A nice subclass of functionally countable spaces

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Abstract. A space X is functionally countable if $f(X)$ is countable for any continuous function $f\colon X \to \mathbb{R}$. We will call a space X exponentially separable if for any countable family $\mathcal{F}$ of closed subsets of X, there exists a countable set $A \subset X$ such that $A \cap \bigcap \mathcal{G} \neq \emptyset$ whenever $\mathcal{G} \subset \mathcal{F}$ and $\bigcap \mathcal{G} \neq \emptyset$. Every exponentially separable space is functionally countable; we will show that for some nice classes of spaces exponential separability coincides with functional countability. We will also establish that the class of exponentially separable spaces has nice categorical properties: it is preserved by closed subspaces, countable unions and continuous images. Besides, it contains all Lindelöf P-spaces as well as some wide classes of scattered spaces. In particular, if a scattered space is either Lindelöf or ω-bounded, then it is exponentially separable.

Keywords: countably compact space; Lindelöf space; Lindelöf P-space; functionally countable space; exponentially separable space; retraction; scattered space; extent; Sokolov space; weakly Sokolov space; function space

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1. Introduction

A space X is *Sokolov* (or *has the Sokolov property*) if for any choice of a closed set $F_n \subset X^n$ for every $n \in \mathbb{N}$, there exists a continuous map $f\colon X \to X$ such that $nw(f(X)) \le \omega$ and $f^n(F_n) \subset F_n$ for each $n \in \mathbb{N}$. Sokolov spaces were introduced in the paper [6]; it was proved in [6] that Corson compact spaces are Sokolov; besides, a space X is Sokolov if and only if $C_p(X)$ is Sokolov and if X is a compact Sokolov space, then all iterated function spaces $C_{p,n}(X)$ are Lindelöf. In the paper [8] the class of Sokolov spaces was studied systematically and it was proved, among other things, that every Sokolov space is collectionwise normal, ω-stable, ω-monolithic and has countable extent.

In the paper [12] the Sokolov property in Lindelöf P-spaces was studied. It was proved in [12] that some Lindelöf P-spaces fail to be Sokolov but every Lindelöf P-space X has a weaker version of the Sokolov property, namely, for any countable family $\mathcal{F}$ of closed subspaces of X there exists a retraction $r\colon X \to X$ such that the set $r(X)$ is countable and we have the inclusion $r(F) \subset F$ for any $F \in \mathcal{F}$. Another property of Lindelöf P-spaces proved in [12] is what we call *exponential separability* in this paper. A space X is *exponentially separable* if for any countable

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family $\mathcal{F}$ of closed subsets of X, there exists a countable set $A \subset X$ such that $A \cap \bigcap \mathcal{G} \neq \emptyset$ whenever $\mathcal{G} \subset \mathcal{F}$ and $\bigcap \mathcal{G} \neq \emptyset$.

In this paper we show that the class $\mathcal{ES}$ of exponentially separable spaces turns out to have some nice categorical properties: if $X \in \mathcal{ES}$, then all closed subspaces and continuous images of X belong to $\mathcal{ES}$. Besides, any countable union of spaces from $\mathcal{ES}$ belongs to $\mathcal{ES}$. We will also establish that the class $\mathcal{ES}$ contains all Lindelöf scattered spaces and all ω-bounded scattered spaces; however, under continuum hypothesis (CH), there exists a scattered countably compact space that fails to be exponentially separable.

Recall that a space X is *functionally countable* if any second countable continuous image of X is countable. We establish that any exponentially separable space is functionally countable. On the other hand, if X is either perfectly normal or countably compact normal space, then functional countability of X is equivalent to its exponential separability. This easily implies that a compact space X is exponentially separable if and only if X is scattered.

2. Notation and terminology

All spaces are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology and $\tau(x, X) = \{U \in \tau(X) \colon x \in U\}$ for any point $x \in X$. The set $\mathbb{R}$ is the real line with its usual topology, $\mathbb{I} = [0,1] \subset \mathbb{R}$ and $\mathbb{N} = \{1, 2, \ldots\} \subset \mathbb{R}$. We denote by $\mathbb{D}$ the set $\{0,1\}$ with the discrete topology. A space X is *scattered* if every nonempty subspace of X has an isolated point. We say that X is a *P-space* if every G_δ-subset of X is open. The space X is *ω-bounded* if $\overline{A}$ is compact for any countable set $A \subset X$. Say that X is a *Lindelöf p-space* if there exists a perfect map of X onto a second countable space. The space X is *Lindelöf* Σ (or has the *Lindelöf Σ-property*) if X is a continuous image of a Lindelöf p-space. Recall that $A \subset X$ is a *zero-subset of* X if there exists a continuous function $f \colon X \to \mathbb{R}$ such that $A = f^{-1}(0)$.

A map $f \colon X \to Y$ is a *condensation* if f is a continuous bijection; in this case it is said that X *condenses onto* Y. If $\varphi \colon X \to Y$ is a map then $\varphi^n \colon X^n \to Y^n$ is defined by the formula $\varphi(x) = (\varphi(x_1), \ldots, \varphi(x_n))$ for any point $x = (x_1, \ldots, x_n) \in X^n$ and $n \in \mathbb{N}$. A family $\mathcal{N}$ of subsets of a space X is called a *network in* X if every $U \in \tau(X)$ is the union of a subfamily of $\mathcal{N}$. The cardinal $\mathrm{nw}(X) = \min\{|\mathcal{N}| \colon \mathcal{N}$ is a network of $X\}$ is called the *network weight* of X and $\mathrm{ext}(X) = \sup\{|D| \colon D$ is a closed discrete subset of $X\}$ is *the extent* of the space X. The cardinal $\mathrm{iw}(X) = \min\{\kappa \colon$ the space X has a weaker Tychonoff topology of weight less than or equal to $\kappa\}$ is called the *i-weight* of X.

For any spaces X and Y the set $C(X, Y)$ consists of continuous functions from X to Y; if it has the topology induced from Y^X, then the respective space is denoted by $C_p(X, Y)$. We write $C_p(X)$ instead of $C_p(X, \mathbb{R})$. Given a space X let $C_{p,0}(X) = X$ and $C_{p,n+1}(X) = C_p(C_{p,n}(X))$ for all $n \in \omega$, i.e., $C_{p,n}(X)$ is the nth iterated function space of X.

The rest of our topological notation is standard and follows the book [1]. For unreferenced notions of C_p-theory, see the books [9]–[11].

3. Scattered spaces and exponential separability

Our main purpose is to show that in many scattered spaces every countable family of closed subsets has a property that looks like separability. In particular, this is true for Lindelöf scattered spaces and for scattered ω-bounded spaces.

Definition 3.1. Suppose that X is a space and $\mathcal{F}$ is a family of subsets of X. Say that a set $A \subset X$ is *strongly dense in* $\mathcal{F}$ if $A \cap \bigcap \mathcal{F}' \neq \emptyset$ for any family $\mathcal{F}' \subset \mathcal{F}$ such that $\bigcap \mathcal{F}' \neq \emptyset$. The family $\mathcal{F}$ will be called *strongly separable* if some countable subset of X is strongly dense in $\mathcal{F}$. The space X will be called *exponentially separable* if every countable family of closed subsets of X is strongly separable.

The proof of the following statement is straightforward and can be left to the reader.

Proposition 3.2. (a) *Any countable space is exponentially separable.*

(b) *If a space X is exponentially separable, then every closed subspace of X is exponentially separable.*

(c) *If a space X is exponentially separable, then every continuous image of X is exponentially separable.*

(d) *If X is a space, $X_n \subset X$ is exponentially separable for any $n \in \omega$ and $X = \bigcup_{n\in\omega} X_n$, then X is exponentially separable.*

The following theorem was established in [12] in a different terminology.

Theorem 3.3. *Every Lindelöf P-space is exponentially separable.*

Proposition 3.4. *Every Lindelöf scattered space is exponentially separable.*

Proof: If X is a Lindelöf scattered space, then let Y be the set X with the topology generated by all G_δ-subsets of X. It is evident that Y is a P-space and its topology is stronger that the topology of X; besides Y has to be Lindelöf by a theorem of V. V. Uspenskij, see [13]. Therefore Y is exponentially separable by Theorem 3.3 and hence we can apply Proposition 3.2 to conclude that X is also exponentially separable being a continuous image of Y. □

Proposition 3.5. *If X is a second countable exponentially separable space, then X is countable.*

Proof: Assume that X is uncountable and fix a countable base $\mathcal{B}$ in the space X. If $\mathcal{F} = \{\overline{B}\colon B \in \mathcal{B}\}$, then $\mathcal{F}$ is a countable family of closed subsets of X and every point of X is the intersection of a subfamily of $\mathcal{F}$. Therefore each strongly dense set for $\mathcal{F}$ must be uncountable being equal to X which is a contradiction. □

Corollary 3.6. *If a space X is exponentially separable and* $\mathrm{iw}(X) \leq \omega$, *then* $|X| \leq \omega$.

Proof: If X condenses onto a second countable space Y, then Y is exponentially separable by Proposition 3.2 and hence countable by Proposition 3.5. Therefore X is also countable. $\square$

Recall that a space X is *functionally countable* if any second countable continuous image of X is countable. It is not difficult to see that a space X is functionally countable if and only if $f(X)$ is countable for any continuous function $f\colon X \to \mathbb{R}$. The following fact is immediate from Proposition 3.2 and Proposition 3.5.

Corollary 3.7. *Any closed subspace of an exponentially separable space is functionally countable.*

We will show next that functional countability is closer to exponential separability than it seems at the first sight.

Theorem 3.8. *A space X is functionally countable if and only if every countable family of zero-subsets of X is strongly separable.*

Proof: To abridge notation, let us temporarily say that X is an FC-space if every countable family of zero-subsets of X is strongly separable; we must prove that X is an FC-space if and only if it is functionally countable. Observe first that the FC-property is trivially preserved by continuous images and assume that X is an FC-space. If M is a second countable image of X, then M is an FC-space by our observation. Since all closed subsets of M are zero-sets, the space M is exponentially separable and hence we can apply Proposition 3.5 to see that M must be countable and hence every FC-space is functionally countable.

Now assume that X is a functionally countable space and $\mathcal{F}$ is a countable family of zero-subsets of X. Choose a continuous function $g_F\colon X \to \mathbb{R}$ such that $F = g_F^{-1}(0)$ for every $F \in \mathcal{F}$. The diagonal product $g = \Delta\{g_F\colon F \in \mathcal{F}\}$ maps X into the second countable space $\mathbb{R}^{\mathcal{F}}$ and hence the set $Y = g(X)$ is countable. Let $p_F\colon Y \to \mathbb{R}$ be the projection of Y onto the factor determined by F, i.e., $p_F(g(x)) = g_F(x)$ for each $x \in X$.

Take a countable set $A \subset X$ such that $g(A) = Y$ and let $\mathcal{G} \subset \mathcal{F}$ be a subfamily of $\mathcal{F}$ with $G = \bigcap \mathcal{G} \neq \emptyset$. There exists a point $a \in A$ such that $g^{-1}(g(a)) \cap G \neq \emptyset$. Given any $F \in \mathcal{G}$, observe that it follows from the equalities $F = g_F^{-1}(0)$ and $g_F = p_F \circ g$ that $F = g^{-1}(g(F))$ and therefore $g^{-1}(g(a)) \subset F$. This implies that $g^{-1}(g(a)) \subset \bigcap \mathcal{G}$ and hence $a \in \bigcap \mathcal{G}$, i.e., A is strongly dense in $\mathcal{F}$. $\square$

Corollary 3.9. *A perfectly normal space is exponentially separable if and only if it is functionally countable.*

Proof: Since necessity is provided by Corollary 3.7, assume that X is a perfectly normal functionally countable space. Then every countable family of zero-subsets of X is strongly separable by Theorem 3.8. Since every closed subset of X is a zero-set, every countable family of closed subsets of X is strongly separable, i.e., X is exponentially separable. $\square$

Example 3.10. The hereditarily Lindelöf non-separable space L constructed by Moore in ZFC is functionally countable (see Theorem 7.18 of the paper [3]); since hereditarily Lindelöf spaces are perfectly normal, L is exponentially separable by Corollary 3.9. Therefore a hereditarily Lindelöf exponentially separable space need not be countable.

It is worth noting that every Lindelöf space with a G_δ-diagonal has countable i-weight (see Problem 318 of the book [11]). This implies that any exponentially separable space X such that $X \times X$ is hereditarily Lindelöf, must have countable i-weight and hence X must be countable by Corollary 3.6.

Proposition 3.11. *If X is exponentially separable, then* $\operatorname{ext}(X) \le \omega$.

PROOF: If $D \subset X$ is a closed discrete subset of X with $|D| = \omega_1$, then any injective map of D in $\mathbb{R}$ shows that D is not functionally countable, which is a contradiction with Corollary 3.7. □

Corollary 3.12. *Any exponentially separable space is zero-dimensional.*

PROOF: It is a widely known fact that every functionally countable space is zero-dimensional; Corollary 3.7 does the rest. □

Proposition 3.13. *If a pseudocompact space X is exponentially separable, then X is scattered.*

PROOF: If M is a second countable space and $f\colon \beta X \to M$ is a continuous onto map, then $f(X) = M$ because $f(X)$ is a compact dense subspace of M. The space M must be countable by Corollary 3.7; this proves that βM is functionally countable. If βX is not scattered, then there exists a continuous onto map $f\colon \beta X \to \mathbb{I}$ (see [10, Problem 133]) which is a contradiction. Therefore βX is scattered and hence so is X. □

Corollary 3.14. *A compact space X is exponentially separable if and only if X is scattered.*

PROOF: Just apply Proposition 3.13 and Proposition 3.4. □

Example 3.15. Let M be a Mrowka space whose one-point compactification coincides with its Stone–Čech compactification βM (see Corollary 3.11 of the paper [4]). Recall that $M = D \cup F$ where D is a countable set, all points of D are isolated and $M = \overline{D}$. Furthermore, $F = M \setminus D$ is an uncountable closed discrete subset of M and hence M is not exponentially separable. However, βM is a scattered compact space so it is functionally countable; this easily implies that M is functionally countable as well. Therefore a functionally countable pseudocompact space can fail to be exponentially separable. Note also that M is perfect being the countable union of closed discrete subspaces so normality cannot be omitted in Corollary 3.9. This example also shows that a functionally countable space with a G_δ-diagonal is not necessarily countable.

Example 3.16. For any infinite cardinal κ, the Cantor cube $\mathbb{D}^\kappa$ turns out to have a dense σ-compact exponentially separable subspace. This can be easily seen if we consider the σ-product $S = \{x \in \mathbb{D}^\kappa \colon x^{-1}(1) < \omega\}$ in the space $\mathbb{D}^\kappa$. It is known (and easy to prove) that S is the countable union of scattered compact spaces, so S is a dense subspace of $\mathbb{D}^\kappa$ which is exponentially separable by Proposition 3.2 and Corollary 3.14. This example shows, among other things, that a σ-compact exponentially separable space need not be scattered.

Example 3.17. Under CH, there exists a countably compact scattered space X which is not exponentially separable.

Proof: In the paper [2], V. Kannan and M. Rajagopalan constructed under CH a countably compact scattered space X that can be mapped continuously onto $\mathbb{I}$. Corollary 3.7 shows that X is not exponentially separable. □

Theorem 3.18. *If X is a countably compact space, then X is exponentially separable if and only if so is $\overline{A}$ for any countable $A \subset X$.*

Proof: By Proposition 3.2 we only have to prove sufficiency so assume that X is a countably compact space such that $\overline{B}$ is exponentially separable for any countable set $B \subset X$. Given a countable family $\mathcal{F}$ of closed subsets of X take a countable set $B \subset X$ such that $B \cap \bigcap \mathcal{F}' \neq \emptyset$ whenever $\mathcal{F}'$ is a finite subfamily of $\mathcal{F}$ with nonempty intersection. We claim that $\overline{B}$ is strongly dense in $\mathcal{F}$.

Indeed, if $\mathcal{G} \subset \mathcal{F}$ and $G = \bigcap \mathcal{G} \neq \emptyset$, then $G = \bigcap\{G_n \colon n \in \omega\}$ where $G_{n+1} \subset G_n$ and G_n is the intersection of a finite subfamily of $\mathcal{F}$ for each $n \in \omega$. Therefore $\mathcal{H} = \{G_n \cap \overline{B} \colon n \in \omega\}$ is a decreasing family of nonempty closed subsets in the countably compact space $\overline{B}$. Therefore $H = \bigcap \mathcal{H}$ is a nonempty set and $H \subset \bigcap \mathcal{G} \cap \overline{B}$ so H is the witness of strong density of $\overline{B}$ in $\mathcal{F}$.

Since $\overline{B}$ is exponentially separable, we can pick a countable set $A \subset \overline{B}$ which is strongly dense in the family $\{F \cap \overline{B} \colon F \in \mathcal{F}\}$. It is straightforward that A is also strongly dense in $\mathcal{F}$ so X is exponentially separable. □

Corollary 3.19. *If X is an ω-bounded scattered space, then X is exponentially separable.*

Proof: It is trivial that X is countably compact. If A is a countable subset of X, then the set $\overline{A}$ is compact and scattered, so it is exponentially separable by Corollary 3.14. Therefore we can apply Theorem 3.18 to conclude that X is exponentially separable. □

Corollary 3.20. *If X is a countably compact subspace of an ordinal, then X is exponentially separable.*

Proof: Just observe that X is scattered and $\overline{A}$ is countable and hence compact for any countable set $A \subset X$; Corollary 3.19 does the rest. □

Corollary 3.21. *Every ordinal is exponentially separable.*

Proof: Given an ordinal μ observe first that μ is scattered; besides, μ is either σ-compact or countably compact depending on its cofinality. If μ is σ-compact, then it is exponentially separable by Proposition 3.4. If μ is countably compact, then we can apply Corollary 3.20 to see that μ is exponentially separable. □

Definition 3.22. Call a space X *weakly Sokolov* if for any countable family $\mathcal{F}$ of closed subsets of X, there exists a continuous map $f\colon X \to X$ such that $\mathrm{nw}(f(X)) \le \omega$ and $f(F) \subset F$ for any $F \subset \mathcal{F}$.

It follows from [11, Problem 153] that Sokolov spaces are weakly Sokolov and Corollary 3.14 of the paper [12] shows that weakly Sokolov spaces are not necessarily Sokolov.

Proposition 3.23. *Suppose that X is a space and $\mathcal{F}$ is a countable family of closed subsets of X. If $f\colon X \to X$ is a continuous map such that $f(F) \subset F$ for any $F \in \mathcal{F}$, then $Y = f(X)$ is strongly dense in $\mathcal{F}$.*

PROOF: If $\mathcal{G} \subset \mathcal{F}$ and $\bigcap \mathcal{G} \neq \emptyset$, then pick any point $x \in \bigcap \mathcal{G}$. For any $F \in \mathcal{G}$, the point $y = f(x)$ belongs to $f(F) \subset F$ and therefore $y \in Y \cap \bigcap \mathcal{G}$, i.e., Y is strongly dense in $\mathcal{F}$. $\square$

Corollary 3.24. *If X is a weakly Sokolov space, then* $\mathrm{ext}(X) \le \omega$.

PROOF: Suppose that there exists a closed discrete subset $D \subset X$ such that $|D| = \omega_1$. Let $\mathcal{B}$ be a countable base for a topology on the set D and choose a continuous map $f\colon X \to X$ such that $\mathrm{nw}(f(X)) \le \omega$ while $f(D) \subset D$ and $f(B) \subset B$ for every $B \in \mathcal{B}$. If $g = f|D$, then $g\colon D \to D$ and $A = g(D)$ is a countable set while $g(B) \subset B$ for any $B \in \mathcal{B}$. By Proposition 3.23 the set A is strongly dense in $\mathcal{B}$. However, every point of D is the intersection of a subfamily of $\mathcal{B}$ so the countable set A must be equal to D which is a contradiction. $\square$

Corollary 3.25. *A weakly Sokolov space is exponentially separable if and only if it is functionally countable.*

PROOF: We must only prove sufficiency so take any countable family $\mathcal{F}$ of closed subsets of the space X. There exists a continuous map $f\colon X \to X$ such that $\mathrm{nw}(f(X)) \le \omega$ and $f(F) \subset F$ for any $F \in \mathcal{F}$. Functional countability of X easily implies that the set $Y = f(X)$ is countable. By Proposition 3.23 the set Y is strongly dense in $\mathcal{F}$ so X is exponentially separable. $\square$

Corollary 3.26. *If $C_p(X)$ is a Lindelöf Σ-space, then X is exponentially separable if and only if every closed subspace of X is functionally countable.*

PROOF: By Corollary 3.7 we only have to prove sufficiency so assume that every closed subspace of X is functionally countable. Since discrete functionally countable spaces are countable, this implies that $\mathrm{ext}(X) \le \omega$ so X is Lindelöf because it embeds in $C_p(C_p(X))$ (see [9, Problem 167] and [10, Problem 269]). The space $\upsilon X = X$ must be Lindelöf Σ by [11, Problem 206] so both X and $C_p(X)$ are Lindelöf Σ-spaces. This makes it possible to apply Corollary 5.5 of the paper [5] to conclude that X is Sokolov and hence weakly Sokolov. Finally, apply Corollary 3.25 to see that X is exponentially separable. $\square$

Theorem 3.27. *Suppose that κ is an uncountable cardinal and consider the σ-product $S = \{x \in \mathbb{D}^\kappa\colon |x^{-1}(1)| < \omega\}$ in the Cantor cube $\mathbb{D}^\kappa$; let $u \in S$ be the* *function equal to zero at all points of κ. Then the space $X = S \setminus \{u\}$ has the following properties:*

(a) *the set X is C-embedded in S;*
(b) *the space $C_p(X)$ has the Lindelöf Σ-property;*
(c) *the space X is functionally countable;*
(d) $\mathrm{ext}(X) = \kappa > \omega$ *and hence X is not exponentially separable.*

In particular, in Corollary 3.26 *it is not possible to omit the assumption about exponential separability of all closed subspaces.*

Proof: Let $\varphi\colon X \to \mathbb{R}$ be a continuous function. Since X is dense in $\mathbb{D}^\kappa$, we can apply [9, Problem 299] to see that there exists a countable set $A \subset \kappa$ and a continuous function $\xi\colon p_A(X) \to \mathbb{R}$ such that $\varphi = \xi \circ (p_A|X)$; here $p_A\colon S \to \mathbb{D}^A$ is the natural projection. Fix an ordinal $\beta \in \kappa \setminus A$ and define a function $v \in X$ by the equalities $v(\alpha) = 0$ for all $\alpha \in \kappa \setminus \{\beta\}$ and $v(\beta) = 1$. Then $v \in X$ and $\pi_A(v) = \pi_A(u)$ which shows that $\pi_A(X) = \pi_A(S)$ so the function $\xi \circ p_A$ is a continuous extension of φ over S; this proves (a).

(b) Observe first that $C_p(S)$ is a Lindelöf Σ-space by Problem 356 of the book [11]. If $\pi\colon C_p(S) \to C_p(X)$ is the restriction map, then $\pi(C_p(S)) = C_p(X)$ because X is C-embedded in S by (a). Therefore $C_p(X)$ is a Lindelöf Σ-space being a continuous image of $C_p(S)$.

(c) It is standard to see that S is the countable union of scattered compact spaces so S is exponentially separable and hence functionally countable by Corollary 3.7, Proposition 3.2 and Corollary 3.14. If $f\colon X \to \mathbb{R}$ is a continuous function, then there exists a continuous function $g\colon S \to \mathbb{R}$ such that $g|X = f$. Since $g(S)$ is countable, the set $f(X) \subset g(S)$ is also countable.

(d) If $K = \{x \in \mathbb{D}^\kappa\colon |x^{-1}(1)| \leq 1\}$, then it is standard to see that K is compact and u is the unique non-isolated point of K. Therefore $D = K \setminus \{u\}$ is a closed discrete subset of X such that $|D| = \kappa$. Since $w(X) \leq w(\mathbb{D}^\kappa) = \kappa$, we proved that $\mathrm{ext}(X) = \kappa > \omega$ and hence X is not exponentially separable by Proposition 3.11. □

Observation 3.28. If X is a functionally countable Lindelöf p-space, then it is a perfect preimage of a countable space so $X = \bigcup_{n\in\omega} K_n$ where every K_n is compact. Since every K_n is C-embedded in X, it must also be functionally countable and hence scattered (see [10, Problem 133]). This, together with Proposition 3.2 (d), shows that a Lindelöf p-space is exponentially separable if and only if it is the countable union of scattered compact subspaces. The author could not find out whether the same is true for Lindelöf Σ-spaces; this sounds like an interesting conjecture.

The following lemma might be known but it is presented here with a complete proof because the author could not find the respective reference.

Lemma 3.29. *Suppose that X is a normal space and $F_1, \dots, F_n$ are closed subsets of X. If $F = F_1 \cap \ldots \cap F_n$, then $\mathrm{cl}_{\beta X}(F) = \mathrm{cl}_{\beta X}(F_1) \cap \ldots \cap \mathrm{cl}_{\beta X}(F_n)$.*

Proof: The statement of the lemma is trivially true if $n = 1$. Proceeding by induction assume that our lemma holds for some $n \in \mathbb{N}$ and take any closed sets $F_1, \dots, F_n, F_{n+1}$ in the space X. Let $F = F_1 \cap \dots \cap F_{n+1}$; we must only prove that $\mathrm{cl}_{\beta X}(F_1) \cap \dots \cap \mathrm{cl}_{\beta X}(F_{n+1}) \subset \mathrm{cl}_{\beta X}(F)$.

Suppose that $x \in \bigcap_{i \le n+1} \mathrm{cl}_{\beta X}(F_i)$ but $x \notin \mathrm{cl}_{\beta X}(F)$ for some $x \in \beta X$. Fix a set $U \in \tau(x, \beta X)$ such that $\mathrm{cl}_{\beta X}(U) \cap F = \emptyset$. If $G_i = F_i \cap \mathrm{cl}_{\beta X}(U)$, then G_i is a closed subset of X and $x \in \mathrm{cl}_{\beta X}(G_i)$ for every $i \le n+1$; besides, $\bigcap_{i \le n+1} G_i = \emptyset$. By the induction hypothesis, the point x belongs to the closure of the set $G = G_1 \cap \dots \cap G_n$. Therefore G and G_{n+1} are disjoint closed subsets of the normal space X whose closures in βX contain the point x; this contradiction completes the proof of the induction step. $\square$

Theorem 3.30. *If X is a countably compact normal space, then X is exponentially separable if and only if it is functionally countable.*

Proof: We must only prove sufficiency so assume that X is functionally countable. A moment's reflection shows that βX is also functionally countable so it is scattered by [10, Problem 133]. Given a countable family $\mathcal{F}$ of closed subsets of X apply Corollary 3.14 to find a countable set $B \subset \beta X$ that is strongly dense in the family $\mathcal{E} = \{\mathrm{cl}_{\beta X}(F) \colon F \in \mathcal{F}\}$. For every $b \in B$ let $\mathcal{Q}_b = \{F \in \mathcal{F} \colon b \in \mathrm{cl}_{\beta X}(F)\}$ and let $\{F_n^b \colon n \in \omega\}$ be an enumeration of the family $\mathcal{Q}_b$.

Since $b \in \bigcap\{\mathrm{cl}_{\beta X}(F_i^b) \colon i \le n\}$, it follows from Lemma 3.29 that the set $\bigcap\{F_i^b \colon i \le n\}$ is nonempty for every $n \in \omega$ so $F^b = \bigcap\{F_n^b \colon n \in \omega\} = \bigcap \mathcal{Q}_b \neq \emptyset$ by countable compactness of X. Choose a point $a_b \in F^b$ for every $b \in B$.

Take any subfamily $\mathcal{G}$ of the family $\mathcal{F}$ such that $\bigcap \mathcal{G} \neq \emptyset$. Then it follows from $\bigcap\{\mathrm{cl}_{\beta X}(G) \colon G \in \mathcal{G}\} \neq \emptyset$ and our choice of B that there exists $b \in B$ such that $b \in \bigcap\{\mathrm{cl}_{\beta X}(G) \colon G \in \mathcal{G}\}$ and hence $\mathcal{G} \subset \mathcal{Q}_b$. Therefore $a_b \in \bigcap \mathcal{Q}_b \subset \bigcap \mathcal{G}$ so the countable set $A = \{a_b \colon b \in B\}$ is strongly dense in $\mathcal{F}$. $\square$

4. Open questions

There are still a lot of interesting open questions about functionally countable and exponentially separable spaces. The most intriguing one is whether every countably compact functionally countable space is exponentially separable.

Question 4.1. Suppose that X is a functionally countable countably compact space. Must X be exponentially separable?

Question 4.2. Suppose that X is a countably compact space in which every closed subspace is functionally countable. Must X be exponentially separable?

Question 4.3. Let X be an exponentially separable space with a G_δ-diagonal. Must X be countable?

Question 4.4. Suppose that X is a space in which every closed subspace is functionally countable. Must X be exponentially separable?

Question 4.5. Suppose that X is an exponentially separable space. Must $X \times X$ be exponentially separable?

Question 4.6. Suppose that X is a Lindelöf exponentially separable space. Must $X \times X$ be exponentially separable?

Question 4.7. Let X be a P-space with $\mathrm{ext}(X) \leq \omega$. Is it true that X is exponentially separable?

Question 4.8. Let X be a normal P-space with $\mathrm{ext}(X) \leq \omega$. Is it true that X is exponentially separable?

Question 4.9. Suppose that X is finite-like in the sense of R. Telgársky, see [7]. Must X be exponentially separable?

Question 4.10. Assume that X is an exponentially separable space. Must the Hewitt realcompactification of X be exponentially separable?

Question 4.11. Let X be a functionally countable Lindelöf space. Must X be exponentially separable?

Question 4.12. Assume that X is a functionally countable Lindelöf Σ-space. Is X the countable union of Lindelöf scattered spaces?

Question 4.13. Suppose that X is a functionally countable Lindelöf Σ-space. Must X be exponentially separable?

Question 4.14. Suppose that X and $C_p(X)$ are Lindelöf Σ-spaces and X is exponentially separable. Is X the countable union of Lindelöf scattered spaces?

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