_0$, and
(2) basic neighborhoods of points $x \in B_n$, $n \in \mathbb{N}$, are all subsets S of X_n such that $x \in S$ and $|X_n \backslash S| < \aleph_0$.

It is easy to see that each member of $\mathcal{A}$ is an open subset of $\mathbf{X}$. We claim that $\mathcal{A}$ is locally finite. To see this fix $x \in X$. If $x \in A_n$ (or $x \in B_n$, respectively) for some $n \in \mathbb{N}$ then A_n (or X_n, respectively) is a neighborhood of x meeting finitely many members of $\mathcal{A}$. Thus $\mathcal{A}$ is locally finite as claimed and $\mathbf{X}$ is not countably lightly compact, hence not lightly compact also.

We show next that $\mathbf{X}$ satisfies the (C_3) condition. Assume, aiming for a contradiction, that $\mathcal{G}$ is an infinite, pairwise disjoint, locally finite family of closed subsets of $\mathbf{X}$. Since $\mathcal{A}$ has no partial multiple choice set it follows that neither $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ does. Hence, for every closed set G of $\mathbf{X}$ each of the sets

$$\{i \in \mathbb{N} : G \cap A_i \neq \emptyset \text{ is finite}\} \quad \text{and} \quad \{i \in \mathbb{N} : G \cap B_i \neq \emptyset \text{ is finite}\}$$

is finite. Therefore, if G is an infinite closed subset of $\mathbf{X}$ then there exists the least integer k such that $G \cap A_k$ is infinite, or $G \cap B_k$ is infinite. We observe that in case $G \cap B_k$ is infinite then for every $n \geq k$, $B_n \subseteq G$. Since $\mathcal{G}$ is pairwise disjoint it follows that $\mathcal{G}$ can contain at most one element meeting some member of $\mathcal{B}$ in an infinite set. So, by discarding this element of $\mathcal{G}$, we may assume that every $G \in \mathcal{G}$ meets each $B \in \mathcal{B}$ in a finite set and only finitely many nontrivially. Therefore, if $G \in \mathcal{G}$ is infinite then there exists the least $k \in \mathbb{N}$ such that $G \cap A_k$ is infinite. Since G is closed it follows that $B_k \cup A_k \subseteq G$, as well as $B_n \cup A_n \subseteq G$ for every $n \geq k$. Since $\mathcal{G}$ is pairwise disjoint, it follows that there exists at most one member of $\mathcal{G}$ meeting in an infinite set some member of $\mathcal{A}$. So, without loss of generality we may assume that each member of $\mathcal{G}$ is finite. Since $\mathcal{G}$ is locally finite, it follows that for every $n \in \mathbb{N}$,

$$|\{G \in \mathcal{G} \colon G \cap A_n \neq \emptyset\}| < \aleph_0 \quad \text{and} \quad |\{G \in \mathcal{G} \colon G \cap B_n \neq \emptyset\}| < \aleph_0.$$

So, for every $n \in \mathbb{N}$,

$$C_n = \bigcup\{G \in \mathcal{G} \colon G \cap A_n \neq \emptyset\} \quad \text{and} \quad D_n = \bigcup\{G \in \mathcal{G} \colon G \cap B_n \neq \emptyset\}$$

are finite sets. Since $\mathcal{G}$ is infinite, it follows that one of the sets

$$\mathcal{C} = \{C_n \colon n \in \mathbb{N}\}, \qquad \mathcal{D} = \{D_n \colon n \in \mathbb{N}\}$$

is infinite. If $\mathcal{C}$ is infinite then $\mathcal{A}$ has a partial multiple choice set, otherwise $\mathcal{B}$ does. This leads to a contradiction. Hence, $\mathbf{X}$ satisfies the (C_3), and consequently the (C_6) condition also.

(ii) Fix a topological space $\mathbf{X}$ satisfying condition (C_6).

We show first that $\mathbf{X}$ is pseudocompact. Assume the contrary and let $f \colon \mathbf{X} \to \mathbb{R}$ be a continuous unbounded strictly positive function. Via a straightforward induction construct a strictly increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that $f^{-1}[k_n, k_{n+1}] \neq \emptyset$ for all $n \in \mathbb{N}$. By the continuity of f it follows that

$$\mathcal{G} = \{f^{-1}[k_{2n}, k_{2n+1}] \colon n \in \mathbb{N}\}$$

is a pairwise disjoint family of closed subsets of $\mathbf{X}$. It is a routine work to verify that $\mathcal{G}$ is locally finite. Hence, by our hypothesis, X does not satisfy condition (C_6). Contradiction!

We show next that $\mathbf{X}$ satisfies condition (B_6). Assume the contrary and fix a countable open cover $\mathcal{U} = \{U_n \colon n \in \mathbb{N}\}$ of $\mathbf{X}$ and an infinite subset A of X such that for all $n \in \mathbb{N}$, $|\overline{U}_n \cap A| < \aleph_0$. Via a straightforward induction we construct a strictly increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$G_n = (\overline{U}_{k_{n+1}} \backslash \overline{U}_{k_n}) \cap A \neq \emptyset.$$

Since $\overline{U}_{k_{n+1}} \cap A$ is finite and $\mathbf{X}$ is T_1 it follows that G_n is closed. Furthermore, for every $n, m \in \mathbb{N}$, $n < m$, $G_n \subseteq \overline{U}_{k_m}$ and $G_m \cap \overline{U}_{k_m} = \emptyset$. Hence, $\mathcal{G} = \{G_n \colon n \in \mathbb{N}\}$ is a pairwise disjoint family of closed sets of $\mathbf{X}$. We claim that $\mathcal{G}$ is locally finite. To see this, fix $x \in X$ and let t be the least natural number with $x \in U_{k_t}$. Clearly, U_{k_t} is a neighborhood of x avoiding G_n for every $n > t$. Thus, $\mathcal{G}$ is locally finite as claimed. Hence, by condition (C_6), $\mathcal{G}$ is finite. Contradiction! □

In contrast to Example 8 we show next in $(\mathrm{ZF} + \mathrm{NT})$ that a T_4 pseudocompact topological space satisfies (B_6).

Theorem 11 $(\mathrm{ZF} + \mathrm{NT})$**.** *Every T_4 pseudocompact topological space satisfies condition* (B_6).

PROOF: Fix a pseudocompact T_4 space $\mathbf{X}$. We show that $\mathbf{X}$ satisfies condition (B_6). Assume the contrary and fix a countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ and an infinite subset A of $\mathbf{X}$ such that for all $n \in \mathbb{N}$, $|\overline{U_n} \cap A| < \aleph_0$. Let $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ be as in the proof of (v). Clearly, $G = \bigcup \mathcal{G}$ is a closed subset of X and the function $f : G \to \mathbb{R}$, $f(x) = n$, $x \in G_n$, $n \in \mathbb{N}$ is continuous and unbounded. By NT, f extends continuously to $\mathbf{X}$. Hence, $\mathbf{X}$ is not pseudocompact. Contradiction! □

We show next that Theorem 5 is not a theorem of ZF.

Theorem 12. *The statement: "Every Tychonoff pseudocompact and metacompact space is compact" implies* IWDI.
In particular, it is relatively consistent with ZF *the existence of a non-compact, pseudocompact and metacompact topological space.*

Assume the contrary and let X be an infinite weakly Dedekind finite set endowed with the discrete topology. Trivially, X is Tychonoff, metacompact and pseudocompact (if $f : X \to \mathbb{R}$ is unbounded and strictly positive, then $\{f^{-1}(n, \infty) : n \in \mathbb{N}\}$ is a countably infinite subset of $\mathcal{P}(X)$, contradicting the fact that X is weakly Dedekind finite). Thus, by our hypothesis, $\mathbf{X}$ is compact. However, $\mathcal{U} = \{\{x\} : x \in X\}$ is an open cover of $\mathbf{X}$ with no finite subcover meaning that $\mathbf{X}$ is not compact. Contradiction!

4. Summary results

Let LC, IC, CC, PSC and CLC abbreviate lightly compact, ineptly compact, countably compact, pseudocompact and countably lightly compact, respectively. The following table summarizes the ZF implications/non-implications between LC, IC, CC, PSC, CLC, (B_6), (C_3) and (C_6) obtained in this paper and in [4]. The interpretation of the table is as follows: Given $P, Q \in$ {LC, IC, CC, PSC, CLC, (B_6), (C_3), (C_6)}, if in the P-line and Q-row entry there is "$\to$" then in ZF, every topological space satisfying property P satisfies property Q also. In case there is "$\nrightarrow$" then, either there exists a topological space satisfying P but not Q and the argument can be given in ZF, or there is a ZF model including a topological space satisfying property P but not Q.

	IC	LC	CC	PSC	CLC	(B_6)	(C_3)	(C_6)
IC	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$	$\to$
LC	$\nrightarrow$	$\to$	$\nrightarrow$	$\to$	$\to$	$\to$	$\nrightarrow$	$\nrightarrow$
CC	$\nrightarrow$	$\nrightarrow$	$\to$	$\to$	$\to$	$\to$	$\nrightarrow$	$\to$
PSC	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\to$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$
CLC	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\to$	$\to$	$\to$	$\nrightarrow$	$\nrightarrow$
(B_6)	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\to$	$\nrightarrow$	$\nrightarrow$
(C_3)	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\to$	$\nrightarrow$	$\to$	$\to$	$\to$
(C_6)	$\nrightarrow$	$\nrightarrow$	$\nrightarrow$	$\to$	$\nrightarrow$	$\to$	$\nrightarrow$	$\to$

Table 1.

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