On the mappings $\mathcal{Z}_A$ and $\Im_A$ in intermediate rings of $C(X)$

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Abstract. In this article, we investigate new topological descriptions for two well-known mappings $\mathcal{Z}_A$ and $\Im_A$ defined on intermediate rings $A(X)$ of $C(X)$. Using this, coincidence of each two classes of z-ideals, $\mathcal{Z}_A$-ideals and $\Im_A$-ideals of $A(X)$ is studied. Moreover, we answer five questions concerning the mapping $\Im_A$ raised in [J. Sack, S. Watson, *C and C^* among intermediate rings*, Topology Proc. **43** (2014), 69–82].

Keywords: z-ideal; $\mathcal{Z}_A$-ideal; $\Im_A$-ideal; z-filter; $\mathcal{Z}_A$-filter; $\Im_A$-filter; intermediate ring

Classification: 54C30, 46E25

1. Introduction

Throughout this article all topological spaces are assumed to be Tychonoff. For a given topological space X, $C(X)$ denotes the algebra of all real-valued continuous functions on X and $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. A subring of $C(X)$ containing $C^*(X)$ is called an intermediate ring. An intermediate ring which is isomorphic with $C(Y)$ for some Tychonoff space Y is called an intermediate C-ring. The reader is referred to [5] for undefined terms and notations concerning $C(X)$. For each element f of an intermediate ring $A(X)$, we set $S_A(f) = \{p \in \beta X \colon (fg)^*(p) = 0,\ \forall g \in A(X)\}$. As stated in [11], we could easily observe that $S_A(fg) = S_A(f) \cup S_A(g)$, $S_A(f^2+g^2) = S_A(f) \cap S_A(g)$ and $S_A(f^n) = S_A(f)$ for each $f, g \in A(X)$ and each $n \in \mathbb{N}$. Also, $S_C(f) = \mathrm{cl}_{\beta X} Z(f)$ for each $f \in C(X)$ and $S_{C^*}(f) = Z(f^\beta)$ for each $f \in C^*(X)$. Moreover, $\mathrm{cl}_{\beta X} Z(f) \subseteq S_A(f) \subseteq Z(f^*)$ and thus $S_A(f) \cap X = Z(f)$ for each $f \in A(X)$. For each $p \in \beta X$, we use M^p_A (or O^p_A) to denote the set $\{f \in A(X) \colon p \in S_A(f)\}$ ($\{f \in A(X) \colon p \in \mathrm{int}_{\beta X} S_A(f)\}$, respectively). Evidently, $M^p_C = M^p$ (or $O^p_C = O^p$) and $M^p_{C^*} = M^{*p}$ ($O^p_{C^*} = O^{*p}$, respectively). Moreover, we can see that $\mathrm{int}_{\beta X} S_A(f) = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z(f)$ for each $f \in A(X)$ and thus $O^p_A = O^p \cap A(X)$ for each $p \in \beta X$. An ideal I of a commutative ring R is called a z-ideal if $M_f(R) \subseteq I$ whenever $f \in I$ in which $M_f(R)$ denotes the intersection of all the maximal ideals of R containing f. It is well-known that $M_f(C(X)) = \{g \in C(X) \colon Z(f) \subseteq Z(g)\}$ for each $f \in C(X)$. Also, from [9, Proposition 2.7], it follows that $M_f(A(X)) = \{g \in A(X) \colon S_A(f) \subseteq S_A(g)\}$ for each element f of an intermediate ring $A(X)$. Therefore, an ideal I in $C(X)$

DOI 10.14712/1213-7243.2015.249

(or in an intermediate ring $A(X)$) is a z-ideal if and only if whenever $Z(f) \subseteq Z(g)$ (or $S_A(f) \subseteq S_A(g)$), where $f \in I$ and $g \in C(X)$ ($g \in A(X)$, respectively), then $g \in I$. The aim of this paper is to answer the five basic questions concerning the mapping $\Im_A$ which have been raised in [13]. This is done by investigating new topological descriptions for two well-known mappings $\mathcal{Z}_A$ and $\Im_A$ defined on intermediate rings of $C(X)$. This paper consists of 3 sections. Section 1, which is already noticed, is the introduction. In Section 2, we study the mapping $\mathcal{Z}_A$ which was first introduced in [12] and more studied in [3]. Moreover, coincidence of z-ideals of $A(X)$ with $\mathcal{Z}_A$-ideals and z_A-ideals is studied. In Section 3, we study the mapping $\Im_A$ which was first introduced in [8] and further studied in [14] and [13]. We establish a topological description for this mapping. Coincidence of $\Im_A$-ideals with z-ideals and $\mathcal{Z}_A$-ideals are studied. Moreover, the five questions concerning the mapping $\Im_A$ raised in [13] are answered.

2. The mapping $\mathcal{Z}_A$

The mapping $\mathcal{Z}_A$ was first introduced in [12] on an intermediate ring $A(X)$ as follows: for each element f of $A(X)$,

$$\mathcal{Z}_A(f) = \{E \in Z(X) \colon \exists\, g \in A(X),\ fg|_{X \setminus E} = 1\}.$$

By the following statement we provide a new topological description for $\mathcal{Z}_A$.

Theorem 2.1. *For each element f of an intermediate ring $A(X)$, we have*

$$\mathcal{Z}_A(f) = \{E \in Z(X) \colon S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E\}.$$

Proof: If $E \in \mathcal{Z}_A(f)$, then there exists $g \in A(X)$ such that $fg|_{X\setminus E} = 1$. Thus, $(fg)^*|_{\mathrm{cl}_{\beta X}(X\setminus E)} \neq 0$. This means that $S_A(f) \cap \mathrm{cl}_{\beta X}(X \setminus E) = \emptyset$ and thus $S_A(f) \subseteq \beta X \setminus \mathrm{cl}_{\beta X}(X \setminus E) = \beta X \setminus \mathrm{cl}_{\beta X}(\beta X \setminus \mathrm{cl}_{\beta X} E) \subseteq \mathrm{cl}_{\beta X} E$; i.e., $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E$. For the reverse inclusion, let $E \in Z(X)$ be such that $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E$. Thus, $S_A(f) \cap (\beta X \setminus \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E) = \emptyset$. Hence, there exists some $h \in C^*(X)$ such that $h^\beta(S_A(f)) = \{1\}$ and $(\beta X \setminus \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E) \subseteq Z(h^\beta)$. These imply that $S_A(f) \cap S_A(h) = \emptyset$ and $X \setminus E \subseteq X \setminus \mathrm{int}_X E \subseteq Z(h)$. Therefore, $S_A(f^2 + h^2) = S_A(f) \cap S_A(h) = \emptyset$ which implies that there exists some $k \in A(X)$ such that $(f^2 + h^2)k = 1$. Thus, clearly $fk \in A(X)$ and $(f^2k)|_{X\setminus E} = ((f^2 + h^2)k)|_{X\setminus E} = 1$; i.e., $E \in \mathcal{Z}_A(f)$. □

The following properties of the mapping $\mathcal{Z}_A$ follow from Theorem 2.1. Note that for each two z-filters $\mathcal{F}$ and $\mathcal{F}'$, we denote by $\mathcal{F} \wedge \mathcal{F}'$ and $\mathcal{F} \vee \mathcal{F}'$ the meet and join on the lattice of z-filters, respectively.

Proposition 2.2. *The following statements hold for each two elements f, g of an intermediate ring $A(X)$ and each $n \in \mathbb{N}$.*

(a) *If $0 \leq f \leq g$, then $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(g)$.*
(b) *$\bigcap \mathcal{Z}_A(f) = Z(f)$.*

(c) $\mathcal{Z}_A(fg) = \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$.
(d) $\mathcal{Z}_A(f+g) \subseteq \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$.
(e) $\mathcal{Z}_A(f^2+g^2) = \mathcal{Z}_A(f) \vee \mathcal{Z}_A(g)$.
(f) $\mathcal{Z}_A(f^n) = \mathcal{Z}_A(f)$.

Proof: Parts (c) through (f) easily follow from Theorem 2.1.

(a) It suffices to show that $S_A(g) \subseteq S_A(f)$. Let $p \notin S_A(f)$. Thus, $f \notin M_A^p$ and hence $g \notin M_A^p$, i.e., $p \notin S_A(g)$, since $0 \le f \le g$ and from [3, Theorem 2.5] it follows that every maximal ideal in $A(X)$ is a convex ideal. Therefore, $S_A(g) \subseteq S_A(f)$.

(b) It is evident that $Z(f) \subseteq E$ for each $E \in \mathcal{Z}_A(f)$. Thus, $Z(f) \subseteq \bigcap \mathcal{Z}_A(f)$. Let $x \in X$ and $x \notin Z(f)$. Hence, $x \notin S_A(f)$ and thus there exists a zero-set E such that $x \notin \mathrm{cl}_{\beta X} E$ and $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E$. It follows that $E \in \mathcal{Z}_A(f)$, however, $x \notin E$. This implies that $\bigcap \mathcal{Z}_A(f) \subseteq Z(f)$ and the equality follows. $\square$

It is shown in [8, Theorem 1.2] that the mapping $\mathcal{Z}_A$ could characterize $C^*(X)$ among intermediate rings. The following statement shows that the mapping $\mathcal{Z}_A$ could also characterize $C(X)$ among intermediate rings in the case that X is a P-space.

Theorem 2.3. *Let $A(X)$ be an intermediate ring of $C(X)$. Then $\mathcal{Z}_A(f) = (Z(f))$ for each $f \in A(X)$ if and only if X is a P-space and $A(X) = C(X)$.*

Proof: ($\Rightarrow$) As $\mathcal{Z}_A(f) = (Z(f))$ for each $f \in A(X)$, we have $Z(f) \in \mathcal{Z}_A(f)$ and thus $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$. This implies that $S_A(f) = \mathrm{cl}_{\beta X} Z(f) = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)$ for each $f \in A(X)$. Hence, X is a P-space and, by [10, Theorem 2.3], $A(X) = C(X)$

($\Leftarrow$) It is clear that $\mathcal{Z}_C(f) \subseteq (Z(f))$. If $E \in (Z(f))$, then, $Z(f) \subseteq E$ and as X is a P-space, we have $S_C(f) = \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E$. Thus, $(Z(f)) \subseteq \mathcal{Z}_C(f)$. $\square$

For an ideal I of an intermediate ring $A(X)$, we denote by $\mathcal{Z}_A(I)$ the set $\bigcup_{f\in I} \mathcal{Z}_A(f)$. Also, for a z-filter $\mathcal{F}$ on X we denote by $\mathcal{Z}_A^{-1}(\mathcal{F})$ the set $\{f \in A(X)\colon \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$. It is clear that $I \subseteq \mathcal{Z}_A^{-1}\mathcal{Z}_A(I)$ and $\mathcal{Z}_A\mathcal{Z}_A^{-1}(\mathcal{F}) \subseteq \mathcal{F}$. We call an ideal I in $A(X)$ a $\mathcal{Z}_A$-ideal, if $\mathcal{Z}_A^{-1}\mathcal{Z}_A(I) = I$. Also, a z-filter $\mathcal{F}$ on X is called a $\mathcal{Z}_A$-filter, if $\mathcal{Z}_A\mathcal{Z}_A^{-1}(\mathcal{F}) = \mathcal{F}$. Evidently, $\mathcal{Z}_A^{-1}\mathcal{Z}_A(M_A^p) = M_A^p$. Hence, every maximal ideal in $A(X)$ is $\mathcal{Z}_A$-ideal.

Remark 2.4. It easily follows from Theorem 2.1 that $\mathcal{Z}_A(M_A^p) = \mathcal{Z}_A(O_A^p)$ for each $p \in \beta X$. Moreover, $\mathcal{Z}_A^{-1}(\mathcal{U}^p) = M_A^p$ for each $p \in \beta X$. Also, one can easily prove that $\mathcal{Z}_A(M)$ is contained in a unique z-ultrafilter for each maximal ideal M in $A(X)$. These provide a new approach to [3, Theorem 3.2].

It easily follows from Theorem 2.1 that every $\mathcal{Z}_A$-ideal of an intermediate ring $A(X)$ is a z-ideal. However, the converse of this fact does not hold, in general. For example, let $X = \mathbb{R}$ and $A(X) = C^*(X) = C^*(\mathbb{R})$ and $p \in X$; then $O_A^p \subsetneq M_A^p$, easily verifiable. It follows that O_A^p is not a $\mathcal{Z}_A$-ideal in $A(X)$, however, it is clearly a z-ideal. In [7, Theorem 2.14] it is stated that whenever every ideal of an intermediate ring $A(X)$ is a $\mathcal{Z}_A$-ideal, then X is a P-space. The next theorem shows that even when every z-ideal is a $\mathcal{Z}_A$-ideal, then we have $A(X) = C(X)$. Note that we call an ideal I in $A(X)$ a z_A-ideal, if whenever $Z(f) \subseteq Z(g)$ where $f \in I$ and $g \in A(X)$, then $g \in I$. It is easy to see that the ideals O^p_A and $M^p \cap A(X)$, for each $p \in \beta X$, are z_A-ideals in $A(X)$. Also, the ideal M^p_A, for each $p \in \beta X \setminus \upsilon_A X$, is a maximal ideal in $A(X)$ which is not a z_A-ideal, refer to [2], [1], and [10] for more details.

Theorem 2.5. *The following statements are equivalent for an intermediate ring $A(X)$.*

(a) *Every z-ideal in $A(X)$ is a $\mathcal{Z}_A$-ideal.*
(b) *Every z_A-ideal in $A(X)$ is a $\mathcal{Z}_A$-ideal.*
(c) *X is a P-space and $A(X) = C(X)$.*

Proof: (a) $\Rightarrow$ (c) The proof is straightforward by using [7, Theorem 2.5 and Theorem 3.10].

(c) $\Rightarrow$ (a) By our hypothesis and Theorem 2.3, $\mathcal{Z}_C(f) = (Z(f))$ for each $f \in C(X)$. This clearly implies that every z-ideal in $C(X)$ is a $\mathcal{Z}_C$-ideal.

(b) $\Rightarrow$ (c) As O^p_A, for each $p \in \beta X$, is a z_A-ideal, by our hypothesis, O^p_A would be a $\mathcal{Z}_A$-ideal. Hence, $O^p_A = \mathcal{Z}_A^{-1}\mathcal{Z}_A(O^p_A) = \mathcal{Z}_A^{-1}\mathcal{Z}_A(M^p_A) = M^p_A$. This means that $A(X)$ is a regular ring. Thus, by [7, Theorem 2.5 and Theorem 3.3], X is a P-space and $A(X) = C(X)$.

(c) $\Rightarrow$ (b) It is evident that whenever $A(X) = C(X)$, then z_A-ideals coincide with z-ideals in $A(X)$. Moreover, as X is a P-space, by [4, Theorem 3.13], every z_A-ideal is a $\mathcal{Z}_A$-ideal. □

3. The mapping $\Im_A$

The mapping $\Im_A$ on an intermediate ring $A(X)$ was first introduced in [8] as follows: for each $f \in A(X)$

$$\Im_A(f) = \{E \in Z(X)\colon \text{for all zero-sets } H \subseteq X \setminus E,\ \exists\, g \in A(X),\ fg|_H = 1\}.$$

The following statement provides a new topological description to the mapping $\Im_A$ which could be proved similar to Theorem 2.1.

Theorem 3.1. *Let $A(X)$ be an intermediate ring of $C(X)$. For each $f \in A(X)$ we have*

$$\Im_A(f) = \{E \in Z(X)\colon \forall\, H \in Z(X) \text{ and } H \subseteq X \setminus E,\ S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}(X \setminus H)\}.$$

The following question concerning the basic properties of the mapping $\Im_A$ has been raised in [13].

Question 1. Let $f, g \in A(X)$. Which properties analogous to those of Proposition 2.2 hold with $\Im_A$ in place of $\mathcal{Z}_A$?

The answer to Question 1 will be given by the next statement which is an easy consequence of Theorem 3.1.

Proposition 3.2. *For each two elements f, g of an intermediate ring $A(X)$, the following statements hold.*

(a) *If $0 \leq f \leq g$, then $\Im_A(f) \subseteq \Im_A(g)$.*
(b) $\bigcap \Im_A(f) = Z(f)$.
(c) $\Im_A(fg) = \Im_A(f) \wedge \Im_A(g)$.
(d) $\Im_A(f+g) \subseteq \Im_A(f) \vee \Im_A(g)$.
(e) $\Im_A(f^2+g^2) = \Im_A(f) \vee \Im_A(g)$.
(f) $\Im_A(f^n) = \Im_A(f)$ *for each $n \in \mathbb{N}$.*

It is stated in [3, Lemma 1.3] that for a z-filter $\mathcal{F}$ and $f \in A(X)$, we have $\lim_{\mathcal{F}} fg = 0$ for each $g \in A(X)$ if and only if $\mathcal{F} \subseteq \mathcal{Z}_A(f)$. Note that for a z-filter $\mathcal{F}$ on X and $f \in C(X)$, we use $\lim_{\mathcal{F}} f$ to denote the limit of the filter base $f(\mathcal{F})$. The following question has been raised in [13].

Question 2. Let $f \in A(X)$ and $\mathcal{F}$ be a z-filter on X. Is it the case that $\Im_A(f) \subseteq \mathcal{F}$ if and only if $\lim_{\mathcal{F}} fg = 0$ for every $g \in A(X)$?

Answer to Question 2: As stated in [13], if $\Im_A(f) \subseteq \mathcal{F}$, then $\lim_{\mathcal{F}} fg = 0$ for every $g \in A(X)$. We show that the converse of this statement does not hold, in general. Take $X = \mathbb{R}$, $f(x) = x$ for each $x \in \mathbb{R}$ and $\mathcal{F}$ be the z-filter of all zero set neighbourhood of 0 in $\mathbb{R}$. Then for all $h \in C(\mathbb{R})$, $\lim_{\mathcal{F}} fh = 0$ but $\Im_C(f) = \{Z \in Z[X] \colon 0 \in Z\}$ which contains $\mathcal{F}$ properly.

In [7, Theorem 2.8] it is stated that a topological space X is a P-space if and only if, for every intermediate ring $A(X)$, we have $\Im_A(M_A^p) = \Im_A(O_A^p)$ for each $p \in X$. The next theorem extends this fact for each $p \in \beta X$. We use the following lemma which could be proved by a little modification of the arguments of [7, Proposition 2.7] and exploiting the complete regularity of βX.

Lemma 3.3. *Let $A(X)$ be an intermediate ring of $C(X)$. Then $\Im_A(O_A^p) = \mathcal{Z}_A(M_A^p)$ for each $p \in \beta X$.*

Theorem 3.4. *A topological space X is a P-space if and only if for every intermediate ring $A(X)$ we have $\Im_A(M_A^p) = \Im_A(O_A^p)$ for each $p \in \beta X$.*

Proof: ($\Rightarrow$) This easily follows from [7, Theorem 2.8].

($\Leftarrow$) By our hypothesis, $\Im_C(M^p) = \Im_C(O^p)$. Hence, $Z(M_A^p) = \Im_C(O^p) = Z(O^p)$ for each $p \in \beta X$ which clearly implies that X is a P-space. $\square$

For an ideal I of an intermediate ring $A(X)$, we denote by $\Im_A(I)$ the set $\bigcup_{f \in I} \Im_A(f)$. Moreover, for a z-filter $\mathcal{F}$ on X, we use $\Im_A^{-1}(\mathcal{F})$ to denote the set $\{f \in A(X) \colon \Im_A(f) \subseteq \mathcal{F}\}$. An ideal I in $A(X)$ is called a $\Im_A$-ideal, if $\Im_A^{-1}\Im_A(I) = I$. Also, a z-filter $\mathcal{F}$ on X is called a $\Im_A$-filter, if $\Im_A \Im_A^{-1}(\mathcal{F}) = \mathcal{F}$. It easily follows from Theorem 2.1 and Theorem 3.1 that every $\mathcal{Z}_A$-ideal in $A(X)$ is a $\Im_A$-ideal and every $\Im_A$-ideal in $A(X)$ is a z-ideal. However, the converse of these facts does not hold, in general, see the next example.

Example 3.5. (a) Whenever X is not a P-space, then there exists $p \in \beta X$ such that $O^p \neq M^p$. Hence, O^p is clearly a z-ideal and thus is a $\Im_C$-ideal in $C(X)$, however, it is not a $\mathcal{Z}_C$-ideal, since, $\mathcal{Z}_C^{-1}\mathcal{Z}_C(O^p) = M^p$.

(b) Let $X = \mathbb{N}$ and $A(X) = C^*(\mathbb{N})$. It is clear that there exists $p \in \beta\mathbb{N}$ such that $M_A^p \neq O_A^p$. By Theorem 3.4, $\Im_A(M_A^p) = \Im_A(O_A^p)$ for each $p \in \beta\mathbb{N}$. Thus, $\Im_A^{-1}\Im_A(O_A^p) = \Im_A^{-1}\Im_A(M_A^p) = M_A^p \neq O_A^p$ which means that O_A^p is not a $\Im_A$-ideal, however, it is evidently a z-ideal.

The following theorem shows that coincidence of z_A-ideals with $\Im_A$-ideals characterizes $C(X)$ among intermediate rings.

Theorem 3.6. *Let $A(X)$ be an intermediate ring of $C(X)$. Then every $\Im_A$-ideal in $A(X)$ is a z_A-ideal if and only if $A(X) = C(X)$.*

Proof: ($\Rightarrow$) Since $A(X) \neq C(X)$, there exists $f \in C(X) \setminus A(X)$. Take $g = 1/(1 + |f|)$, then g is a non-unit of $A(X)$. Consequently, $g \in M_A^p$ for some $p \in \beta X$. It is clear that, since $Z(g) = \emptyset$, M_A^p is not a z_A-ideal in $A(X)$, but M_A^p being a maximal ideal is a $\Im_A$-ideal in $A(X)$.

($\Leftarrow$) This is evident, since, $\Im_C$-ideals of $C(X)$ are nothing other than z-ideals. $\square$

Remark 3.7. It is stated in [7, Theorem 2.14] that whenever every ideal of an intermediate ring $A(X)$ is a $\Im_A$-ideal, then X is a P-space. We show that this condition implies $A(X) = C(X)$. It is evident that whenever every ideal in $A(X)$ is a $\Im_A$-ideal, then every ideal in $A(X)$ is a z-ideal and thus, by [6, Theorem 1.2], $A(X)$ is a regular ring. Hence, we have $A(X) = C(X)$, since, otherwise, there exists $f \in A(X)$ such that $Z(f) = \emptyset$ and $S_A(f) \neq \emptyset$. Hence, if we choose some $p \in S_A(f)$, then $M^p \cap A(X) \neq M_A^p$ which means that $M^p \cap A(X)$ is a prime ideal in $A(X)$ which is not a maximal ideal and thus $A(X)$ is not a regular ring. Other than this, in the proof of this theorem in [7] it is stated that $\Im_A^{-1}\mathcal{Z}_A(M_A^p) = M_A^p$ for each $p \in X$. We claim that this equality does not hold, in general. For example, if we let $A(X) = C(X)$ where X is not a P-space, then there exists $f \in C(X)$ and $p \in X$ such that $f \in M^p \setminus O^p$. Also, we clearly have $\Im_C^{-1}(\mathcal{F}) = Z^{-1}(\mathcal{F})$ for each z-filter $\mathcal{F}$. It follows that $\Im_C^{-1}\mathcal{Z}_C(M^p) = Z^{-1}\mathcal{Z}_C(M^p) = Z^{-1}Z(O^p) = O^p \neq M^p$. Recall that $\mathcal{Z}_A(M_A^p) = \mathcal{Z}_A(O_A^p)$ and $\mathcal{Z}_A(M_A^p) = \{E \in Z(X) \colon p \in \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}E\}$. Hence, $\mathcal{Z}_C(M^p) = Z(O^p)$ for each $p \in \beta X$.

The following question has been raised in [13].

Question 3. Is it the case that if $\mathcal{F}$ is a z-filter on X, then $\Im_A^{-1}(\mathcal{F})$ is an ideal in $A(X)$?

We answer Question 3 by using Proposition 3.2.

Answer to Question 3: Let $\mathcal{F}$ be a z-filter on X, $f, g \in \Im_A^{-1}(\mathcal{F})$ and $h \in A(X)$. Then, $\Im_A(fh) = \Im_A(f) \wedge \Im_A(h) \subseteq \mathcal{F}$, since, $f \in \Im_A^{-1}(\mathcal{F})$. Hence, $fh \in \Im_A^{-1}(\mathcal{F})$. Moreover, $\Im_A(f + g) \subseteq \Im_A(f) \vee \Im_A(g) \subseteq \mathcal{F}$, since, $f, g \in \Im_A^{-1}(\mathcal{F})$. Thus, $f + g \in \Im_A^{-1}(\mathcal{F})$. Therefore, $\Im_A^{-1}(\mathcal{F})$ is an ideal in $A(X)$ for each z-filter $\mathcal{F}$ on X.

The following question also has been raised in [13].

Question 4: Is it the case that $A(X) = C(X)$ if and only if $\Im_A(M)$ is a z-ultrafilter for every maximal ideal M?

Answer to Question 4. If $A(X) = C(X)$, then for each maximal ideal M of $A(X)$, $\Im_A(M) = Z(M)$ is a z-ultrafilter on X. On the other hand, it is easily inferred from [7, Theorem 2.10] that if X is a non pseudocompact P-space, then there exists an intermediate ring $B(X) \subsetneq C(X)$ for which for any maximal ideal M of $B(X)$, $\Im_B(M)$ is a z-ultrafilter on X.

The following question has been raised in [13].

Question 5. Is it the case that $A(X) = C(X)$ if and only if every z-filter on X is a $\Im_A$-filter?

The answer to Question 5 is negative as it could be inferred from [7, Example 2.13].

Acknowledgment. The author would like to express his deep gratitude to the well-informed referee for giving useful suggestions on this article.

References

[1] Aliabad A. R., Parsinia M., *Remarks on subrings of $C(X)$ of the form $I + C^*(X)$*, Quaest. Math. **40** (2017), no. 1, 63–73.

[2] Aliabad A. R., Parsinia M., *z_R-ideals and z_R°-ideals in subrings of $\mathbb{R}^X$*, to appear in Iranian J. Math. Sci. Inform.

[3] Byun H. L., Watson S., *Prime and maximal ideals in subrings of $C(X)$*, Topology Appl. **40** (1991), no. 1, 45–62.

[4] Domínguez J. M., Gómez Pérez J., *Intersections of maximal ideals in algebras between $C^*(X)$ and $C(X)$*, Iberoamerican Conf. on Topology and Its Applications, Morelia, 1997, Topology Appl. **98** (1999), no. 1–3, 149–165.

[5] Gillman L., Jerison M., *Rings of Continuous Functions*, The University Series in Higher Mathematics, D. Van Nostrand Co., Princeton, New York, 1960.

[6] Mason G., *z-ideals and prime ideals*, J. Algebra **26** (1973), 280–297.

[7] Murray W., Sack J., Watson S., *P-spaces and intermediate rings of continuous functions*, Rocky Mountain J. Math. **47** (2017), no. 8, 2757–2775.

[8] Panman P., Sack J., Watson S., *Correspondence between ideals and z-filters for rings of continuous functions between C^* and C*, Comment. Math. **52** (2012), no. 1, 11–20.

[9] Parsinia M., *Remarks on LBI-subalgebras of $C(X)$*, Comment. Math. Univ. Carolin. **57** (2016), no. 2, 261–270.

[10] Parsinia M., *Remarks on intermediate C-rings of $C(X)$*, Quaest. Math. (online 2017), 8 pages.

[11] Plank D., *On a class of subalgebras of $C(X)$ with applications to $\beta X \backslash X$*, Fund. Math. **64** (1969), 41–54.

[12] Redlin L., Watson S., *Maximal ideals in subalgebras of $C(X)$*, Proc. Amer. Math. Soc. **100** (1987), no. 4, 763–766.

[13] Sack J., Watson S., *C and C^* among intermediate rings*, Topology Proc. **43** (2014), 69–82.
[14] Sack J., Watson S., *Characterizing $C(X)$ among intermediate C-rings on X*, Topology Proc. **45** (2015), 301–313.

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(*Received* November 30, 2017, *revised* May 1, 2018)