Iterated arc graphs

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Abstract. The arc graph $\delta(G)$ of a digraph G is the digraph with the set of arcs of G as vertex-set, where the arcs of $\delta(G)$ join consecutive arcs of G. In 1981, S. Poljak and V. Rödl characterized the chromatic number of $\delta(G)$ in terms of the chromatic number of G when G is symmetric (i.e., undirected). In contrast, directed graphs with equal chromatic numbers can have arc graphs with distinct chromatic numbers. Even though the arc graph of a symmetric graph is not symmetric, we show that the chromatic number of the iterated arc graph $\delta^k(G)$ still only depends on the chromatic number of G when G is symmetric. Our proof is a rediscovery of the proof of [Poljak S., *Coloring digraphs by iterated antichains*, Comment. Math. Univ. Carolin. **32** (1991), no. 2, 209–212], though various mistakes make the original proof unreadable.

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1. Introduction

The arc graph $\delta(G)$ of a digraph G is the digraph with the set $A(G)$ of arcs of G as vertex-set, where the arcs of $\delta(G)$ join consecutive arcs of G. The iterated arc graphs $\delta^k(G)$, $k \geq 1$, are defined recursively by $\delta^k(G) = \delta(\delta^{k-1}(G))$. However it is possible to interpret $\delta^k(G)$ in terms of sequences of vertices of G:

$$\begin{aligned} V(\delta^k(G)) = \{&(u_0, \dots, u_k) \in V(G)^k \colon \\ &(u_i, u_{i+1}) \in A(G) \ \text{ for } \ i = 0, \dots, k-1\}, \\ A(\delta^k(G)) = \{&((u_0, \dots, u_k), (u_1, \dots, u_{k+1})) \colon \\ &(u_0, \dots, u_k), (u_1, \dots, u_{k+1}) \in V(\delta^k(G))\}. \end{aligned}$$

In particular, the iterated arc graphs of complete graphs with loops are the well-known de Bruijn graphs. The iterated arc graphs of transitive tournaments are a folklore construction of graphs with large chromatic numbers and no short odd cycles (see [4]).

We will be investigating chromatic numbers of iterated arc graphs. Here, the chromatic number of a digraph is defined as the minimum number of colours

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needed to colour its vertices so that the endpoints of an arc have different colours. Thus, the direction of an arc has no effect on the chromatic number. In [3], C. C. Harner and R. C. Entriger give the following relations between the chromatic number of a digraph and that of its arc graph.

Theorem 1 ([3])**.**

(i) *If* $\chi(\delta(G)) \leq n$*, then* $\chi(G) \leq 2^n$*.*

(ii) *If* $\chi(G) \leq \binom{n}{\lfloor n/2 \rfloor}$*, then* $\chi(\delta(G)) \leq n$*.*

Inductively, the bound gives $\theta(\log_{(k)}(\chi(G))t)$ behaviour for $\chi(\delta^k(G))$ in terms of $\chi(G)$.

The "undirected" graphs are symmetric digraphs with each edge corresponding to an opposite pair of arcs. In [8], S. Poljak and V. Rödl give a characterization of the chromatic number of the arc graph of a graph.

Theorem 2 ([8])**.** *For any graph* G*,*

$$\chi(\delta(G)) = \min\left\{n \colon \chi(G) \leq \binom{n}{\lfloor n/2 \rfloor}\right\}.$$

In particular, for a graph G, $\chi(\delta(G))$ depends on $\chi(G)$ alone and not on the structure of G. In contrast, digraphs with equal chromatic numbers can have arc graphs with distinct chromatic numbers. For instance, let C be the cyclic tournament on three vertices, and T the transitive tournament on three vertices. Then $\chi(C) = \chi(T) = 3$, while $\chi(\delta(C)) = 3$ and $\chi(\delta(T)) = 2$.

Now, what about iterated arc graphs? Theorem 2 cannot be used to characterize $\chi(\delta^2(G))$ in terms of $\chi(\delta(G))$, since $\delta(G)$ is not symmetric in general. However, it turns out that for all k, $\chi(\delta^k(G))$ is indeed characterized by $\chi(G)$. This seems to have been first noticed by S. Poljak in [7]. There are numerous mistakes in the original exposition, which are corrected here. In addition, the topic is connected to interesting combinatorial structures and problems, which makes it worthy of more attention.

The characterization of $\chi(\delta^k(G))$ uses numbers defined in terms of specific posets. An *ideal* in a poset P is a subset I of P such that $x \leq y \in I$ implies $x \in I$. The set of ideals of P, ordered by inclusion, is a poset which we denote $\mathcal{I}(P)$. Iterating the construction, we get the posets $\mathcal{I}^k(P)$, $k \geq 0$. The complement $\overline{K}_n$ of the complete graph with n vertices can be viewed as a poset, more precisely an antichain of size n. We let $b(n,k)$ be the maximum size of an antichain in $\mathcal{I}^k(\overline{K}_n)$ (i.e., the *width* of $\mathcal{I}^k(\overline{K}_n)$).

Theorem 3. *For any graph* G *and any integer* $k \geq 1$*,*

$$\chi(\delta^k(G)) = \min\{n \colon \chi(G) \leq b(n,k)\}.$$

The posets $\mathcal{I}^k(\overline{K}_n)$ turn out to be interesting objects. The lattice $\mathcal{I}(\overline{K}_n)$ is the boolean lattice with n generators, and $b(n,1) = \binom{n}{\lfloor n/2 \rfloor}$ by Sperner's theorem.

Thus the case $k = 1$ of Theorem 3 is Theorem 2. The lattice $\mathcal{I}^2(\overline{K}_n)$ is the free distributive lattice with n generators. Thus the number of elements in $\mathcal{I}^2(\overline{K}_n)$ are called the Dedekind numbers (sequence A000372 in the on-line Encyclopedia of integer sequences). The largest known antichain in $\mathcal{I}^2(\overline{K}_n)$ consists of the ideals with exactly 2^{n-1} elements, that is, half the elements of $\mathcal{I}(\overline{K}_n)$.

Now for any poset P, the number of elements in an ideal I of P defines its height as an element of $\mathcal{I}(P)$. Thus, $\mathcal{I}(P)$ is always a graded poset. A graded poset which has a maximal antichain consisting of all elements of the same height is called a *Sperner poset*. A conjecture attributed to Richard Stanley states that $\mathcal{I}^2(\overline{K}_n)$ is indeed a Sperner poset. (We thank Dwight Duffus for this information.)

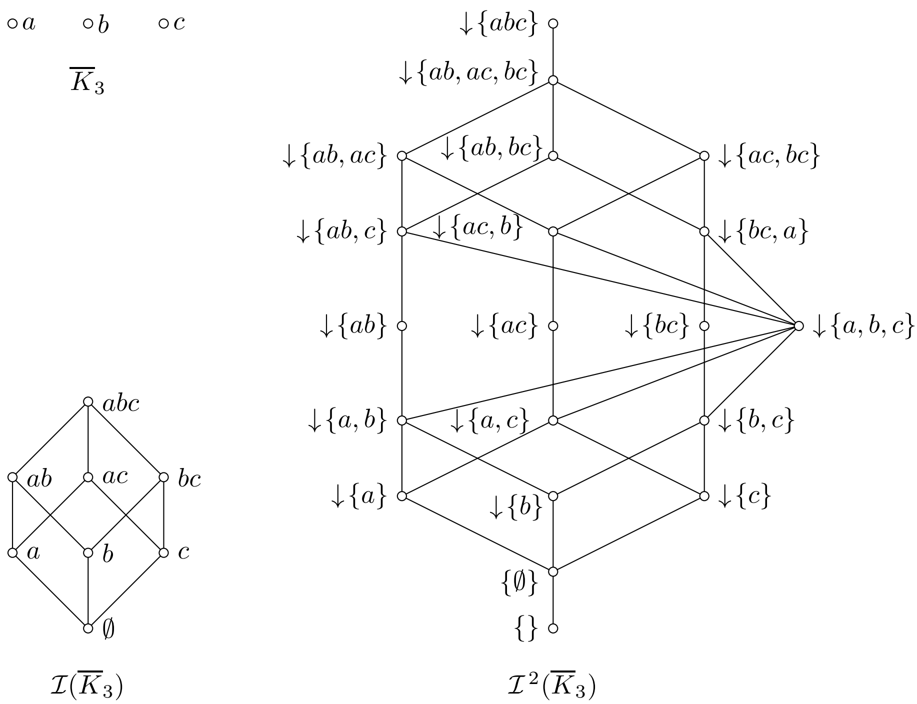

FIGURE 1. $\mathcal{I}^k(\overline{K}_3)$, $k \in \{0, 1, 2\}$.

It seems that the only published statement of Stanley's conjecture is in the thesis [6], where it is verified that for n up to 6, the unique maximum antichain in $\mathcal{I}^2(\overline{K}_n)$ is indeed the one which consists of the ideals of $\mathcal{I}(\overline{K}_n)$ with exactly 2^{n-1} elements. Thus the known values of $b(n, 2)$ are as follows:

$$b(3,2) = 4, \qquad b(4,2) = 24, \qquad b(5,2) = 621, \qquad b(6,2) = 492288.$$

By inspecting the 20-element $\mathcal{I}^2(\overline{K}_3)$—see Figure 1—it is easy to check that $b(3,3) = 7$. The levels 8, 9, 11, 12 of $\mathcal{I}^3(\overline{K}_3)$ each constitute antichains of size 7, but level 10 is an antichain of size 6. Thus the level of maximum size is not necessarily the middle level. Apart from the trivial cases $b(1,k) = 1$ and $b(2,k) = 2$ for all k, no further values $b(n,k)$ are known.

2. Proof of Theorem 3

We will show that for any digraph G, an n-colouring of $\delta^k(G)$ corresponds to a homomorphism from G to a suitably defined digraph $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$. When G is a graph, its edges must be mapped to the symmetric arcs of $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$. These symmetric arcs span a graph which retracts to $K_{b(n,k)}$. The details of this argument are provided below.

2.1 The right adjoint of the arc graph construction. Viewed as a digraph functor, δ admits a kind of "right adjoint". More precisely, there is a construction δ_R such that there exists a homomorphism from $\delta(G)$ to K if and only if there exists a homomorphism from G to $\delta_R(K)$. (Note that our definition of a right adjoint is less restrictive than the standard categorial definition, in which a correspondence between morphisms is required.) Here, a *homomorphism* $\varphi\colon G \to H$ is a map φ from the vertex set of G to that of H such that if (u,v) is an arc of G, then $(\varphi(u), \varphi(v))$ is an arc of H.

For a digraph K, $\delta_R(K)$ is the digraph defined as follows.

- The vertices of $\delta_R(K)$ are the ordered pairs (X,Y) such that X and Y are sets of vertices of K with an arc (x,y) between all vertices x of X and all vertices y of Y.
- The arcs of $\delta_R(K)$ are ordered pairs $((X,Y),(Z,W))$ such that $Y \cap Z \neq \emptyset$.

The sets X, Y of vertices of K used in the definition of $\delta_R(K)$ are allowed to be empty. In particular, if K_0 is the graph with no vertex and no edge, then $\delta_R(K_0)$ is a single vertex, and $\delta_R^2(K_0)$ has three vertices and one arc.

We use the following result.

Lemma 4 ([2])**.** *Given two digraphs G and K, there exists a homomorphism of $\delta(G)$ to K if and only if there exists a homomorphism of G to $\delta_R(K)$.*

Proof: We include the sketch of an elementary proof to make the paper self contained. First note that a homomorphism $\varphi\colon G \to H$ induces homomorphisms $\delta(\varphi)\colon \delta(G) \to \delta(H)$ defined by $\delta(\varphi)(u,v) = (\varphi(u), \varphi(v))$, and $\delta_R(\varphi)\colon \delta_R(G) \to \delta_R(H)$ defined by $\delta_R(\varphi)(X,Y) = (\varphi(X), \varphi(Y))$. Second, note that there are homomorphisms from $\delta(\delta_R(G))$ to G defined by mapping $((X,Y),(Z,W))$ to any element of $Y \cap Z$, and from G to $\delta_R(\delta(G))$ defined by mapping u to (u^-, u^+), where u^- and u^+ are respectively the sets of arcs entering and leaving u.

Therefore, if there exists a homomorphism from $\delta(G)$ to K then there exists a homomorphism from $\delta_R(\delta(G))$ to $\delta_R(K)$, which composed with a homomorphism from G to $\delta_R(\delta(G))$ yields a homomorphism from G to $\delta_R(K)$. Similarly, if there exists a homomorphism from G to $\delta_R(K)$, then there exists a homomorphism from $\delta(G)$ to $\delta(\delta_R(K))$, and the latter admits a homomorphism to K. $\square$

Corollary 5. *Given two digraphs G and K and any $k \geq 1$, there exists a homomorphism of $\delta^k(G)$ to K if and only if there exists a homomorphism of G to $\delta_R^k(K)$.*

A subdigraph H of a digraph G is called a *retract* of G if there exists a homomorphism $\varrho\colon G \to H$ such that the restriction of ϱ to H is the identity.

Lemma 6. *The digraph $\delta_R(K)$ retracts to its subdigraph induced by the vertices (X, Y) such that*

$$Y = \{y \in V(K)\colon (x, y) \in A(K) \text{ for all } x \in X\}$$

and

$$X = \{x \in V(K)\colon (x, y) \in A(K) \text{ for all } y \in Y\}.$$

Proof: Let $\varrho\colon \delta_R(K) \to \delta_R(K)$ be the map defined by $\varrho(X, Y) = (X', Y')$, where X' is the set of common inneighbours of Y, and Y' is the set of common outneighbours of X'. Then ϱ is easily seen to be a retraction on the prescribed subdigraph. $\square$

2.2 Nondomination digraphs of posets. The *nondomination digraph* $\mathcal{N}(P)$ of a poset P is the digraph which has the elements of P for vertices, and for arcs the ordered pairs (u, v) such that u is strictly less than v or u and v are incomparable. In other words, if $G = \mathcal{N}(P)$, then $A(G)$ is the complement in $V(G)^2$ of the relation "$\geq$". Note that $K_n = \mathcal{N}(\overline{K}_n)$, where $\overline{K}_n$ is the antichain of size n. The constructions δ_R, $\mathcal{N}$ and $\mathcal{I}$ connect as follows.

Lemma 7. *For any poset P, $\delta_R(\mathcal{N}(P))$ retracts to $\mathcal{N}(\mathcal{I}(P))$.*

Proof: By Lemma 6, $\delta_R(\mathcal{N}(P)))$ retracts to its subdigraph G induced by the vertices (X, Y) such that Y is the common outneighbourhood of all vertices in X and X is the common inneighbourhood of all vertices in Y. We will show that $G = \mathcal{N}(\mathcal{I}(P))$.

For $(X, Y) \in V(G)$, suppose that there exists a vertex u of $\mathcal{N}(P)$ not contained in X or Y. Since $u \notin X$, u is not a common inneighbour of the vertices in Y. Thus, there exists a vertex $y \in Y$ such that $u \geq y$ in P. Likewise, $u \notin Y$ so there exists a vertex $x \in X$ such that $x \geq u$. By transitivity, $x \geq y$, contradicting the fact that there is an arc from x to y in $\mathcal{N}(P)$. Therefore, $Y = \overline{X}$.

Hence the elements of G are determined by their first coordinates. It is easy to see that these first coordinates are ideals of P, that is, elements of $\mathcal{I}(P)$. Indeed, if $(X, \overline{X}) \in V(G)$ and $x < y \in X$, then (y, x) is not an arc of G, hence $x \notin \overline{X}$. So it only remains to show that adjacency in G corresponds to adjacency in $\mathcal{N}(\mathcal{I}(P))$. By definition $((X, \overline{X}), (Y, \overline{Y}))$ is an arc of G if and only if $\overline{X}$ intersects Y, that is, $X \not\supseteq Y$, which is equivalent to $(X, Y) \in A(\mathcal{N}(\mathcal{I}(P)))$. $\square$

Corollary 8. *For any poset P and any integer k, $\delta_R^k(\mathcal{N}(P))$ retracts to $\mathcal{N}(\mathcal{I}^k(P))$.*

2.3 Symmetric restrictions. Let $[G]$ denote the symmetric restriction of a digraph G, that is, the subgraph spanned by its symmetric arcs.

Lemma 9. *For any poset P, $[\mathcal{N}(P)]$ retracts to a complete subgraph with cardinality equal to the width of P.*

Proof: Any antichain in P is a complete subgraph of $[\mathcal{N}(P)]$. Let $K = [A]$, where A is a maximum antichain in P. Then K is a complete subgraph of $[\mathcal{N}(P)]$ of cardinality equal to the width w of P. By Dilworth's theorem, see [1], there exists a chain partition $\{C_1, \dots, C_w\}$ of P. Each chain C_i is an independent set in $[\mathcal{N}(P)]$. Therefore, the map $\varrho\colon [\mathcal{N}(P)] \to K$ mapping each C_i to its intersection with A is a retraction of $[\mathcal{N}(P)]$ to K. $\square$

To summarize the proof of Theorem 3, the existence of an n-colouring of $\delta^k(G)$ is equivalent to the existence of a homomorphism of G to $\delta^k_R(K_n)$ by Corollary 5. Since $K_n = \mathcal{N}(\overline{K}_n)$, this is equivalent to the existence of a homomorphism of G to $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$ by Corollary 8. When G is a graph, its edges must be mapped to those of $[\mathcal{N}(\mathcal{I}^k(\overline{K}_n))]$, which retracts to $K_{b(n,k)}$ by Lemma 9. This concludes the proof of Theorem 3. $\square$

3. Further comments

For any poset P, the chromatic number of $\mathcal{N}(P)$ is equal to the number of elements in P, since any two elements of P are joined by an arc in at least one direction. In particular, the number of elements of $\mathcal{I}^k(\overline{K}_n)$ is equal to the maximum possible chromatic number of a digraph G with the property that $\chi(\delta^k(G)) = n$. Note that $G = \mathcal{N}(\mathcal{I}^k(\overline{K}_n))$ achieves the bound, as well as any spanning tournament of $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$. We can even require the spanning tournament to be transitive, since any linear extension of $\mathcal{I}^k(\overline{K}_n)$ corresponds to a spanning transitive tournament in $\mathcal{N}(\mathcal{I}^k(\overline{K}_n))$. Specializing to $k = 2$, this implies that the nth Dedekind number is equal to the maximum cardinality of a transitive tournament T such that $\chi(\delta^2(T)) = n$.

As we mentioned, the proof of Theorem 3 given here constitutes a clarification of that of [7]. In that paper, it looks as though the publisher replaced "$\not\geq$" by "$<$" as a symbol for nondomination. Furthermore, $\mathcal{I}(P)$ is obviously isomorphic to $\mathcal{F}(P)$, the containment poset of the filters of P. Both posets can be represented by means of a specific ordering on antichains, but different orderings are used for $\mathcal{I}(P)$ and $\mathcal{F}(P)$. These confusions make it impossible to follow the exposition in [7] without rediscovering the results.

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