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On certain non-constructive properties of infinite-dimensional vector spaces

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Abstract. In set theory without the axiom of choice (AC), we study certain non-constructive properties of infinite-dimensional vector spaces. Among several results, we establish the following:

(i) None of the principles $\mathrm{AC}^{\mathrm{LO}}$ (AC for linearly ordered families of nonempty sets)—and hence $\mathrm{AC}^{\mathrm{WO}}$ (AC for well-ordered families of nonempty sets)—$\mathrm{DC}(<\kappa)$ (where κ is an uncountable regular cardinal), and "for every infinite set X, there is a bijection $f\colon X \to \{0,1\} \times X$", implies the statement "there exists a field F such that every vector space over F has a basis" in ZFA set theory. The above results settle the corresponding open problems from Howard and Rubin "Consequences of the axiom of choice", and also shed light on the question of Bleicher in "Some theorems on vector spaces and the axiom of choice" about the set-theoretic strength of the above algebraic statement.

(ii) "For every field F, for every family $\mathcal{V} = \{V_i \colon i \in I\}$ of nontrivial vector spaces over F, there is a family $\mathcal{F} = \{f_i \colon i \in I\}$ such that $f_i \in F^{V_i}$ for all $i \in I$, and f_i is a nonzero linear functional" is equivalent to the full AC in ZFA set theory.

(iii) "Every infinite-dimensional vector space over $\mathbb{R}$ has a norm" is not provable in ZF set theory.

Keywords: choice principle; vector space; base for vector space; nonzero linear functional; norm on vector space; Fraenkel–Mostowski permutation models of ZFA + ¬AC; Jech–Sochor first embedding theorem

Classification: 03E25, 03E35, 15A03, 15A04

1. Notation and terminology

- ZF denotes the Zermelo–Fraenkel set theory without AC.
- ZFC is ZF + AC.
- ZFA is ZF with the axiom of extensionality weakened to allow the existence of atoms.

Definition 1. Let X and Y be two sets:

(1) $|X| \le |Y|$ if there is an injection (i.e., a one-to-one mapping) $f\colon X \to Y$;
(2) $|X| = |Y|$ if there is a bijection (i.e., a one-to-one and onto mapping) $f\colon X \to Y$;
(3) $|X| < |Y|$ if $|X| \le |Y|$ and $|X| \ne |Y|$.

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Let us also recall here the definition of *alephs*.

Definition 2. By transfinite recursion on ordinals α we define:

$$\omega_0 = \omega \text{ (the set of natural numbers);}$$
$$\omega_{\alpha+1} = H(\omega_\alpha);$$
$$\omega_\alpha = \sup\{\omega_\beta \colon \beta < \alpha\} \ \Big(= \bigcup\{\omega_\beta \colon \beta < \alpha\}\Big) \text{ if } \alpha \text{ is a limit ordinal, } \alpha \neq 0.$$

(Where for a set A, $H(A)$ is the *Hartogs number* of A, i.e., the least ordinal α such that $|\alpha| \nleq |A|$.) As it is customary, ω_α is denoted by $\aleph_\alpha$.

For each ordinal number α, $\aleph_\alpha$ is an infinite well-ordered cardinal number, i.e., an infinite initial ordinal (where an ordinal α is an initial ordinal if for any $\beta < \alpha$, $|\beta| \neq |\alpha|$); in particular, $\aleph_0$ is the first infinite cardinal, and $\aleph_1$ is the first uncountable cardinal.

Definition 3 (Definitions concerning set-theoretic principles/choice forms)**.**

(1) AC is the *axiom of choice* (Form 1 in [5]): Every family of nonempty sets has a choice function.

(2) MC is the *axiom of multiple choice* (Form 67 in [5]): For every family $\mathcal{A}$ of nonempty sets there exists a function F with domain $\mathcal{A}$ such that for all $x \in \mathcal{A}$, $F(x)$ is a nonempty finite subset of x. (The fuction F is called a *multiple choice function* of $\mathcal{A}$.)

It is known that MC is equivalent to AC in ZF, but *not* equivalent to AC in ZFA (see [9, Theorems 9.1 and 9.2]).

(3) $\mathrm{AC}^{\mathrm{LO}}$ (Form 202 in [5]): Every linearly ordered family of nonempty sets has a choice function.

It is known that $\mathrm{AC}^{\mathrm{LO}}$ is equivalent to AC in ZF, but *not* equivalent to AC in ZFA (see [5]).

(4) $\mathrm{AC}^{\mathrm{WO}}$ (Form 40 in [5]): Every well-ordered family of nonempty sets has a choice function.

It is known that $\mathrm{AC}^{\mathrm{WO}}$ is strictly weaker than AC in ZF (see [5]).

(5) $\mathrm{AC}^{\aleph_0}$ is the *axiom of countable choice* (Form 8 in [5]): Every countably infinite family of nonempty sets has a choice function.

(6) $\mathrm{MC}^{\aleph_0}$ is the *axiom of countable multiple choice* (Form 126 in [5]): Every countably infinite family of nonempty sets has a multiple choice function.

(7) CH is the *continuum hypothesis*: $|2^\omega| = |\omega_1|$.

(2^ω is the set of all mappings from ω into $2 = \{0,1\}$; it is part of the foklore that in ZF, $|2^\omega| = |\mathbb{R}| = |\mathcal{P}(\omega)|$, where $\mathcal{P}(\omega)$ is the power set of ω.)

(8) $W_{\aleph_\alpha}$ (Form 71 (α) in [5]): For all x, $|x| \leq \aleph_\alpha \vee \aleph_\alpha \leq |x|$.

It is known that $\forall\,\kappa(W(\kappa))$ is equivalent to AC (see [9, Theorem 8.1]).

(9) $\mathrm{DC}(\aleph_\alpha)$ (Form 87 (α) in [5]): Given a relation R such that for every subset Y of a set X with $|Y| < \aleph_\alpha$, there is an $x \in X$ with $Y\ R\ x$ then there is a function $f \colon \aleph_\alpha \to X$ such that $\{f(\gamma) \colon \gamma < \beta\}\ R\ f(\beta)$ for all $\beta < \aleph_\alpha$.

$DC(\aleph_0)$ is the *principle of dependent choices* DC (Form 43 in [5]).

$DC(< \aleph_\alpha)$: For all $\kappa < \aleph_\alpha(DC(\kappa))$.

It is known that $\forall \kappa(DC(\kappa))$ is equivalent to AC (see [9, Theorem 8.1]).

(10) AC_{fin} (Form 62 in [5]): Every family of nonempty finite sets has a choice function.

It is known (see [5, Form 62 E]) that AC_{fin} is equivalent to the *Kinna–Wagner selection principle for families of finite sets*: For every family $\mathcal{A}$ of finite sets there is a function f such that for all $x \in \mathcal{A}$, if $|x| > 1$ then $f(x)$ is a nonempty proper subset of x.

(11) BPI is the *Boolean prime ideal theorem* (Form 14 in [5]): Every nontrivial Boolean algebra has a prime ideal.

It is known (see [5]) that AC_{fin} is strictly weaker than BPI in ZF.

(12) Form 3 in [5]: For every infinite set X, $|X| = |2 \times X|$.

It is known that $AC^{LO} \not\Rightarrow$ (Form 3) in ZFA, and (Form 3) $\not\Rightarrow$ AC in ZF (see [5]). Furthermore, Form 3 strictly implies Form 9 in ZF (Form 9 in [5]: "Every Dedekind-finite set is finite", where a set X is called *Dedekind-finite* if there is no injection $f\colon \omega \to X$).

Definition 4 (Definitions concerning vector spaces)**.** Let $(V, +, \cdot)$ be a vector space over a field F.

(1) If $X \subseteq V$, then $\langle X \rangle$ denotes the *linear span* of X, i.e., the subspace of V which consists of all finite linear combinations of elements of X.

(2) A set $B \subseteq V$ is called a *basis* for V if B is linearly independent and $V = \langle B \rangle$. (If B is a basis for V, then every vector $v \in V$ can be expressed uniquely as a finite linear combination of elements of B.)

(3) The vector space V is called *finite-dimensional* if V is finitely generated (i.e., V is spanned by a finite set of vectors). Otherwise, V is called *infinite-dimensional*.

(4) If $F \in \{\mathbb{R}, \mathbb{Q}\}$, then a mapping $\|\cdot\|\colon V \to \mathbb{R}^+ \cup \{0\}$ is called a *norm* on V if it has the following properties:

(N1) $\|x\| = 0 \Rightarrow x = 0_V$.

(N2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in F$ and $x \in V$.

(N3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

In the subsequent notations, the parameter F represents a field.

Definition 5 (Definitions concerning linear-algebraic principles)**.**

(1) $B(F)$: Every vector space over F has a basis. The statements "$\exists F(B(F))$" and "$\forall F(B(F))$" are, respectively, Form 428 and Form 66 in [5].

(2) $S(F)$ (Direct summand): For every vector space V over F and every subspace W of V there is a subspace W' of V such that $V = W \oplus W'$. (Every $v \in V$ can be written uniquely in the form $v = w + w'$ where $w \in W$ and $w' \in W'$.)

$S(F)$ is $AL21(F)$ in [13] and is Form 95 (F) in [5]. The following notation is also used in [5]: Form [67 AD] for "$\exists F(S(F))$", Form [67 AE] for "$\exists F$ of characteristic 0 ($\mathrm{S}(F)$)", Form [67 AF] for $\mathrm{S}(\mathbb{Q})$, Form [218 A] for "$\forall F(\mathrm{S}(F))$".

(3) $\mathrm{D}(F)$: For every nontrivial vector space V over F, there is a nonzero linear functional $f\colon V \to F$.

(4) $\mathrm{MD}(F)$: For every family $\mathcal{V} = \{V_i \colon i \in I\}$ of nontrivial vector spaces over F, there is a family $\mathcal{F} = \{f_i \colon i \in I\}$ such that $f_i \in F^{V_i}$ and f_i is a nonzero linear functional for all $i \in I$,.

(5) $\mathrm{ACVS}(F)$: For every family $\mathcal{V} = \{V_i \colon i \in I\}$ of nontrivial vector spaces over F, there is a choice function of the family $\mathcal{W} = \{V_i \setminus \{0_{V_i}\} \colon i \in I\}$.

(6) $\mathrm{PIDSub}(F)$: Every infinite-dimensional vector space over F has a proper infinite-dimensional subspace.

(7) $\mathrm{ILI}(F)$: Every infinite-dimensional vector space over F has an infinite linearly independent subset.

(8) For $F \in \{\mathbb{R}, \mathbb{Q}\}$, $\mathrm{N}(F)$: Every infinite-dimensional vector space over F has a norm.

2. Introduction, known results and aims

It is part of the folklore that AC implies that for every field F, every infinite-dimensional vector space V over F has a basis (using either *Zorn's lemma* or the *well-ordering theorem*; each of which is equivalent to AC (see [5])). (Recall also the standard result, taught in every undergraduate linear algebra course, that every finitely generated vector space has a basis, without invoking any form of choice.)

A. Blass in [2] showed that if for every field F, every vector space V over F has a basis, then, in ZF, the axiom of multiple choice MC is true. Since in ZF, MC is equivalent to AC (see [9, Theorem 9.1]), A. Blass established the following result.

Theorem 1. *In* ZF, $\forall F(\mathrm{B}(F))$ *is equivalent to* AC.

In [2], a multiple choice function is constructed for a given family $\mathcal{A}$ of nonempty sets using $\mathrm{B}(F)$ for a field F which depends on the family $\mathcal{A}$. If one considers a *specific field* F (for instance, the field $\mathbb{Q}$ of rational numbers), then it is an *open problem* whether the statement "every vector space over F (respectively, $\mathbb{Q}$) has a basis" implies AC.

Moreover, M. N. Bleicher in [3] asked whether or not AC is essential in proving $\exists F(\mathrm{B}(F))$ (Form 428 in [5]), and if it is essential, is its full strength essential, that is, if the aforementioned algebraic statement is equivalent to AC or to some weak form of AC.

P. Howard and E. Tachtsis in [6] provided an answer, in the setting of ZFA, to the first part of Bleicher's question by establishing the following result, which shows that AC is indeed essential in proving $\exists F(\mathrm{B}(F))$.

Theorem 2. *The statement* $\exists F(\mathrm{B}(F))$ *is false in the Dawson–Howard permutation model* $\mathcal{N}29$ *of* [5]. *Moreover, since* BPI *is true in* $\mathcal{N}29$, $\mathrm{BPI} \not\Rightarrow \exists F(\mathrm{B}(F))$ *in* ZFA.

With regard to Bleicher's questions on the set-theoretic strength of the statement $\exists F(\mathrm{B}(F))$, our *aim* in this paper is to supply *considerably further information* by establishing (in Section 3.1) that

$$\mathrm{AC}^{\mathrm{LO}} \not\Rightarrow \exists F(\mathrm{B}(F)) \qquad \text{in ZFA},$$

and consequently

$$\mathrm{AC}^{\mathrm{WO}} \not\Rightarrow \exists F(\mathrm{B}(F)) \qquad \text{in ZFA}.$$

(We note here that $\mathrm{AC}^{\mathrm{WO}}$ is false in $\mathcal{N}29$; see [5]. Recall also that $\mathrm{AC}^{\mathrm{LO}}$ is equivalent to AC in ZF, but not equivalent to AC in ZFA.)

Furthermore, we shall prove that for any uncountable regular cardinal $\aleph_\alpha$

$$\mathrm{DC}(< \aleph_\alpha) \not\Rightarrow \exists F(\mathrm{B}(F)) \qquad \text{in ZFA},$$

and also that

$$\text{“for every infinite set } X,\ |X| = |2 \times X|\text{”} \not\Rightarrow \exists F(\mathrm{B}(F)) \qquad \text{in ZFA}.$$

By P. Howard and J. E. Rubin in [5], it is stated as *unknown* whether any of $\mathrm{AC}^{\mathrm{LO}}$, $\mathrm{AC}^{\mathrm{WO}}$, $\mathrm{DC}(\aleph_\alpha)$ and “for every infinite set X, $|X| = |2 \times X|$” implies $\exists F(\mathrm{B}(F))$. Our aforementioned results to be proved in the sequel, *settle these open questions* in the setting of ZFA.

Moreover, it is apparent that the above results indicate that $\exists F(\mathrm{B}(F))$ is a *strong axiom*.

A closely related subject that we will study in this paper (see Section 3.2) concerns the existence of nonzero linear functionals on nontrivial vector spaces. It is clear that for any field F, $\mathrm{B}(F)$ implies $\mathrm{D}(F)$. The set-theoretic strength of the principle $\mathrm{D}(F)$ for a given F, as well as of $\forall F(\mathrm{D}(F))$ and $\exists F(\mathrm{D}(F))$, has been investigated thoroughly by P. Howard and E. Tachtsis in [6] and by M. Morillon in [12]. In [6], among several results that we list below for the reader's convenience, it is shown that $\forall F(\mathrm{D}(F))$ implies none of AC and $\forall F(\mathrm{B}(F))$ in ZFA, and BPI implies “$\forall$ finite $F(\mathrm{D}(F))$”. The question of whether $\forall F(\mathrm{D}(F))$ implies AC in ZF is *still open.* Characterizations of $\mathrm{D}(F)$, where F is any field, as well as the deductive strength of $\mathrm{D}(\mathbb{Q})$ and $\mathrm{D}(\mathbb{Z}_p)$, where p is a prime number, are given by M. Morillon in [12] and by P. Howard and E. Tachtsis in [6].

Now, it is clear that AC implies $\forall F(\mathrm{MD}(F))$ (the latter principle being *introduced here*), which in turn implies $\forall F(\mathrm{D}(F))$, and hence the natural question which arises here is whether or not any of the previous two implications is reversible.

We shall establish (in Section 3.2) that $\forall F(\mathrm{MD}(F))$ is equivalent to the full AC in ZFA, and therefore using the result from [6] that $\forall F(\mathrm{D}(F))$ does not imply AC in ZFA, we shall obtain that $\forall F(\mathrm{D}(F))$ is strictly weaker than $\forall F(\mathrm{MD}(F))$

in ZFA. Moreover, we shall prove that $\forall F(\mathrm{ACVS}(F))$ is equivalent to AC in ZFA, hence $\forall F(\mathrm{MD}(F))$ is equivalent to $\forall F(\mathrm{ACVS}(F))$.

Let us recall here (most of) the results from [12] and [6] on $\mathrm{D}(F)$.

Theorem 3 ([12]). *For any field F, the following statements are pairwise equivalent:*

(i) $\mathrm{D}(F)$;

(ii) $\mathrm{DE}(F)$: *for every nontrivial vector space V over F, for every vector subspace W of V and for every linear functional $f\colon W \to F$ there exists a linear functional $g\colon V \to F$ such that $f \subseteq g$;*

(iii) $\mathrm{DS}(F)$: *for every nontrivial vector space V over F and every $a \in V \setminus \{0\}$, there exists a linear functional $f\colon V \to F$ such that $f(a) = 1$;*

(iv) *multiple* $\mathrm{DS}(F)$: *for every family $\mathcal{V} = \{V_i\colon i \in I\}$ of nontrivial vector spaces over F, for every family $\mathcal{A} = \{a_i\colon i \in I\}$ such that a_i is a nonzero element of V_i for all $i \in I$, there exists a family $\mathcal{F} = \{f_i\colon i \in I\}$ such that $f_i\colon V_i \to F$ is a linear functional and $f_i(a_i) = 1$ for all $i \in I$.*

Theorem 4 ([6]). *The following statements hold:*

(i) BPI *implies the statement "for every finite field F, $\mathrm{D}(F)$". Hence, the latter statement is strictly weaker than* AC *in* ZF*; in particular, "for every finite field F, $\mathrm{D}(F)$" is true in the Basic Cohen model (Model $\mathcal{M}1$ of* [5]*) of* $\mathrm{ZF} + \mathrm{BPI} + \neg\mathrm{AC}$.

(ii) *For any field F,* $\mathrm{S}(F)$ *implies* $\mathrm{D}(F)$.

(iii) *For any field F,* $\mathrm{D}(F)$ *is equivalent to "for every system S of linear equations over F, S has a solution (in F) if and only if every finite subsystem of S has a solution (in F)" (the latter statement is Form 284 in* [5]*).*

(iv) MC *(which is equivalent to $\exists F(\mathrm{S}(F))$) implies $\exists F(\mathrm{D}(F))$. Therefore, $\exists F(\mathrm{D}(F))$ does not imply* AC *in* ZFA.

(v) *The statement $\forall F(\mathrm{D}(F))$ does not imply* AC *in* ZFA. *In particular, $\forall F(\mathrm{S}(F))$ is true in Lévy's permutation model $\mathcal{N}6$ (in* [5]*) of* $\mathrm{ZFA} + \neg\mathrm{AC}$, *and hence (by* (ii)*) $\forall F(\mathrm{D}(F))$ is also true in $\mathcal{N}6$.*

(vi) *If F is a field of characteristic* 0, *then, in* ZFA, MC *implies* $\mathrm{D}(F)$.

(vii) ([12]) $\mathrm{D}(\mathbb{Q})$ *implies van Douwen's choice principle (i.e., every family $\mathcal{A} = \{(A_i, \leq_i)\colon i \in I\}$ of linearly ordered sets isomorphic with $(\mathbb{Z}, \leq)$, where "$\leq$" is the usual ordering on $\mathbb{Z}$, has a choice function). Further,* $\mathrm{D}(\mathbb{Q})$ *does not imply* $\mathrm{B}(\mathbb{Q})$.

(ix) ([12]) *For every prime natural number p, $\mathrm{D}(\mathbb{Z}_p)$ implies $C(p)$ (i.e., for every family $\{X_i\colon i \in I\}$ of nonempty finite sets, there is a function F with domain I such that $F(i) \subseteq X_i$ and p does not divide the cardinal number $|F(i)|$ of $F(i)$ for all $i \in I$).*

For any field F, $\mathrm{D}(F)$ is also related to $\mathrm{PIDSub}(F)$ ("every infinite-dimensional vector space over F has a proper infinite-dimensional subspace"). In particular, we will show (in Section 3.4) that for any field F, $\mathrm{D}(F)$ implies $\mathrm{PIDSub}(F)$ (and hence, by Theorem 4 (v), we will deduce that $\forall F(\mathrm{PIDSub}(F))$ does not imply AC in ZFA) and that the latter principle is not provable in ZF. We do not know what choice forms are implied by $\mathrm{PIDSub}(F)$ for specific fields F, or whether $\forall F(\mathrm{PIDSub}(F))$ implies AC in ZF. However, in view of Theorem 4 (i) and the fact that $\mathrm{D}(F)$ implies $\mathrm{PIDSub}(F)$, we shall obtain that for any finite field F, $\mathrm{PIDSub}(F) \not\Rightarrow \mathrm{AC}$ in ZF. In addition to the above results, we shall also prove that if F is any well-orderable field and if κ is any infinite well-ordered cardinal number with $\kappa > |F|$, then W_κ implies $\mathrm{PIDSub}(F)$. In particular, if $\kappa > \aleph_1$, then $\mathrm{CH} + W_\kappa$ implies $\mathrm{PIDSub}(\mathbb{R})$.

A ZFC-property of infinite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{Q}$, which is a consequence of the existence of a basis, is the *existence of norms* on such spaces. Indeed, if V is an infinite-dimensional vector space over F (where $F \in \{\mathbb{R}, \mathbb{Q}\}$), then firstly, we may let (by AC) B be a basis for V. Then for every element $v \in V$ there exists a unique finite set $\{b_1, \dots, b_n\} \subseteq B$ and scalars $\lambda_1, \dots, \lambda_n$ from F such that

$$v = \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_n b_n.$$

The mappings $\|\cdot\|_1 \colon V \to F^+ \cup \{0\}$ and $\|\cdot\|_2 \colon V \to F^+ \cup \{0\}$ defined by

$$\|v\|_1 = \max\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_n|\}$$

and

$$\|v\|_2 = \sum_{i=1}^{n} |\lambda_i|,$$

are easily seen to be norms on V. Since the above argument was carried out in ZFC (in particular, the nonprovable (in ZF, see [12]) principle $\mathrm{B}(F)$ suffices as an assumption), the most natural question that emerges here is whether $\mathrm{N}(F)$, where $F \in \{\mathbb{R}, \mathbb{Q}\}$, is provable in set theory without choice. We answer this question (in Section 3.3) in the negative by showing that there are permutation models in which $\mathrm{N}(\mathbb{R})$ and $\mathrm{N}(\mathbb{Q})$ are false; the latter results are transferred to ZF via the Jech–Sochor first embedding theorem. We also show that it is relatively consistent with ZF that there exists an infinite-dimensional vector space over $\mathbb{R}$ which has a norm, but has no basis.

In the concluding Section 3.5 of the paper, we investigate the question of the placement of the principle $\forall F(\mathrm{ILI}(F))$ ("for every field F, every infinite-dimensional vector space over F has an infinite linearly independent subset") in the hierarchy of weak choice forms, and prove that the latter statement lies in strength between the weak choice forms $\mathrm{AC}^{\aleph_0}$ and $\mathrm{MC}^{\aleph_0}$. The above result enhances the result of P. Howard and E. Tachtsis in [8, Theorem 6.1] that the stronger principle "for every field F, every infinite-dimensional vector space over F has a countably infinite linearly independent subset" also lies in strength between the aforementioned weak choice principles. In addition, we prove that $\mathrm{ILI}(\mathbb{R})$ implies "every countably infinite family $\mathcal{A} = \{A_n \colon n \in \omega\}$ of finite sets each having at least two elements, has a partial Kinna–Wagner selection function" (actually, in Section 3.3, it is shown that the above implication holds if $\mathbb{R}$ is replaced by any linearly orderable field of characteristic 0), and hence the implication "$\forall F(\mathrm{ILI}(F)) \Rightarrow \mathrm{MC}^{\aleph_0}$" is not reversible in ZFA set theory (the Second Fraenkel model—Model $\mathcal{N}2$ in [5]—witnesses $\mathrm{MC} + \neg\,\mathrm{ILI}(\mathbb{R})$).

3. Main results

This section is divided into five subsections. The first one deals with Bleicher's question from [3] about the set-theoretic strength of $\exists F(\mathrm{B}(F))$.

3.1 On Bleicher's question about the set-theoretic strength of $\exists F(\mathrm{B}(F))$**.** We start this part with the proof that $\mathrm{AC}^{\mathrm{LO}}$ (and hence $\mathrm{AC}^{\mathrm{WO}}$) does not imply $\exists F(\mathrm{B}(F))$ in ZFA set theory.

Theorem 5. (i) $\mathrm{AC}^{\mathrm{LO}}$ *implies* $\exists F(\mathrm{B}(F))$ *in* ZF.

(ii) $\mathrm{AC}^{\mathrm{LO}}$ *(and hence* $\mathrm{AC}^{\mathrm{WO}}$*) does not imply* $\exists F(\mathrm{B}(F))$ *in* ZFA.

Proof: (i) This follows from the fact that, in ZF, $\mathrm{AC}^{\mathrm{LO}} \Leftrightarrow \mathrm{AC}$.

(ii) For our ZFA independence result, we shall use the permutation model $\mathcal{N}12(\aleph_1)$ of [5], whose description is as follows: We start with a model M of ZFA + AC with an $\aleph_1$-sized set $A = \{a_i : i < \aleph_1\}$ of atoms. Let G be the group of all permutations of A. For any element x of M, $\mathrm{fix}_G(x)$ denotes the subgroup $\{\varphi \in G : \forall t \in x(\varphi(t) = t)\}$ of G and $\mathrm{Sym}_G(x)$ denotes the subgroup $\{\varphi \in G : \varphi(x) = x\}$ of G. Let Γ be the normal filter of subgroups of G generated by $\{\mathrm{fix}_G(E) : E \subset A \text{ and } |E| \leq \aleph_0\}$. An element x of M is called *symmetric* if $\mathrm{Sym}_G(x) \in \Gamma$, and hence x is symmetric if there is some countable (finite or countably infinite) set $E \subset A$ such that $\mathrm{fix}_G(E) \subseteq \mathrm{Sym}_G(x)$. Under these circumstances, E is called a *support* of x. The element x of M is called *hereditarily symmetric* if x and every element in the transitive closure of x is symmetric. $\mathcal{N}12(\aleph_1)$ is the FM model determined by M, G and Γ, that is, $\mathcal{N}12(\aleph_1)$ consists exactly of all the hereditarily symmetric elements of M.

It is known that $\mathrm{AC}^{\mathrm{LO}}$ is true in $\mathcal{N}12(\aleph_1)$ (see [5]). Thus, we only need to show that $\exists F(\mathrm{B}(F))$ is false in $\mathcal{N}12(\aleph_1)$.

To this end, let $(F, +, \cdot)$ be any field in $\mathcal{N}12(\aleph_1)$, and also let

$$Z = \bigcup \{X \times \{X\} : X \in [A]^\omega\},$$
$$W = \{f : Z \to F : |\{t \in Z : f(t) \neq 0\}| < \aleph_0\}$$

and

$$V = \{f \in W : \forall X \in [A]^\omega, \sum_{t \in X \times \{X\}} f(t) = 0\}.$$

For each $u \in Z$, let χ_u be the characteristic function of $\{u\}$. Clearly, W is a vector space over F with basis $B_W = \{\chi_u : u \in Z\}$ and V is a subspace of W. Let E_0 be a support for $(F, +, \cdot)$; then E_0 is a support for W, B_W and V, so W, B_W and V are in $N12(\aleph_1)$.

We shall prove that V does not have a basis in the model $\mathcal{N}12(\aleph_1)$. By way of a contradiction, assume that V has a basis B in $\mathcal{N}12(\aleph_1)$ with support E and without loss of generality assume that $E_0 \subseteq E$. Since $|A| = \aleph_1$ in the ground model M and E is countable, there exists an element $j \in \aleph_1$ which is greater than $\sup\{k \in \aleph_1 : a_k \in E\}$. Let $X \in [A]^\omega$ such that $a_k \in X \Rightarrow k \geq j$. Clearly, $X \cap E = \emptyset$. Since B is a basis for V, there is an element $a \in X$ and an element

$f \in B$ such that $f(a') \neq 0$, where $a' = (a, X)$ and since $\sum_{t \in X \times \{X\}} f(t) = 0$, there is also an element $b \in X \setminus \{a\}$ such that $f(b') \neq 0$, where $b' = (b, X)$. Since $f \in W$, we have $\{t \in Z \colon f(t) \neq 0\}$ is finite (and includes a' and b'). Assume that $\{t \in Z \colon f(t) \neq 0\} \setminus \{a', b'\} = \{r_1, r_2, \ldots, r_m\}$. We may now write f as a linear combination of elements of B_W as follows:

$$f = f(a') \cdot \chi_{a'} + f(b') \cdot \chi_{b'} + \sum_{i=1}^{m} f(r_i) \cdot \chi_{r_i}. \tag{1}$$

Choose a positive integer n for which $n^2 > 2n + m$, and also choose two disjoint subsets $S_{a'}$ and $S_{b'}$ of X (hence, $(S_{a'} \cup S_{b'}) \cap E = \emptyset$) such that $|S_{a'}| = |S_{b'}| = n$ and $(S_{a'} \cup S_{b'}) \cap \{a', b', r_1, \ldots, r_m\} = \emptyset$.

For each ordered pair $(s, t) \in S_{a'} \times S_{b'}$, let $\varphi_{(s,t)} = (a', s)(b', t)$ (i.e., $\varphi_{(s,t)}$ is the product of the transpositions (a', s) and (b', t), so that $\varphi_{(s,t)}(u) = u$ for all $u \in Z \setminus \{a', b', s, t\}$). Clearly, $\varphi_{(s,t)} \in \mathrm{fix}_G(E)$ for all $(s, t) \in S_{a'} \times S_{b'}$, hence $\varphi_{(s,t)}(B) = B$ and $\varphi_{(s,t)}((F, +, \cdot)) = (F, +, \cdot)$ for all $(s, t) \in S_{a'} \times S_{b'}$.

It follows that for all $(s, t) \in S_{a'} \times S_{b'}$ and for all i, $1 \leq i \leq m$, $\varphi_{(s,t)}(r_i) = r_i$ and hence $\varphi_{(s,t)}(\chi_{r_i}) = \chi_{r_i}$. Furthermore, $\varphi_{(s,t)}(\chi_{a'}) = \chi_s$ and $\varphi_{(s,t)}(\chi_{b'}) = \chi_t$. Therefore applying $\varphi_{(s,t)}$ to both sides of equation (1) we obtain

$$\varphi_{(s,t)}(f) = \varphi_{(s,t)}(f(a')) \cdot \chi_s + \varphi_{(s,t)}(f(b')) \cdot \chi_t + \sum_{i=1}^{m} \varphi_{(s,t)}(f(r_i)) \cdot \chi_{r_i}. \tag{2}$$

Since, the restriction of $\varphi_{(s,t)}$ to F for all $(s, t) \in S_{a'} \times S_{b'}$ is a field automorphism, and $f(a') \neq 0$ and $f(b') \neq 0$, it follows (from equation (2)) that $\varphi_{(s,t)}(f)(s) = \varphi_{(s,t)}(f(a')) \neq 0$ and $\varphi_{(s,t)}(f)(t) = \varphi_{(s,t)}(f(b')) \neq 0$. However, $\varphi_{(s,t)}(f)(u) = 0$ for every $u \in (S_{a'} \cup S_{b'}) \setminus \{s, t\}$. Hence, if (s_1, t_1) and (s_2, t_2) are two distinct elements of $S_{a'} \times S_{b'}$, then $\varphi_{(s_1,t_1)}(f) \neq \varphi_{(s_2,t_2)}(f)$. It follows that the set $D = \{\varphi_{(s,t)}(f) \colon (s, t) \in S_{a'} \times S_{b'}\}$ has cardinality $n^2 = |S_{a'} \times S_{b'}|$.

Since $\varphi_{(s,t)} \in \mathrm{fix}_G(E)$ and E is a support of the basis B, we have $D \subseteq B$, hence D is linearly independent. However, by equation (2), D is a subset of the subspace of W spanned by $\{\chi_{r_i} \colon 1 \leq i \leq m\} \cup \{\chi_s \colon s \in S_{a'}\} \cup \{\chi_t \colon t \in S_{b'}\}$, which has $m + 2n$ elements. Since $n^2 > 2n + m$, we have that D is not linearly independent, which is a contradiction. □

Remark 1. In order to provide further insight to the reader, we note here that $\exists\, F(\mathrm{B}(F))$ is also false in a model recently constructed by Howard and Tachtsis in [7], whose description is as follows: We start with a model $\mathcal{M}$ of ZFA + AC with an $\aleph_1$-sized set A of atoms, which is a disjoint countably infinite union of $\aleph_1$-sized sets so that $A = \bigcup\{A_i \colon i \in \omega\}$, where $A_i = \{a_{i,j} \colon j < \aleph_1\}$. Let G be the group of all permutations of A, which fix A_i for all $i \in \omega$. Let Γ be the normal filter of subgroups of G generated by $\{\mathrm{fix}_G(E) \colon E \subset A \text{ and } |E| \leq \aleph_0\}$. Let $M_{\aleph_1}$ be the FM model determined by $\mathcal{M}$, G and Γ. (In [7], it has been shown that if instead of G, one uses as a group of permutations the set of all permutations of A which fix each A_i but move only countably many atoms, then the resulting permutation model is equal to $M_{\aleph_1}$.)

In [7], it is shown that $\mathrm{AC}^{\mathrm{LO}}$ is true in $M_{\aleph_1}$. On the other hand, the fact that for every field F in $M_{\aleph_1}$ there exists a vector space V over F in $M_{\aleph_1}$ which has no basis, can be established similarly to the proof of Theorem 5. Indeed, if $(F,+,\cdot)$ is any field in $M_{\aleph_1}$, then we let

$$Z = \bigcup\{X \times \{(i,X)\} \colon i \in \omega,\ X \in [A_i]^\omega\},$$
$$W = \{f \colon Z \to F \colon |\{t \in Z \colon f(t) \neq 0\}| < \aleph_0\}$$

and

$$V = \{f \in W \colon \forall\, i \in \omega,\ \forall X \in [A_i]^\omega, \sum_{t \in X \times \{(i,X)\}} f(t) = 0\}.$$

Then, the vector space V over F has no basis in $M_{\aleph_1}$. We leave the details to the interested reader.

Theorem 6. *Let κ be an uncountable regular cardinal number. Then the principle* $\mathrm{DC}(<\kappa)$ *does not imply* $\exists\, F(\mathrm{B}(F))$ *in* ZFA.

PROOF: The required model is a generalization of $\mathcal{N}\,12\,(\aleph_1)$ which was used in the proof of Theorem 5. So, one starts with a ground model M of ZFA + AC whose set of atoms, A, has cardinality κ. Let G be the group of all permutations of A, and let $\mathcal{F}$ be the normal filter on G which is generated by the subgroups $\mathrm{fix}_G(E)$ of G, where $E \subset A$ has cardinality less than κ. Let $\mathcal{N}$ be the permutation model which is determined by M, G and $\mathcal{F}$.

The fact that the principle $\mathrm{DC}(\lambda)$ holds in $\mathcal{N}$ for every infinite cardinal $\lambda < \kappa$ follows from T. J. Jech, see [9, Lemma 8.4], which states that if $\beta < \kappa$ and g is a function on β with values in $\mathcal{N}$, then $g \in \mathcal{N}$. On the other hand, the proof that $\exists\, F(\mathrm{B}(F))$ is false in $\mathcal{N}$ is similar to the proof of Theorem 5 and is left to the reader. $\square$

We note that for the proof of Theorem 6, we could also use the corresponding generalization of the permutation model given in Remark 1.

Theorem 7. *"For every infinite set X, $|X| = |2 \times X|$" does not imply $\exists\, F(\mathrm{B}(F))$ in* ZFA.

PROOF: For our independence result we shall use the Halpern–Howard Model $\mathcal{N}\,9$ in [5]: We start with a ground model M of ZFA + AC with a set A of atoms which has the structure of the set

$$\omega^{(\omega)} = \{s \colon \omega \to \omega \wedge\ \exists\, n\, \forall\, j > n,\ s(j) = 0\}.$$

We identify A with the latter set to simplify the description of the group G.

A group G is the group of all permutations φ of A such that $\{a \in A \colon \varphi(a) \neq a\}$ (the support of φ) is bounded, that is the pseudo lengths of elements (for $s \in A$, the *pseudo length* of s is the least number k such that for all $l \geq k$, $s(l) = 0$, see [4]) of the support of φ have a finite bound.

The normal ideal $\mathcal{I}$ of supports is generated by $S = \{A^n_{\mathbf{0}} : n \in \omega\}$, where $\mathbf{0}$ is the sequence which is identically 0 and $A^n_{\mathbf{0}} = \{s \in A : \forall j \geq n,\ s(j) = 0\}$. A set x has support $A^n_{\mathbf{0}}$ if all permutations φ in G leaving

(a) $A^n_{\mathbf{0}}$ pointwise fixed,

(b) $\{A^n_s : s \in A\}$, where $A^n_s = \{t \in A : \forall j \geq n,\ t(j) = s(j)\}$ (called "the n-block containing s" in [4]), fixed, i.e., $\varphi(A^n_s) = A^n_{\varphi(s)}$, and

(c) the first n coordinates of each $s \in A$ pointwise fixed, also leave x fixed. $\mathcal{N}9$ is the Fraenkel–Mostowski permutation model which is determined by M, G and $\mathcal{I}$.

J. D. Halpern and P. E. Howard in [4] have shown that, in $\mathcal{N}9$, the principle "for every infinite set X, $|X| = |2 \times X|$" is true.

Therefore, we only need to show that $\exists F(\mathrm{B}(F))$ is false in $\mathcal{N}9$. To this end, let $(F, +, \cdot)$ be any field in $\mathcal{N}9$ and also let (as in the proof of Theorem 5)

$$Z = \bigcup \{X \times \{X\} : X \in [A]^\omega\},$$

$$W = \{f : Z \to F : |\{t \in Z : f(t) \neq 0\}| < \aleph_0\}$$

and

$$V = \{f \in W : \forall X \in [A]^\omega, \sum_{t \in X \times \{X\}} f(t) = 0\}.$$

(Note that $Z \in \mathcal{N}9$ since it has empty support, i.e., any permutation of A in G fixes Z. Furthermore, recall that "for every infinite set X, $|X| = |2 \times X|$" (strictly) implies "every Dedekind-finite set is finite" (in ZF).)

Then, W is a vector space over F with basis $B_W = \{\chi_u : u \in A\}$ and V is a subspace of W. Let E_0 be a support for $(F, +, \cdot)$. As in the proof of Theorem 5, we have that W, B_W and V are in $\mathcal{N}9$.

We show that V does not have a basis in $\mathcal{N}9$. By way of a contradiction, assume that V has a basis B in $\mathcal{N}9$ with support an n-block $E = A^n_{\mathbf{0}}$ (containing $\mathbf{0}$) for some $n \in \omega$. Without loss of generality, we may assume that $E_0 \subseteq E$. Let $\mathbf{r} = (r_0, \ldots, r_{n-1}) \in \omega^n$, and also let

$$X = \{s \in A : s \restriction n = \mathbf{r},\ s(n) \in \omega \setminus \{0\}, \text{ and for all } m > n,\ s(m) = 0\}.$$

Then $X \in \mathcal{N}9$ and is countably infinite in $\mathcal{N}9$. Indeed, X is an infinite subset of the $(n+1)$-block $A^{n+1}_{\mathbf{0}}$ containing $\mathbf{0}$, which is countably infinite in $\mathcal{N}9$; $|A^{n+1}_{\mathbf{0}}| = |\omega^{n+1}| = \aleph_0$ in the ground model M, and $A^{n+1}_{\mathbf{0}}$ is a support of any enumeration of its elements by natural numbers—see conditions **(a)**, **(b)**, and **(c)** above for an m-block $A^m_{\mathbf{0}}$ containing $\mathbf{0}$ to be a support for a set.

Furthermore, note that $X \cap E = \emptyset$ (since $E = A^n_{\mathbf{0}} = \{s \in A : \forall j \geq n,\ s(j) = 0\}$ and for all $s \in X$, $s(n) \in \omega \setminus \{0\}$) and that for any two distinct elements s and t of X, $s \restriction n = t \restriction n = \mathbf{r}$ and s, t have the same pseudo length, namely $n + 1$.

We may now continue identically to the proof of Theorem 5 in order to derive the required contradiction. Thus, we conclude that $\exists F(\mathrm{B}(F))$ is false in $\mathcal{N}9$, finishing the proof of the theorem. $\square$

3.2 On the axiom of choice for families of sets of nonzero linear functionals for nontrivial vector spaces over the same field. Clearly, AC ($\Leftrightarrow$ $\forall F(\mathrm{B}(F))$ in ZF) implies $\forall F(\mathrm{MD}(F))$. So the natural question that arises is whether $\forall F(\mathrm{MD}(F))$ is equivalent to AC.

We show next that, in ZFA, the answer to the above question is in the *affirmative*. Furthermore, we prove that also $\forall F(\mathrm{ACVS}(F))$ is equivalent to the full AC in ZFA, thus $\forall F(\mathrm{MD}(F))$ and $\forall F(\mathrm{ACVS}(F))$ express the same truth in ZFA. Our proof comprises two steps; firstly, we show that the above statements imply MC, and secondly, that they imply $\mathrm{AC}_{\mathrm{fin}}$ in ZFA.

Theorem 8. *In* ZFA, *the following statements are true:*
(i) $\forall F(\mathrm{MD}(F))$ *implies* MC.
(ii) $\forall F(\mathrm{ACVS}(F))$ *implies* MC.

Proof: (i) Assume $\forall F(\mathrm{MD}(F))$ is true. Let $\mathcal{X} = \{X_i : i \in I\}$ be a family of nonempty sets. Without loss of generality, we assume that $\mathcal{X}$ is disjoint. Let $X = \bigcup \mathcal{X}$ and let F be any field which is disjoint from X. Let $F(X)$ be the field of all rational functions with indeterminates from X and coefficients in F. (Every element $u \in F(X)$ is of the form $(p_1 + \cdots + p_n)/(q_1 + \cdots + q_m)$, where p_i and q_i are monomials, i.e., of the form $a \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot \ldots \cdot x_k^{n_k}$ where $a \in F$, $x_r \in X$ (with $1 \leq r \leq k$), and $q_1 + \ldots + q_m \neq 0$.) For every $i \in I$, the *i-degree* of a monomial $p = a \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot \ldots \cdot x_k^{n_k}$ is defined as $\sum_{x_r \in X_i} n_r$. A rational function $u \in F(X)$ is called *i-homogeneous of degree* 0 if all monomials appearing in the quotient expression of u have the same i-degree. Let K be the subfield of $F(X)$ consisting of all rational functions in $F(X)$ that are i-homogeneous of degree 0 for all $i \in I$. Then $F(X)$ is a vector space over K.

For each $i \in I$, let V_i be the subspace of $F(X)$ which is generated by X_i, i.e., V_i is the linear span $\langle X_i \rangle$. Then for each $i \in I$, V_i is a finite-dimensional vector space over K. Indeed, let x be any element of X_i. Then for all $y \in X_i$ with $y \neq x$, we have $y = (y/x) \cdot x$ and $(y/x) \in K$. It follows that $V_i = \langle x \rangle$.

By our hypothesis, we have that for the family $\mathcal{V} = \{V_i : i \in I\}$ there is a family $\mathcal{F} = \{f_i : i \in I\}$ such that for all $i \in I$, $f_i : V_i \to K$ is a nonzero linear functional.

Claim 1. For every $i \in I$, there exists a unique element $v_i \in V_i$ such that $f_i(v_i) = 1_K$ (where 1_K denotes the multiplicative identity of the field K).

Proof: We prove the claim by contradiction. So assume that there exists an index $i \in I$ such that there are at least two elements $u, w \in V_i$ with $f_i(u) = f_i(w) = 1_K$. Clearly, $u \neq 0_{V_i}$ and $w \neq 0_{V_i}$. Since $V_i = \langle x \rangle$ for all $x \in X_i$, we have that $u = \lambda w$ for some $\lambda \in K \setminus \{0\}$. Then we have

$$1_K = f_i(u) = f_i(\lambda w) = \lambda f_i(w) = \lambda \cdot 1_K = \lambda.$$

We conclude that $u = w$, which is a contradiction. This completes the proof of the claim. $\square$

For each $i \in I$, fix the unique element $v_i \in V_i$ guaranteed by Claim 1. We define a mapping F on I by requiring for all $i \in I$,

$$F(i) = \{x \in X_i : x \text{ appears in the quotient expression of } v_i \text{ in reduced form}\}.$$

It is clear that F is a multiple choice function of $\mathcal{X}$. Thus, MC holds as desired.

(ii) The proof is similar to the proof of part (i), so we leave it to the interested reader. □

Theorem 9. *In* ZFA*, the following statements are true:*
(i) $\forall F(\mathrm{MD}(F))$ *implies* $\mathrm{AC_{fin}}$.
(ii) $\forall F(\mathrm{ACVS}(F))$ *implies* $\mathrm{AC_{fin}}$.

Proof: (i) It suffices to show that $\forall F(\mathrm{MD}(F))$ implies the Kinna–Wagner selection principle for families of finite sets, each having at least two elements. To this end, assume that $\forall F(\mathrm{MD}(F))$ is true. Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of finite sets such that for all $i \in I$, $|A_i| > 1$. For each $i \in I$, consider the set $\mathbb{R}^{A_i}$ equipped with pointwise addition and scalar multiplication, and also let

$$W_i = \Big\{ f \in \mathbb{R}^{A_i} : \sum_{x \in A_i} f(x) = 0 \Big\}.$$

We identify $\mathbb{R}^{A_i}$ with the set $U_i = \big\{ \sum_{x \in A_i} a_x x : a_x \in \mathbb{R} \big\}$, i.e., with the set of all formal sums with indeterminates from A_i and coefficients in $\mathbb{R}$, which can be considered to be a vector space over $\mathbb{R}$ under the usual conventions about addition and scalar multiplication. Consequently, we identify W_i with the set $\big\{ \sum_{x \in A_i} a_x x : a_x \in \mathbb{R}, \sum_{x \in A_i} a_x = 0 \big\}$. Clearly, W_i is a nontrivial vector subspace of U_i for all $i \in I$. For each $i \in I$, we let

$$V_i = \langle S_i \rangle, \qquad \text{where } S_i = \Big\{ \Big(\sum_{x \in A_i \setminus \{z\}} x \Big) - (|A_i| - 1)z : z \in A_i \Big\},$$

that is, V_i is the (nontrivial) vector subspace of W_i spanned by S_i. (Note that $|S_i| = |A_i|$.) It is not hard to verify that S_i is a linearly dependent subset of the vector space W_i for all $i \in I$, and also that any subset of S_i with cardinality $|A_i| - 1$ is a basis for V_i.

From our hypothesis, we may let for each $i \in I$, $f_i \in \mathbb{R}^{V_i}$ be a nonzero linear functional. It follows that for all $i \in I$, there exists $s \in S_i$ with $f_i(s) \neq 0$; otherwise if there exists $i \in I$ such that for all $s \in S_i$, $f_i(s) = 0$, then f_i is zero on a basis of V_i which is contained in S_i. But this is absurd since f_i is a nonzero function. Furthermore, by the previous observation, as well as the fact that f_i is a linear mapping, it follows that f_i is not constant on the set S_i for all $i \in I$.

Therefore, let r_i for all $i \in I$ be the least real number r for which there exists $s \in S_i$ such that $f_i(s) = r$. For each $i \in I$, we let

$$M_i = \{s \in S_i : f_i(s) = r_i\},$$

and we also let

$$N_i = \{z \in A_i \colon \text{ "}-(|A_i|-1)z\text{" appears in the expression of some } s \in M_i\}.$$

From the above arguments, we conclude a nonempty proper subset N_i of A_i for all $i \in I$. Thus, the mapping

$$g = \{(i, N_i) \colon i \in I\}$$

is a Kinna–Wagner selection function of the family $\mathcal{A} = \{A_i \colon i \in I\}$, and consequently $\mathrm{AC}_{\mathrm{fin}}$ is true, as required. This completes the proof of (i).

(ii) As in the proof of part (i) of the current theorem, it suffices to show that $\forall F(\mathrm{ACVS}(F))$ implies the Kinna–Wagner selection principle for families of finite sets, each having at least two elements. To this end, assume that $\forall F(\mathrm{ACVS}(F))$ is true. Let $\mathcal{A} = \{A_i \colon i \in I\}$ be a family of finite sets such that $|A_i| > 1$ for all $i \in I$. For each $i \in I$, consider the nontrivial vector subspace $V_i = \left\{f \in \mathbb{R}^{A_i} \colon \sum_{x \in A_i} f(x) = 0\right\}$ of $\mathbb{R}^{A_i}$ (i.e., the set of functions from A_i to $\mathbb{R}$ equipped with pointwise addition and scalar multiplication with scalars from $\mathbb{R}$).

By our hypothesis, we may let $F = \{(i, f_i) \colon i \in I\}$ be a choice function of the family $\mathcal{U} = \{V_i \setminus \{0_{V_i}\} \colon i \in I\}$. Since $f_i \neq 0_{V_i}$ and $\sum_{x \in A_i} f_i(x) = 0$ for all $i \in I$, it follows that

$$\emptyset \neq \{x \in A_i \colon f_i(x) > 0\} \subsetneq A_i \qquad \text{for all } i \in I.$$

We let

$$g = \{(i, \{x \in A_i \colon f_i(x) > 0\}) \colon i \in I\}.$$

Then g is a Kinna–Wagner selection function of $\mathcal{A}$. Thus, $\mathrm{AC}_{\mathrm{fin}}$ holds, finishing the proof of the theorem. $\square$

Corollary 1. *In* ZFA, *the following statements are true:*
(i) $\forall F(\mathrm{MD}(F))$ *is equivalent to* AC.
(ii) $\forall F(\mathrm{ACVS}(F))$ *is equivalent to* AC.

Corollary 2. *The statement* $\forall F(\mathrm{D}(F))$ *is strictly weaker than* $\forall F(\mathrm{MD}(F))$ *(and hence strictly weaker than* $\forall F(\mathrm{ACVS}(F))$*) in* ZFA.

Proof: This follows immediately from Corollary 1 and Theorem 4 (v). $\square$

3.3 On the existence of norms on vector spaces over the fields $\mathbb{R}$ and $\mathbb{Q}$. In this section, we establish that for $F \in \{\mathbb{R}, \mathbb{Q}\}$, the principle $\mathrm{N}(F)$ ("every infinite-dimensional vector space over F has a norm") is *not* a theorem of ZF. Recall that $\mathrm{B}(F) \Rightarrow \mathrm{N}(F)$ and that $\mathrm{B}(F)$ is not provable in ZF. It is *unknown* whether the above implications (for $F = \mathbb{R}$ and $F = \mathbb{Q}$) are reversible in ZF. It is also *unknown* whether any of $\mathrm{B}(F)$ and $\mathrm{N}(F)$ implies $\mathrm{AC}(\mathbb{R})$, i.e., AC restricted to families of nonempty sets of reals.

For use in the proof of the forthcoming Theorem 10, we include here the following well-known ZF-result of Lemma 1 below.

Lemma 1 (ZF). *Let $(X, \|\cdot\|)$ be a normed vector space over F, where $F \in \{\mathbb{R}, \mathbb{Q}\}$. If Z is a proper vector subspace of X, then $\operatorname{int}(Z) = \emptyset$ (i.e., Z has empty interior).*

Proof: Assume the contrary, then there exist $z \in Z$ and $\epsilon > 0$ such that the open ball $B(z, \epsilon) = \{x \in X \colon \|z - x\| < \epsilon\}$ is contained in Z. Let x be any element of X. Then there is a positive rational number a such that $\|ax\| < \epsilon$. It follows that $z + ax \in B(z, \epsilon) \subseteq Z$, so $x = a^{-1}((z + ax) - z) \in Z$. Thus, $X = Z$, which contradicts the fact that Z is a proper subspace of X. □

Theorem 10. *Let $F \in \{\mathbb{R}, \mathbb{Q}\}$. Then there is a permutation model $\mathcal{N}$ of* ZFA *in which there is an infinite-dimensional vector space $(A, +, \cdot)$ over F which does not have a norm. The result can be transferred to* ZF*.*

Proof: We start with a ground model M of ZFA + AC with a set A of atoms such that $|A| \geq |F|$. Endow A with operations "$+$" and "$\cdot$" so that $(A, +, \cdot)$ is an infinite-dimensional vector space over F. Let G be the group of all vector space automorphisms of $(A, +, \cdot)$, and let $\mathcal{F}$ be the finite support (normal) filter, i.e., $\mathcal{F}$ is generated by the subgroups $\operatorname{fix}_G(E) = \{\varphi \in G \colon \forall\, e \in E(\varphi(e) = e)\}$, $E \subset A$ is finite. Let $\mathcal{N}$ be the Fraenkel–Mostowski model determined by M, G and $\mathcal{F}$. (If $F = \mathbb{Q}$ and $|A| = \aleph_0$, then $\mathcal{N}$ is exactly Läuchli's permutation model presented in the proof of [9, Theorem 10.11], see also [10].)

In the model $\mathcal{N}$, A is an infinite-dimensional vector space over F which does not have a basis; in particular, as in the proof of Jech's Theorem 10.11 of [9], it can be shown that if $B \in \mathcal{N}$ is a linearly independent subset of A, then B is finite, and thus A (being infinite-dimensional) has no basis in $\mathcal{N}$.

We now prove the stronger result in $\mathcal{N}$, namely that A does not have a norm in $\mathcal{N}$. Towards a contradiction, we assume that A has a norm $\|\cdot\|$ in $\mathcal{N}$ and we let $\mathcal{T}$ be the topology on A induced by $\|\cdot\|$ (i.e., if d is the metric on A induced by the norm $\|\cdot\|$, then $\mathcal{T}$ is the metric topology on A which is induced by d). Then $(A, \mathcal{T})$ is clearly Hausdorff. (Note also that $(A, \mathcal{T})$ is dense-in-itself, that is, there are no isolated points; indeed, if there exists an element $a \in A$ such that $\{a\} \in \mathcal{T}$, then let $H = \langle\{a\}\rangle$ be the subspace of A spanned by $\{a\}$. Since H is a proper subspace of A, it follows by Lemma 1, that $\operatorname{int}(H) = \emptyset$, which contradicts the fact that $\{a\} \subset H$ and $\{a\} \in \mathcal{T}$.)

Claim 2. If O and Q are two nonempty disjoint subsets of A, then at least one of the subspaces $\langle O\rangle$ and $\langle Q\rangle$ of A is finite-dimensional.

Proof: Assume the hypothesis. If any of O and Q is finite, then the conclusion of the claim is straightforward. So assume that each of O and Q is infinite. Towards a contradiction, we assume that both of the subspaces $\langle O\rangle$ and $\langle Q\rangle$ of A are infinite-dimensional.

Let $E \subset A$ be a finite support of (O, Q), and also let $W = \langle E\rangle$ be the subspace of A spanned by E. Since W is finite-dimensional, we have $O \nsubseteq W$ and $Q \nsubseteq W$.

We assert that $A \setminus W$ (which is nonempty since W is finite-dimensional and A is infinite-dimensional) is a subset of both O and Q so that $O \cap Q \neq \emptyset$, which is a contradiction. Let $q \in O \setminus W$ and also let $r \in A \setminus W$. If $r = q$, then $r \in O$, so we assume that $r \neq q$. Let E' be a maximal linearly independent subset of E. In the ground model M which satisfies AC, pick a basis B for A which extends the linearly independent set $\{q\} \cup E'$. Consider a one-to-one mapping $\varphi\colon B \to A$ such that $\varphi(q) = r$ and $\varphi \in \mathrm{fix}_G(E')$, and then take the unique vector space automorphism of $(A, +, \cdot)$, say ψ, which extends φ. It is fairly easy to see that $\psi \in \mathrm{fix}_G(E)$, and hence $\psi(O) = O$. Furthermore,

$$q \in O \Rightarrow \psi(q) \in \psi(O) \Rightarrow r \in O.$$

Therefore, $A \setminus W \subseteq O$, and similarly we have $A \setminus W \subseteq Q$. We have thus reached a contradiction, and hence at least one of $\langle O \rangle$ and $\langle Q \rangle$ is finite-dimensional as claimed. $\square$

To complete the proof that $(A, +, \cdot)$ does not have a norm in $\mathcal{N}$, we first let O and Q be two nonempty disjoint open subsets of A ($|A| > 1$ and $(A, \mathcal{T})$ is Hausdorff, and note that O and Q are infinite). By Claim 2, we have that at least one of the subspaces $\langle O \rangle$ and $\langle Q \rangle$ of A is finite-dimensional, and thus at least one of them is a proper subspace of A. Since O and Q are open sets, each of the above two possibilities contradicts the result of Lemma 1. Hence A does not admit a norm in $\mathcal{N}$ as desired.

The above ZFA independence result can be transferred to ZF, since for $F \in \{\mathbb{R}, \mathbb{Q}\}$, the statement "there exists an infinite-dimensional vector space $(A, +, \cdot)$ over F without a norm" is a boundable statement (see [9, Chapter 6, Problem 1 on page 94] and [5, Note 103]), hence the Jech–Sochor first embedding theorem (see [9, Theorem 6.1]) applies in order to obtain a symmetric model $\mathcal{M}$ of ZF, in which there is an infinite-dimensional vector space $(A, +, \cdot)$ over F without a norm. This completes the proof of the theorem. $\square$

Next, we shall prove that if $F \in \{\mathbb{R}, \mathbb{Q}\}$, then it is relatively consistent with ZF that there exists an infinite-dimensional vector space over F which has a norm, but has no basis. For use in the proof, we first establish the subsequent Lemma 2 and Theorem 11.

Lemma 2. *Let F be any field and let $\{V_n : n \in \omega\}$ be a family of nontrivial finite-dimensional vector spaces over F. Let $V = \bigoplus_n V_n$ be the weak direct product of the V_n (which is an infinite-dimensional vector space over F).*[1] *For each $n \in \omega$, let $V(n) = \left(\prod_{i \leq n} V_i\right) \times \left(\prod_{i > n} \{0\}\right)$ (which is isomorphic to the vector space $\prod_{i \leq n} V_i$ equipped with pointwise operations, and hence is a finite-dimensional subspace of V).*

If there exists an infinite linearly independent subset S of V, then $\mathcal{V} = \{V(n)\colon n \in \omega\}$ has a partial multiple choice function; in particular, there is a sequence

[1] That is, $V = \left\{f \in \prod_{i \in \omega} V_i \colon |\{i \in \omega\colon f(i) \neq 0\}| < \aleph_0\right\}$ and V is equipped with pointwise operations.

$(L_{n_i})_{i\in\omega}$, where $(n_i)_{i\in\omega}$ is a strictly increasing sequence of natural numbers, such that

$$L_{n_0} = V(n_0) \cap S$$

and for $i > 0$

$$L_{n_i} = (V(n_i) \cap S) \setminus V(n_{i-1}).$$

Proof: Assume that V has an infinite linearly independent subset, say S. Clearly, $f \neq 0$ for all $f \in S$. On the basis that S is a linearly independent subset of V and via mathematical induction, we will construct the required sequence $(L_{n_i})_{i\in\omega}$, and thus a multiple choice function of $\mathcal{V}$.

Let n_0 be the least $n \in \omega$ such that $L_n := V(n) \cap S \neq \emptyset$. Since $V(n_0)$ is finite-dimensional and L_{n_0} is linearly independent, it follows that L_{n_0} is finite.

For the inductive step, assume that we have chosen natural numbers $n_0 < n_1 < \ldots < n_k$ and nonempty finite sets $L_{n_i} = (V(n_i) \cap S) \setminus V(n_{i-1})$, $1 \leq i < k+1$. Since $V(n_k)$ is a finite-dimensional vector space and S is an infinite linearly independent subset of V, it follows that $S \not\subseteq V(n_k)$. Let n_{k+1} be the least natural number $n > n_k$ such that $(V(n) \cap S) \setminus V(n_k) \neq \emptyset$. Let $L_{n_{k+1}} = (V(n_{k+1}) \cap S) \setminus V(n_k)$, then $L_{n_{k+1}}$ is a nonempty finite set. This completes the inductive step.

Then, the function $g = \{(i, L_{n_i}) \colon i \in \omega\}$ is the required partial multiple choice function of $\mathcal{V}$. □

Theorem 11. *Let $(F, \leq)$ be a linearly ordered field of characteristic 0. Then $\mathrm{ILI}(F)$ implies "every countably infinite family $\mathcal{A} = \{A_n \colon n \in \omega\}$ of finite sets each having at least two elements, has a partial Kinna–Wagner selection function".*

Proof: Assume the hypothesis on F and that $\mathrm{ILI}(F)$ is true. Let $\mathcal{A} = \{A_n \colon n \in \omega\}$ be a countably infinite family of finite sets such that $|A_i| \geq 2$ for all $n \in \omega$. For each $n \in \omega$, let

$$V_n = \left\{ f \in F^{A_n} \colon \sum_{x \in A_n} f(x) = 0 \right\}$$

equipped with pointwise operations "+" and "·". Then $(V_n, +, \cdot)$ is a nontrivial finite-dimensional vector space over F. Let $V = \bigoplus_n V_n$ be the weak direct product of the V_n's, which is an infinite-dimensional vector space over F. For each $n \in \omega$, we consider the finite-dimensional subspace of V, $V(n) = \left(\prod_{i \leq n} V_i\right) \times \left(\prod_{i > n} \{0\}\right)$. By $\mathrm{ILI}(F)$, let S be an infinite linearly independent subset of V; then by Lemma 2, there exists a sequence $(L_{n_i})_{i\in\omega}$ (where $(n_i)_{i\in\omega}$ is a strictly increasing sequence of natural numbers) of finite subsets of S such that $L_{n_0} = V(n_0) \cap S$ and for $i > 0$, $L_{n_i} = (V(n_i) \cap S) \setminus V(n_{i-1})$. Since for each $i \in \omega$, L_{n_i} is finite, and the addition "+" on V is commutative, we may let for each $i \in \omega$,

$$l_{n_i} = \sum_{l \in L_{n_i}} l.$$

As $\bigcup_{i\in\omega} L_{n_i} \subseteq S$ and S is linearly independent, we conclude that

$$l_{n_i} \neq 0 \qquad \text{for all } i \in \omega.$$

Furthermore, for each $i \in \omega$, $l_{n_i} \in V(n_i)$ and $L = \{l_{n_i} : i \in \omega\}$ is a countably infinite linearly independent subset of V.

Now, let k_0 be the least integer k such that $\pi_k(l_{n_0}) \neq 0$ (where π_k is the canonical projection of V onto V_k, and 0 is the function on A_k which is identically zero). Let $z_{k_0} = \leq$-min(range($\pi_{k_0}(l_{n_0})$)) (note that range($\pi_{k_0}(l_{n_0})$) is a finite subset of F since $\pi_{k_0}(l_{n_0}) \in F^{A_{k_0}}$ and A_{k_0} is finite, and also recall that "$\leq$" is a linear order on F) and also let

$$G_{k_0} = \{x \in A_{k_0} : \pi_{k_0}(l_{n_0})(x) = z_{k_0}\}.$$

Since $\pi_{k_0}(l_{n_0}) \neq 0$, $\sum_{x\in A_n} \pi_{k_0}(l_{n_0})(x) = 0$ and F is of characteristic 0, it follows that G_{k_0} is a nonempty proper subset of A_{k_0}.

Assume that we have chosen integers $j_0 = 0 < j_1 < \ldots < j_r$ and $k_0 < k_1 < \ldots < k_r$ such that for all $m = 0, 1, \ldots, r$, $\pi_{k_m}(l_{n_{j_m}}) \neq 0$ and $G_{k_m} = \{x \in A_{k_m} : \pi_{k_m}(l_{n_{j_m}})(x) = z_{k_m}\}$, where $z_{k_m} = \leq$-min(range($\pi_{k_m}(l_{n_m})$)), is a nonempty proper subset of A_{k_m}. Since L is a countably infinite linearly independent set, there exist integers $j > j_r$ and $k > k_r$ such that $\pi_k(l_{n_j}) \neq 0$. (If there are no such integers j and k, then the infinite linearly independent set $\{l_{n_i} : i \in \omega,\ i > j_r\}$ is contained in the finite-dimensional space $V(k_r)$, which is impossible.) Let j_{r+1} and k_{r+1} be the least such integers, and also let $z_{k_{r+1}} = \leq$-min(range($\pi_{k_{r+1}}(l_{n_{j_{r+1}}})$)) and

$$G_{k_{r+1}} = \{x \in A_{k_{r+1}} : \pi_{k_{r+1}}(l_{n_{j_{r+1}}})(x) = z_{k_{r+1}}\}.$$

Then (as with G_{k_0}) we have $G_{k_{r+1}}$ is a nonempty proper subset of $A_{k_{r+1}}$. This concludes the inductive step.

Let $H = \{(i, G_{k_i}) : i \in \omega\}$. Then H is a Kinna–Wagner selection function of the infinite subfamily $\mathcal{B} = \{A_{k_i} : i \in \omega\}$ of $\mathcal{A}$, and thus a partial Kinna–Wagner selection function of $\mathcal{A}$, finishing the proof the theorem. □

Theorem 12. *Let $F \in \{\mathbb{R}, \mathbb{Q}\}$. It is relatively consistent with* ZF *that there exists an infinite-dimensional vector space over F which has a norm, but has no basis.*

Proof: Assume the hypothesis on F. Since the statement "there exists an infinite-dimensional vector space over F with a norm, but without a basis" is boundable, it suffices in view of the Jech–Sochor first embedding theorem to find a permutation model of ZFA + ¬AC in which the above statement is true. To this end, we shall use the Second Fraenkel Model (Model $\mathcal{N}2$ in [5]): The set A of atoms is a countable disjoint union $A = \bigcup\{A_n : n \in \omega\}$, where for each $n \in \omega$, $A_n = \{a_n, b_n\}$. The group G is the group of all permutations φ of A such that $\varphi(A_n) = A_n$ for all $n \in \omega$. $\mathcal{F}$ is the finite support normal filter, that is, $\mathcal{F}$ is the filter generated by the subgroups $\mathrm{fix}_G(E)$, $E \subset A$ is finite. The Second Fraenkel Model $\mathcal{N}2$ is the permutation model which is determined by A, G and $\mathcal{F}$.

It is known (see [5]) that in $\mathcal{N}2$, MC is true and the countable family $\mathcal{A} = \{A_n : n \in \omega\}$ does not have a partial choice function.

Let $U = \{f \in F^A : |\{a \in A : f(a) \neq 0\}| < \aleph_0\}$ equipped with pointwise operations "+" and "$\cdot$". Then $(U, +, \cdot)$ is an infinite-dimensional vector space over F for which $B = \{\chi_{\{a\}} : a \in A\}$, where $\chi_{\{a\}}$ is the characteristic function of $\{a\}$, is a basis. It follows that U, as well as any vector subspace of U, has a norm (see Section 2). Consider the subspace of U, $V = \{f \in U : \forall n \in \omega,\ \sum_{a \in A_n} f(a) = 0\}$. Then V is an infinite-dimensional vector space which has a norm. (Note that for every positive integer n, $V(n) = \{f \in V : \forall m > n,\ f \restriction A_m = 0\}$ is a finite-dimensional subspace of V.) Furthermore, V has no infinite linearly independent subsets in $\mathcal{N}2$; otherwise if S were such a subset of V, then as in the proof of Theorem 11, S would give rise to a partial choice function of $\mathcal{A}$, which is impossible in the model $\mathcal{N}2$. Therefore, V has no basis in $\mathcal{N}2$, finishing the proof of the theorem. $\square$

3.4 On the existence of proper infinite-dimensional subspaces. In this section, we prove that for any field F, PIDSub(F) is not provable in ZF. Furthermore, we prove that if F is any well-orderable field and κ any infinite well-ordered cardinal number such that $\kappa > |F|$, then W_κ implies PIDSub(F); in particular, if $\kappa > \aleph_1$, then CH + W_κ implies PIDSub($\mathbb{R}$). We also observe that for any field F, D(F) $\Rightarrow$ PIDSub(F), and that BPI implies "for all finite field F, PIDSub(F)", hence the latter statement is strictly weaker than AC in ZF. Indeed, we have the following theorem.

Theorem 13. *The following hold:*

(i) *For any field F,* D(F) *implies* PIDSub(F).

(ii) BPI *implies "for every finite field F,* PIDSub(F)*". Hence, "for every finite field F,* PIDSub(F)*" does not imply* AC *in* ZF.

(iii) *The principle "$\forall F$(*PIDSub(F)*)" does not imply* AC *in* ZFA. *In particular, "$\forall F$(*PIDSub(F)*)" is true in Lévy's permutation model $\mathcal{N}6$ (in* [5]*) of* ZFA + ¬AC.

(iv) *Let F be any well-orderable field and let κ be any infinite well-ordered cardinal number with $\kappa > |F|$. Then W_κ implies* PIDSub(F). *In particular, if $\kappa > \aleph_1$, then* CH + W_κ *implies* PIDSub($\mathbb{R}$).

(v) *For any field F,* PIDSub(F) *is not provable in* ZF.

Proof: (i) The conclusion follows from Theorem 4.15 of [6], which states that for any field F, D(F) is equivalent to the statement "for every nontrivial vector space V over F and every $u \in V \setminus \{0\}$ there is a subspace W of V such that $V = W \oplus \langle u \rangle$" (where "$V = W \oplus \langle u \rangle$" means that V is the direct sum of W and $\langle u \rangle$, that is, every $u \in W$ is written uniquely as $u = x + y$ where $x \in W$ and $y \in \langle u \rangle$); W is essentially the subspace $\ker(f)$ of V, i.e., the kernel of the linear functional $f : V \to F$ with $f(u) \neq 0$.

(ii) This follows from (i) and Theorem 4 (i).

(iii) This follows from (i) and Theorem 4 (v).

(iv) Let F and κ be as in the statement of (iv) and assume that W_κ is true. Let V be an infinite-dimensional vector space over F. By W_κ we have that $|V| \leq \kappa$ or $\kappa \leq |V|$. If the first possibility occurs, then V is well-orderable. By transfinite induction, we may then construct a basis B for V, which in turn gives rise to a nonzero linear functional $f\colon V \to F$. But then, for any $v \in V \setminus \{0\}$ such that $f(v) \neq 0$, we have (from Theorem 4.15 of [6]) that $V = \ker(f) \oplus \langle v \rangle$, and so $\ker(f)$ is the required proper infinite-dimensional subspace of V. (Note that once B is constructed, one may avoid to define a nonzero linear functional. Indeed, as V is infinite-dimensional, it follows that for some infinite cardinal $\lambda \leq \kappa$, there is a bijection $f\colon \lambda \to B$. Then $W = \langle \{f(2i)\colon i < \lambda\} \rangle$ is a proper infinite-dimensional subspace of V.)

If the second possibility occurs, that is, $\kappa \leq |V|$, then V has a well-orderable subset H such that $|H| = \kappa$. Let $W = \langle H \rangle$. Since H can be well-ordered, we may effectively (i.e., without choice) construct a maximal linearly independent subset of H, thus a basis for W, say B. Then for any element $w \in W$, there is a unique pair $(\{a_1, \ldots, a_n\}, \{b_1, \ldots, b_n\})$, where $n \in \omega \setminus \{0\}$, $\{a_1, \ldots, a_n\} \subseteq F$ and $\{b_1, \ldots, b_n\} \subseteq B$, such that $w = \sum_{i=1}^{n} a_i b_i$. It follows that

$$\kappa = |H| \leq |W| \leq |[F]^{<\omega} \times [B]^{<\omega}|. \tag{3}$$

If B is finite, then since κ is infinite, we must have that F is infinite, and since F is well-orderable, we have (in ZF) that $|[F]^{<\omega}| = |F|$ (see [11, Proposition 4.21]). Then, from the inequalities given by (3), we infer that $\kappa \leq |F \times [B]^{<\omega}| = |F|$ (since for any $\aleph$ and any nonempty finite set x, $|\aleph \times x| = \aleph$; see [11, Proposition 3.25]). This contradicts our assumption that $\kappa > |F|$. Therefore, B is infinite, and consequently W is infinite-dimensional. As in the first possibility, it follows that there is a nonzero linear functional $f\colon W \to F$, which yields a proper infinite-dimensional subspace of W, and hence of V. This completes the proof of (iv).

(v) Let F be any field. Let $\mathcal{N}$ be the permutation model of the proof of Theorem 10. (We consider F to be an element of the kernel $V = \mathcal{P}^\infty(\emptyset)$ of the ground model M of ZFA + AC, hence F is well-orderable in $\mathcal{N}$. Furthermore, if $|F| = \aleph_0$, then we consider that A (the set of atoms) has cardinality $\aleph_0$ in M.)

We prove that every nonempty proper subspace of A is finite-dimensional. To this end, let W be a nonempty proper subspace of A; hence we may let $a \in A \setminus W$. Then the sets W and $W + a$ are nonempty and disjoint, and thus by Claim 2 of the proof of Theorem 10, we have that at least one of the subspaces W and $W + \langle a \rangle$ of A is finite-dimensional. Since W is a subspace of $W + \langle a \rangle$, it follows that W is necessarily finite-dimensional. Hence, PIDSub(F) is false in $\mathcal{N}$ as required.

The above independence result can be transferred to ZF via the Jech–Sochor first embedding theorem, see [9, Theorem 6.1]. This completes the proof of (v) and of the theorem. $\square$

3.5 On the existence of infinite linearly independent subsets. We first note that given a field F, $\mathrm{ILI}(F)$ is not a theorem of ZF, since there is a Läuchli-type permutation model in which there is an infinite-dimensional vector space $(A,+,\cdot)$ over F such that every linearly independent subset of A is finite (see the proof of Theorem 10 or [9, Theorem 10.11]).

In Theorem 6.1 of [8], it has been shown that the stronger than $\forall F(\mathrm{ILI}(F))$ principle "for every field F, every infinite-dimensional vector space over F has a countably infinite linearly independent subset" lies in strength between $\mathrm{AC}^{\aleph_0}$ and $\mathrm{MC}^{\aleph_0}$. We shall prove next that this is also the case with the principle $\forall F(\mathrm{ILI}(F))$, and thus we *strengthen* the result of Theorem 6.1 of [8].

Theorem 14. *The following hold:*

(i) *In* ZF, $\mathrm{AC}^{\aleph_0} \Rightarrow (\forall F(\mathrm{ILI}(F))) \Rightarrow \mathrm{MC}^{\aleph_0}$.

(ii) $\mathrm{ILI}(\mathbb{R})$ *implies "every countably infinite family $\mathcal{A}=\{A_n : n\in\omega\}$ of finite sets each having at least two elements, has a partial Kinna–Wagner selection function". Thus,* MC *(and hence* $\mathrm{MC}^{\aleph_0}$*) does not imply* $\forall F(\mathrm{ILI}(F))$ *in* ZFA.

(iii) *In the Basic Fraenkel Model, the statement "For every linearly orderable field F, $\mathrm{ILI}(F)$" is true, whereas* $\mathrm{MC}^{\aleph_0}$*, and thus (by* (i)*)* $\forall F(\mathrm{ILI}(F))$*, is false.*

Proof: (i) The first implication follows from Theorem 6.1 of [8].

For the second implication, assume that "$\forall F(\mathrm{ILI}(F))$" is true. Let $\mathcal{X} = \{X_i : i\in\omega\}$ be a countably infinite family of nonempty sets which, without loss of generality, we assume that it is disjoint. Since $\mathrm{MC}^{\aleph_0}$ is equivalent to its partial version, i.e., is equivalent to "every countably infinite family of nonempty sets has an infinite subfamily with a multiple choice function" (see [5]), it suffices to show that $\mathcal{X}$ has a partial multiple choice function. To this end, let $X=\bigcup\mathcal{X}$ and let F be any field which is disjoint from X. Let $F(X)$ be the field of all rational functions with indeterminates from X and coefficients in F. Then $F(X)$ is a vector space over the subfield K of all rational functions in $F(X)$ which are i-homogeneous of degree 0 for all $i\in\omega$.

For each $i\in\omega$, let $V_i=\langle X_i\rangle$ be the subspace of $F(X)$ spanned by X_i. Then for each $i\in\omega$, V_i is finite-dimensional (see the proof of Theorem 8). Let $V=\bigoplus_{i\in\omega}V_i$ be the weak direct product of the V_i's (i.e., $V=\{f\in\prod_{i\in\omega}V_i : |\{i\in\omega : f(i)\neq 0\}|<\aleph_0\}$ equipped with pointwise operations). Then V is an infinite-dimensional vector space over K. For each $n\in\omega$, we consider the finite-dimensional subspace of V, $V(n)=\left(\prod_{i\leq n}V_i\right)\times\left(\prod_{i>n}\{0\}\right)$.

By our hypothesis, there exists an infinite linearly independent subset of V, say S. By Lemma 2 (see Section 3.3), there exists a sequence $(L_{n_i})_{i\in\omega}$ (where $(n_i)_{i\in\omega}$ is a strictly increasing sequence of natural numbers) of finite subsets of S such that $L_{n_0}=V(n_0)\cap S$ and for $i>0$, $L_{n_i}=(V(n_i)\cap S)\setminus V(n_{i-1})$.

For every $f\in V\setminus\{0\}$ and for every $i\in\mathrm{supp}(f)$, where $\mathrm{supp}(f)=\{i\in\omega : f(i)\neq 0\}$ is the support of f, let $A_{f(i)}$ be the nonempty finite set of all elements $x\in X_i$ which appear in the quotient expression (in reduced form) of $f(i)$. By induction, we construct now a partial multiple choice function of $\mathcal{X}$. Let $M_{n_0}=\{i : i\in(n_0+1)\cap\mathrm{supp}(f)$ for some $f\in L_{n_0}\}$. For every $i\in M_{n_0}$, let $F_i=$

$\bigcup\{A_{f(i)}\colon f \in L_{n_0}$ such that $i \in \operatorname{supp}(f)\}$. Clearly, F_i is a nonempty finite subset of X_i for all $i < M_{n_0}$.

Assume that for $i < k+1$ we have defined sets $M_{n_i} \subseteq n_i + 1$ and for every $j \in M_{n_i}$ a nonempty finite subset F_j of X_j. Let $M_{n_{k+1}} = \{i\colon i \in (n_k, n_{k+1}] \cap \operatorname{supp}(f)$ for some $f \in L_{n_{k+1}}\}$ and for every $i \in M_{n_{k+1}}$, let $F_i = \bigcup\{A_{f(i)}\colon f \in L_{n_{k+1}}$ such that $i \in \operatorname{supp}(f)\}$. This completes the inductive step. Then the function

$$G = \left\{(i, F_i)\colon i \in \bigcup_{j\in\omega} M_{n_j}\right\}$$

is a partial multiple choice function of $\mathcal{X}$ as required. Thus, $\mathrm{MC}^{\aleph_0}$ holds, finishing the proof of (i).

(ii) The implication follows immediately from Theorem 11.

The second part of (ii) (that is, $\mathrm{MC} \not\Rightarrow \forall F(\mathrm{ILI}(F))$ in ZFA) follows from the first one and the fact that in the Second Fraenkel Model (Model $\mathcal{N}2$ in [5]), MC is true, whereas there is a countably infinite family of unordered pairs of atoms which does not have a partial choice function in the model.

(iii) We recall first the description of the Basic Fraenkel Model; Model $\mathcal{N}1$ in [5]: We start with a ground model M of ZFA + AC with a countably infinite set A of atoms. Let G be the group of all permutations of A and let $\mathcal{F}$ be the finite support normal filter of subgroups of G. Then the Basic Fraenkel Model $\mathcal{N}1$ is the permutation model determined by M, G and $\mathcal{F}$.

It is known that $\mathrm{MC}^{\aleph_0}$ is false in $\mathcal{N}1$ (see [5]), thus (by (i)) "for every field F, every infinite-dimensional vector space over F has a countably infinite linearly independent subset" is also false in $\mathcal{N}1$. We also recall here that A is an amorphous set in $\mathcal{N}1$, that is, the only subsets of A in $\mathcal{N}1$ are the finite and the cofinite ones (see [5], [9, Section 4.6, Problem 7]), and that in $\mathcal{N}1$, every linearly ordered set can be well-ordered (see [5] and Form 90 therein).

Now we show that "for every linearly orderable field, $\mathrm{ILI}(F)$" is true in $\mathcal{N}1$. To this end, let F be any linearly orderable field in $\mathcal{N}1$, and also let $(V, +, \cdot)$ be an infinite-dimensional vector space over F in $\mathcal{N}1$. By the discussion in the previous paragraph, we have F is well-orderable. If V is well-orderable, then via transfinite induction we may easily construct an infinite linearly independent subset of V. So we may assume that V is not well-orderable in $\mathcal{N}1$. Let $E \subset A$ be a finite support for $(V, +, \cdot)$ and for each element of F (F is well-orderable in $\mathcal{N}1$, and hence there is a finite subset K of A such that $\mathrm{fix}_G(K) \subseteq \mathrm{fix}_G(F)$).

Since V is not well-orderable in $\mathcal{N}1$, it follows from a result of [1] that V contains a copy of an infinite subset of A, say B. Without loss of generality, we assume that $B \subset A$. Then B is a cofinite subset of A and we let $C = B \setminus E$. Since E is finite, we have that C is infinite and we assert that C is a linearly independent subset of the vector space V. Assume the contrary, then there exist scalars $\lambda_1, \ldots, \lambda_n \in F$, not all of them zero, and elements $c_1, \ldots, c_n \in C$ such that

$$\lambda_1 c_1 + \cdots + \lambda_n c_n = 0.$$

It follows that for some integer i with $1 \leq i \leq n$, we have

$$c_i = \sum_{j \in \{1,\dots,n\} \setminus \{i\}} \mu_j c_j, \tag{4}$$

where for $j \in \{1,\dots,n\} \setminus \{i\}$, $\mu_j = -\lambda_j/\lambda_i$. Pick an atom $a \in C \setminus (E \cup \{c_j \colon 1 \leq j \leq n\})$ and consider the transposition $\psi = (a, c_i)$, that is, ψ interchanges a and c_i, but fixes all the other atoms. Then, $\psi \in \mathrm{fix}_G(E) \subseteq \mathrm{fix}_G(F) \cap \mathrm{Sym}_G((V,+,\cdot))$. It follows that

$$\begin{aligned}
&((\mu_1 c_1, \dots, \mu_{i-1}c_{i-1}, \mu_{i+1}c_{i+1}, \dots, \mu_n c_n), c_i) \in + \\
\Rightarrow\ &\psi(((\mu_1 c_1, \dots, \mu_{i-1}c_{i-1}, \mu_{i+1}c_{i+1}, \dots, \mu_n c_n), c_i)) \in \psi(+) \\
\Rightarrow\ &((\mu_1 c_1, \dots, \mu_{i-1}c_{i-1}, \mu_{i+1}c_{i+1}, \dots, \mu_n c_n), a) \in +.
\end{aligned}$$

Thus, $a = \sum_{j \in \{1,\dots,n\} \setminus \{i\}} \mu_j c_j$, and consequently by equation (4), we have $a = c_i$, which is a contradiction. Therefore, C is an infinite linearly independent subset of V, finishing the proof of (iii) and of the theorem. □

From the proof of Theorem 14 (iii), we obtain the following corollary.

Corollary 3. *It is relatively consistent with* ZFA *that for every linearly orderable field F, every infinite-dimensional vector space over F has an amorphous (and hence infinite and Dedekind-finite) linearly independent subset. In particular, the latter statement is true in the Basic Fraenkel Model of* ZFA.

Remark 2. We would like to point out here that the proof of Theorem 11 (see Section 3.3) yields an *enhancement* of the result of Theorem 6.3 (i) of [8], which states that if every infinite-dimensional Banach space has a countably infinite linearly independent subset then every countably infinite family of finite sets each having at least two elements has a partial Kinna–Wagner selection function. In particular, the proof of Theorem 11 suggests that the weaker statement "every infinite-dimensional Banach space has an infinite linearly independent subset" implies the above weak choice principle. Indeed, assume the hypothesis, and let $\mathcal{A} = \{A_n \colon n \in \omega\}$ be a countably infinite family of finite sets each having at least two elements. Let $A = \bigcup \mathcal{A}$ and $H = \big\{f \in \mathbb{R}^A \colon |\{x \in A \colon f(x) \neq 0\}| < \aleph_0$ and $\forall\, n \in \omega\ \ \sum_{x \in A_n} f(x) = 0\big\}$. It is clear that H equipped with pointwise operations is a vector space over $\mathbb{R}$ which is isomorphic to the vector space $V = \bigoplus_{n \in \omega} V_n$ of the proof of Theorem 11 (where the linearly ordered field $(F, \leq)$ of characteristic 0 therein, is replaced by the field $\mathbb{R}$ with its usual ordering). Moreover, $\|f\| = \Big(\sum_{x \in A}(f(x))^2\Big)^{1/2}$ is a norm on H, and furthermore, $(H, \|\cdot\|)$ is a Banach space; see the proof of Theorem 6.3 (i) of [8]. We may now follow exactly the proof of Theorem 11 in order to construct a partial Kinna–Wagner selection function for the family $\mathcal{A}$.

4. Open questions

(1) Does $\exists F(\mathrm{B}(F))$ imply AC? Does $\mathrm{B}(\mathbb{R})$ imply $\mathrm{AC}(\mathbb{R})$ (i.e., AC for families of nonempty sets of reals)?
(2) Does $\forall F(\mathrm{D}(F))$ imply AC in ZF?
(3) Does $\mathrm{N}(F)$, where $F \in \{\mathbb{R}, \mathbb{Q}\}$, imply $\mathrm{AC}(\mathbb{R})$?
(4) Does $\mathrm{N}(\mathbb{R})$ imply $\mathrm{B}(\mathbb{R})$ or $\mathrm{D}(\mathbb{R})$? Does $\mathrm{D}(\mathbb{R})$ imply $\mathrm{N}(\mathbb{R})$?
(5) Does BPI imply $\mathrm{N}(\mathbb{R})$? Is $\mathrm{N}(\mathbb{R})$ false in the Dawson–Howard Model $\mathcal{N}29$ of [5]? (Recall that by Theorem 2, $\mathrm{B}(\mathbb{R})$ is false in $\mathcal{N}29$.)
(6) What is the deductive strength of $\mathrm{PIDSub}(F)$ for specific fields F?
(7) Does $\forall F(\mathrm{PIDSub}(F))$ imply AC in ZF?
(8) Given a field F, what is the relationship between $\mathrm{PIDSub}(F)$ and $\mathrm{ILI}(F)$?
(9) Does $\forall F(\mathrm{ILI}(F))$ imply $\mathrm{AC}^{\aleph_0}$ or "every Dedekind-finite set is finite"?

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