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Monotonically normal e-separable spaces may not be perfect

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Abstract. A topological space X is said to be e-separable if X has a σ-closed-discrete dense subset. Recently, G. Gruenhage and D. Lutzer showed that e-separable PIGO spaces are perfect and asked if e-separable monotonically normal spaces are perfect in general. The main purpose of this article is to provide examples of e-separable monotonically normal spaces which are not perfect. Extremely normal e-separable spaces are shown to be stratifiable.

Keywords: monotonically normal space; σ-closed-discrete dense set; e-separable space; perfect space; perfectly normal space; point network; perfect images of generalized ordered space

Classification: 54G20, 54B10, 54D15

1. Introduction

All topological spaces are assumed to be T_1. For simplicity, we adopt the terminology in [6] and call a space with a σ-closed-discrete dense subset *e-separable.* A space X is *perfect* if every closed subset of X is a G_δ-subset of X. A topological space is *monotonically normal* if for each pair of disjoint closed subsets H, K there is an open set $U(H,K)$ satisfying

(1) $H = U(H,K) \subset \overline{U(H,K)} \subset X \setminus K$, and
(2) if $H \subset H'$ and $K' \subset K$, then $U(H,K) \subset U(H',K')$.

Recently, G. Gruenhage and D. Lutzer in [7] defined the class of perfect images of generalized ordered (PIGO) spaces to be the perfect images of GO-spaces. G. Gruenhage and D. Lutzer showed that perfect images of generalized ordered PIGO spaces are monotonically normal and e-separable PIGO spaces are perfect extending well-known results on GO-spaces. This prompted G. Gruenhage and D. Lutzer to ask the following question.

Question 1.1 ([7]). Must every e-separable monotonically normal space be perfect?

In Section 2 we describe a machine that embeds any topological space into an e-separable space $E(X)$. This machine will preserve many properties from the original topological space. In particular, we show that if X is perfect or monotonically normal, then $E(X)$ will be perfect or monotonically normal, respectively, thus providing counterexamples to Question 1.1. In Section 2 we also

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provide counterexamples to Question 1.1 that have a point network in the sense of Z. Balogh, see [1]. We leave open the problem of classifying when e-separable monotonically normal spaces must be perfect.

Problem 1.2. Characterize perfectly normal, e-separable, monotonically normal spaces.

The article closes by showing that e-separable extremely normal spaces are stratifiable in Section 3, providing another class of e-separable monotonically normal spaces that are perfect.

2. Counterexamples

Let X be any topological space. Let τ be the usual product topology on $X \times (\omega + 1)$. Endow $E(X) = X \times (\omega + 1)$ with the topology generated by $\tau \cup \{\{(x,n)\}: x \in X \text{ and } n \in \omega\}$. That is, the points in $D = \{(x,n): x \in X$ and $n \in \omega\}$ of $E(X)$ are isolated, and a basis for the points in $X \times \{\omega\}$ are the open neighborhoods from the product topology on $X \times (\omega+1)$. Clearly, X is homeomorphic to $X \times \{\omega\}$ with the subspace topology inherited from $E(X)$.

Theorem 2.1. *For any topological space X, $E(X)$ is e-separable.*

Proof: Note that $D_n = \{(x,n): x \in X\}$ is a closed-discrete subset of X for each $n \in \omega$. Hence, $D = \bigcup_{n\in\omega} D_n$ is a σ-closed-discrete dense subset of $E(X)$. □

Theorem 2.2. *$E(X)$ is perfect if and only if X is perfect.*

Proof: If $E(X)$ is perfect, then so is its subspace X. Conversely, suppose X is perfect, and let K be a closed subset of $E(X)$. Let $H_K = \{x \in X : (x,\alpha) \in K \setminus D(K)\}$ where $D(K) = D \cap K$. Then H_K is a closed subset of X. Since X is perfect, there are open sets $V_n \subset X$, $n \in \omega$, such that $H_K = \bigcap_{n\in\omega} V_n$. For each $n \in \omega$, let $U_n = (V_n \times [n,\omega]) \cup D(K)$ which is an open subset of $E(X)$. Then $K = \bigcap_{n\in\omega} U_n$, and $E(X)$ is perfect. □

To show that $E(X)$ is monotonically normal whenever X is monotonically normal, we use the following equivalent version of monotone normality.

Theorem 2.3 ([2])**.** *A topological space is monotonically normal if and only if for every $x \in U$, where U is open, there is an open set $\mu(x,U)$ such that*

(1) *$x \in \mu(x,U) \subset U$, and*
(2) *if $\mu(x,U) \cap \mu(y,V) \neq \emptyset$, then either $x \in V$ or $y \in U$.*

Theorem 2.4. *$E(X)$ is monotonically normal if and only if X is monotonically normal.*

Proof: Suppose $E(X)$ is monotonically normal. Then X is monotonically normal since X is homeomorphic to $X \times \{\omega\}$ and monotone normality is a hereditary property, see [9].

Suppose X is monotonically normal with monotone normality operator μ. For each $(x,n) \in D$, define $\mu_E((x,n),U) = \{(x,n)\}$ for every open set U of $E(X)$ which contains (x,n). Let $(x,\omega) \in X \times \{\omega\}$, and let U be any open subset of $E(X)$ containing (x,ω). Let n_U be the least ordinal such that $(x,\omega) \in W \times [n_U,\omega] \subset U$ for some open subset W for X. Let $W_U = \bigcup\{W \colon W \subset X$ is open and $W \times [n_U,\omega] \subset U\}$. Then $(x,\omega) \in W_U \times [n_U,\omega] \subset U$. Define $\mu_E((x,\omega),U) = \mu(x,W_U) \times [n_U,\omega]$. Clearly, $\mu_E((x,\omega),U)$ is an open set in $E(X)$ with $(x,\omega) \in \mu_E((x,\omega),U) \subset U$.

To show that the operator μ_E is a monotone normality operator for $E(X)$, suppose $\mu_E((x,\alpha),U) \cap \mu_E((y,\beta),V) \neq \emptyset$. If either $(x,\alpha) \in D$ or $(y,\beta) \in D$, then $(x,\alpha) \in \mu_E((y,\beta),V)$ or $(y,\beta) \in \mu_E((x,\alpha),U)$, respectively. Suppose $\alpha = \beta = \omega$. Then $\mu(x,W_U) \cap \mu(y,W_V) \neq \emptyset$ and either $x \in W_V$ or $y \in W_U$. This implies that $(x,\alpha) \in W_V \times [n_V,\omega] \subset V$ or $(y,\beta) \in W_U \times [n_U,\omega] \subset U$. Hence, μ_E is a monotone normality operator for $E(X)$, and $E(X)$ is monotonically normal. □

The following example answers Question 1.1 in the negative.

Example 2.5. Let ω_1 be the countable ordinals with the usual order topology. Then $E(\omega_1)$ is first countable, e-separable, monotonically normal space which is not perfectly normal.

Proof: Since ω_1 is first countable, monotonically normal and not perfect, $E(\omega_1)$ will be first countable, e-separable, and monotonically normal space which is not perfect. □

Note that the density of $E(\omega_1)$ is ω_1 and cannot be improved. A. J. Ostaszewki in [12] showed that separable, monotonically normal spaces are hereditarily Lindelöf and hence perfectly normal.

Before we present the next class of e-separable monotonically normal spaces that fail to be perfect, we remind the reader the link between monotonically normal spaces and the Collins-Roscoe structuring mechanism. A T_1 space X has $\mathcal{W}$ satisfying (F), see [5], if $\mathcal{W} = (\mathcal{W}(x) \colon x \in X)$ where each $\mathcal{W}(x)$ consists of subsets of X containing x and

(F) if $x \in U$ and U is open, then there exists an open set $V = V(x,U)$ containing x such that $x \in W \subset U$ for some $W \in \mathcal{W}(y)$ whenever $y \in V$.

If each $\mathcal{W}(x)$ is totally ordered by set inclusion, then $\mathcal{W}$ is said to satisfy chain (F), see [5]. The following establishes the relationship between the Collins-Roscoe structuring mechanism and monotonically normal spaces. A space is *acyclic monotonically normal*, see [11], if it has a monotone normality operator μ which also satisfies $\bigcap_{i<n} \mu(x_i, X \setminus \{x_{i+1}\}) = \emptyset$ whenever $n \geq 2$, $x_0, \ldots, x_{n-1}$ are distinct, and $x_n = x_0$.

Theorem 2.6 ([11])**.** *A topological space X has $\mathcal{W}$ satisfying chain* (F) *if and only if X is acyclic monotonically normal.*

If each $\mathcal{W}(x)$ is countable, then $\mathcal{W}$ is said to satisfy property (G), see [5]. Furthermore, if each $\mathcal{W}(x) = \{W_m : m \in \omega\}$ where $W_{m+1} \subset W_m$ for all $m \in \omega$, then $\mathcal{W}$ is said to satisfy decreasing (G), see [5]. Clearly, any $\mathcal{W}$ satisfying decreasing (G) also satisfies chain (F). Thus, any space X that has $\mathcal{W}$ satisfying decreasing (G) is monotonically normal. We use Balogh's terminology and say such spaces possess a point network if X has $\mathcal{W}$ satisfying decreasing (G). Before establishing the relationship between point networks and monotonically normal spaces, recall the definition of stratifiable spaces.

Definition 2.7. A T_1 topological space is semi-stratifiable if and only if one can assign to each closed set H a decreasing sequence $U_n(H)$, $n \in \omega$, of open sets satisfying

(1) $H = \bigcap_{n\in\omega} U_n(H)$, and
(2) if $H \subset K$, then $U_n(H) \subset U_n(K)$ for each $n \in \omega$.

A T_1 topological space is stratifiable if, in addition, X satisfies

(3) $H = \bigcap_{n\in\omega} \overline{U_n(H)}$

Monotonically normal semi-stratifiable spaces are stratifiable and stratifiable spaces are monotonically normal, see [9]. Recall that the pseudocharacter of a point x in a topological space X is defined to be $\psi(x, X) = \min\bigl\{|\mathcal{U}| : \mathcal{U}$ is a family of open subsets of X with $\{x\} = \bigcap \mathcal{U}\bigr\} + \omega$. The pseudocharacter of a topological space X is defined to be $\psi(X) = \sup\{\psi(x, X) : x \in S\}$.

Theorem 2.8 ([1], [4])**.** *A topological space X is stratifiable if and only if X has countable pseudocharacter and a point network.*

Monotone normality on its own is not enough to ensure stratifiability in the class of perfect spaces even for separable spaces. For example, the Alexadroff double arrow space and the Sorgenfrey line are examples of a perfectly normal, monotonically normal spaces that are not stratifiable since stratifiable GO-spaces are metrizable, see [10].

We are ready to describe counterexamples to Question 1.1 that have a point network. Let κ be a cardinal and $\mathrm{cf}(\kappa)$ be the cofinality of κ. Let $X = [0, \kappa]$ with the topology in which each point of $[0, \kappa)$ is isolated and $\{\{\kappa\} \cup A : (\kappa \setminus A) \in [\kappa]^{<\kappa}\}$ is a neighborhood base for κ. Let $Y(\kappa) = (X \times (\omega + 1)) \setminus \{(\kappa, n) : n \in \omega\}$ with the topology inherited from the product topology. Let $I = \{(\alpha, i) : \alpha \in \kappa$ and $i \in \omega\}$, and $T = \{(\alpha, \omega) : \alpha \in \kappa\}$. Note that I is a dense subset of Y and $Y = \{(\kappa, \omega)\} \cup I \cup T$.

For $(\alpha, \omega) \in T$, define $B_n(\alpha, \omega) = \{(\alpha, i) : n \le i \le \omega\}$. For $\alpha \in \kappa$ and $n \in \omega$, define $B(\alpha, n) = \{(\kappa, \omega)\} \cup \{(\beta, i) : \alpha \le \beta < \kappa$ and $n \le i \le \omega\}$. Note that $\{B_n(\alpha, \omega) : n \in \omega\}$ is a basis for (α, ω) and $\{B(\alpha, n) : \alpha \in \kappa$ and $n \in \omega\}$ is a basis for (κ, ω) in Y.

Theorem 2.9. *For every cardinal κ, $Y(\kappa)$ is e-separable and $\psi(X) = \mathrm{cf}(\kappa)$.*

Proof: To show that Y is e-separable, note that for $n \in \omega$, $D_n = \{(\alpha, n) : (\alpha, n) \in I\}$ is a closed-discrete subset of Y, $I = \bigcup_{n\in\omega} D_n$ is a σ-closed-discrete dense subset of Y.

To show $\psi(X) = \mathrm{cf}(\kappa)$, note that $\psi(y, Y) = \omega$ for every $y \in Y$ with $y \neq (\kappa, \omega)$. We show that $\psi((\kappa, \omega), X) = \mathrm{cf}(\kappa)$. Let $\{\alpha_\delta : \delta \in \mathrm{cf}(\kappa)\}$ be a cofinal subset of κ. Then $\{\kappa, \omega\} = \bigcap_{\delta\in\mathrm{cf}(\kappa)} B(\alpha_\delta, 0)$. Hence, $\psi((\kappa, \omega), X) \leq \mathrm{cf}(\kappa)$.

Let $\mathcal{U}$ be a family of open subsets of Y with $(\kappa, \omega) \in \bigcap \mathcal{U}$. Suppose $|\mathcal{U}| < \mathrm{cf}(\kappa)$. For each $U \in \mathcal{U}$, choose $\alpha_U \in \kappa$ and $n_U \in \omega$ such that $B(\alpha_U, n_U) \subset U$. Since $|\mathcal{U}| < \mathrm{cf}(\kappa)$, $\gamma = \sup\{\alpha_U : U \in \mathcal{U}\} < \kappa$. Then $(\gamma, \omega) \in \bigcap \mathcal{U}$. Therefore, $\psi((\kappa, \omega), Y) = \mathrm{cf}(\kappa)$ and the proof is complete. □

If κ is a cardinal with $\mathrm{cf}(\kappa) > \omega$, then $Y(\kappa)$ is an e-separable space that is not perfectly normal. We now show that $Y(\kappa)$ has a point network and hence monotonically normal.

Theorem 2.10. *Let κ be a cardinal. Then $Y(\kappa)$ has a point network.*

Proof: For $(\alpha, i) \in I$, let $\mathcal{W}((\alpha, i)) = \{W_n(\alpha, i) : n \in \omega\}$ where

$$W_n(\alpha, i) = \begin{cases} \{(\alpha, i), (\alpha, \omega), (\kappa, \alpha)\} & \text{if } n = 0; \\ \{(\alpha, i), (\alpha, \omega)\} & \text{if } n = 1; \\ \{(\alpha, i)\} & \text{if } n > 1. \end{cases}$$

For any open neighborhood U of (α, i), let $V(U, (\alpha, i)) = \{(\alpha, i)\}$.

For $(\alpha, \omega) \in T$, let $\mathcal{W}((\alpha, \omega)) = \{W_n(\alpha, \omega) : n \in \omega\}$ where

$$W_n(\alpha, \omega) = \begin{cases} \{(\alpha, \omega), (\kappa, \alpha)\} & \text{if } n = 0; \\ \{(\alpha, \omega)\} & \text{if } n > 0. \end{cases}$$

For any open neighborhood U of (α, ω), let $V(\alpha, \omega) = \{(\alpha, k) : i(U) \leq k \leq \omega\}$ where $i(U)$ be the least ordinal such that $\{(\alpha, i) : i(U) \leq i\} \subset U$.

Let $\mathcal{W}((\kappa, \omega)) = \{(\kappa, \omega)\}$. For any open neighborhood U of (κ, ω), let α_U be the minimum ordinal such that $\{(\beta, i) \in Y : \alpha_U \leq \beta < \kappa \text{ and } n \leq i \leq \omega\} \subset U$ for some $n \in \omega$. Let n_U be the least ordinal such that $\{(\beta, i) \in Y : \alpha_U \leq \beta < \kappa \text{ and } n_U \leq i \leq \omega\} \subset U$. Define $V(\kappa, \omega) = \{(\kappa, \omega)\} \cup \{(\beta, i) \in Y : \alpha_U \leq \beta < \kappa \text{ and } n_U \leq i \leq \omega\}$.

It is easy to check that $\mathcal{W} = (\mathcal{W}(y) : y \in Y)$ is a point network for Y. □

Corollary 2.11. *If κ is a cardinal with $\mathrm{cf}(\kappa) > \omega$, then $Y(\kappa)$ is an e-separable monotonically normal space with uncountable pseudocharacter (and hence not perfect).*

3. Extreme normal spaces

In this section, we show e-separable, extremely normal spaces are stratifiable.

Definition 3.1 ([13]). A space is extremely normal if it has a monotone normality operator μ satisfying if $x \neq y$ and $\mu(x,U) \cap \mu(y,V) \neq \emptyset$ then either $\mu(x,U) \subset V$ or $\mu(y,V) \subset U$.

Extremely normal spaces are monotonically normal which includes proto-metrizable spaces and spaces with one non-isolated point, see [13].

Theorem 3.2. *Extremely normal, e-separable spaces are stratifiable.*

Proof: Let X be an extremely normal space with extremely normal operator μ. By applying the definition of monotone normality to μ, we may assume that $\mu(x,V) \subset \mu(x,U)$ whenever $V \subset U$.

Let $D = \bigcup_{n\in\omega} D_n$ be a dense subset of x where each D_n is a closed-discrete subset of X and define $E_n = \bigcup_{k\leq n} D_k$. Note that E_n is a closed discrete subset of X for each $n \in \omega$. For each closed subset H of X, let $E_n(H) = \{x \in E_n : x \notin H\}$. Clearly, $E_n(H)$ is a closed subset of X.

Let $H \subset X$ be a closed subset of X and $n \in \omega$. Define the set $U_n(H) = \bigcup\{\mu(x, X \setminus E_0(H)) : x \in H\}$. Since $\mu(x,V) \subset \mu(x,U)$ whenever $V \subset U$, $U_{n+1}(H) \subset U_n(H)$ for each $n \in \omega$. Suppose there exists $y \in \left(\bigcap_{n\in\omega} U_n\right) \setminus H$ and consider $\mu(y, X \setminus H)$. Since $y \in \bigcap_{n\in\omega} U_n$, there is $y_n \in H$ such that $\mu(y, X\setminus H) \cap \mu(y_n, (X \setminus E_n(H)) \neq \emptyset$ for each $n \in \omega$. Since $y_n \in H$ and μ is an extreme monotone normality operator, we must have $\mu(y, X \setminus H) \subset X \setminus E_n(H)$ for every $n \in \omega$. Since $\mu(y, X\setminus H)$ is open, $\mu(y, X\setminus H) \cap D_{n_0} \neq \emptyset$ for some $n_0 \in \omega$ which is a contradiction. Hence $H = \bigcap_{n\in\omega} U_n(H)$.

If H and K are closed subsets of X with $H \subset K$, then $E_n(K) \subset E_n(H)$ and $\mu(x, (X \setminus E_n(H)) \subset \mu(x, (X \setminus E_n(H))$. This implies $U_n(H) \subset U_n(K)$ for each $n \in \omega$ which shows that X is semi-stratifiable. Since monotonically normal, semi-stratifiable spaces are stratifiable, see [9], X is stratifiable. □

Corollary 3.3. *If a protometrizable space is e-separable, then it is metrizable.*

Proof: Protometrizable spaces are precisely the extremely normal spaces where each point has a basis which is linearly ordered by set inclusion, see [13], and stratifiable protometrizable spaces are metrizable, see [8]. □

Corollary 3.4 ([8]). *Separable, protometrizable spaces are metrizable.*

Example 3.5. There are extremely normal stratifiable spaces that are not metrizable.

Proof: Similar to Section 2, let κ be a cardinal with $\mathrm{cf}(\kappa) > \omega$. Let $X = [0,\kappa]$ where each point of $[0,\kappa)$ is isolated and let $\{\kappa\} \cup A : (\kappa \setminus A) \in [\kappa]^{<\kappa}$ be a neighborhood base for κ. Let $Z = (X \times (\omega+1)) \setminus (\{(\kappa,n) : n \in \omega\} \cup \{(\alpha,\omega) : \alpha \in \kappa\})$, that is, Z is the point (κ,ω) along with the isolated points of $(X \times (\omega+1))$ with the topology inherited from the product topology. Since the character of X

is $\mathrm{cf}(\kappa) > \omega$, X is not metrizable. Since (κ, ω) is the only non-isolated point of Z and is a G_δ subset of Z, Z is stratifiable. $\square$

Added in proof. A *magnetic base system*, see [3], for a space X is a collection $\{\mathcal{B}_x : x \in X\}$ where each $\mathcal{B}_x$ is a base for the neighborhoods of x, with the following property: if $B_x \in \mathcal{B}_x$, $B_y \in \mathcal{B}_y$, and $B_x \cap B_y$, then either $x \in \overline{B_y}$ or $y \in \overline{B_x}$. A magnetic base system is *open* (*closed, clopen*) if each member of each $\mathcal{B}_x$ is open (closed, clopen, respectively). A space is utterly normal provided that X is regular and X has a magnetic base system. P. Cairns, H. Junilla, and P. Nyikos in [3] showed utterly normal spaces are monotonically normal.

The referee asked how e-separability and being perfect are related in utterly normal spaces. It is clear that a magnetic base system is a hereditary property. If $E(X)$ has an open (closed, clopen) magnetic base system, then X is utterly normal. Conversely, if $\{\mathcal{B}_x : x \in X\}$ is an open (closed, clopen) magnetic base system, then it is routine to check that $\mathcal{B}_{(x,n)} = \{x, n\}$ (for $x \in X$ and $n \in \omega$) and $\mathcal{B}_{(x,\omega)} = \{B \times [n, \omega] : B \in \mathcal{B}_x,\ n \in \omega\}$ will be a (open, closed, clopen) magnetic base system for $E(X)$. Similarly, one can show that $\{\{(\alpha, i)\} : \alpha \in \kappa,\ i \in \omega\} \cup \{\mathcal{B}_{\alpha,\omega} = \{B_n(\alpha, \omega) : n \in \omega\} : \alpha \in \kappa\} \cup \{\mathcal{B}_{\kappa,\omega} = \{B(\alpha, n) : \alpha \in \kappa,\ n \in \omega\}\}$ is a clopen magnetic base system for $Y(\kappa)$.

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