Optimal control of a frictionless contact problem with normal compliance

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Abstract. We consider a mathematical model which describes a contact between an elastic body and a foundation. The contact is frictionless with normal compliance. The goal of this paper is to study an optimal control problem which consists of leading the stress tensor as close as possible to a given target, by acting with a control on the boundary of the body. We state an optimal control problem which admits at least one solution. Next, we establish an optimality condition corresponding to a regularization of the model. We also introduce the regularized control problem for which we study the convergence when the regularization parameter tends to zero.

Keywords: optimal control; variational inequality; linear elastic frictionless contact; regularized problem

Classification: 49J40, 47J20, 74M10

1. Introduction

In our daily life and same in industry, the problem of contact between deformable bodies plays an important role in structural and mechanical systems, contact models are of great interest. To get a background in contact mechanics from mathematical or engineering point of view the reader can consult for instance [14], [23], [25], [27]. At now considerable efforts have been made in its modeling and numerical simulations. The theory of the optimal control of variational inequalities is very elaborated, see for instance [3], [5], [9], [15], [19], [20], [21]. Despite their mechanical relevance, optimal control problems for contact models are not too much developed and represent challenging task, see [1], [4], [6], [7], [8], [11], [17].

In [21], we find the study of optimal control linear or non linear elliptic problems and variational inequalities. Also the optimal control of both contact problems was studied in [17], [18]; in [17] the author considers a bilateral contact between a deformable body and a rigid foundation while in [18] the authors consider a frictional contact with normal compliance condition between an elastic body and a deformable foundation. Recall that the normal compliance contact condition was first introduced in [22] and then it was used in many publications, see [2], [10], [12], [13], [16], [24], [27] and references therein.

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As in [28], in this paper we study an optimal control of a contact problem. Indeed, we consider a linear elastic body which is in frictionless contact with normal compliance with a deformable foundation. We establish a variational formulation of the mechanical problem Problem (P_1) and prove the existence and uniqueness result of solution. The optimal control problem concerning this model is denoted by (POC1). It consists of minimizing a cost functional which is quadratic and nonconvex. As the standard results on the convexity cannot be used, we replace it by an indirect method related to the study of variational inequality. However, we are interested in leading the stress tensor field as close as possible to a given target when we act with a control on a part of the boundary. We prove the existence of solution of (POC1) and we study the regularization of the state problem Problem (P_δ). Next, we introduce the regularized optimal control problem (POC2). Moreover, we deduce an optimality condition with help of an abstract theorem due to J.-L. Lions. Finally, we study the convergence of solution of Problem to the solution of Problem (P_1) and the solution of the regularized optimal control problem (POC2) to the solution of Problem (POC1).

The paper is organized as follows. In Section 2, we describe the mechanical model and discuss its weak solvability. In Section 3, we state the optimal control (POC1) and prove that it has at least one solution. In Section 4 we state and analyze a regularized optimal control problem. In Section 5, we deliver an optimality condition involving the regularized optimal control. In Section 6, some convergence results are established.

2. Variational formulation

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a domain occupied by a linear elastic body. The body Ω is supposed to be open, bounded, with a sufficiently regular boundary partitioned into three measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that $\mathrm{meas}(\Gamma_1) > 0$. The body is acted upon by a volume force of density φ_0 on Ω and a surface traction of density φ_2 on Γ_2. Along Γ_3, the body is in frictionless contact with normal compliance with a deformable foundation. Thus, the classical formulation of the mechanical problem in terms of displacement is written as follows.

Problem (P_1). Find a displacement field $u\colon \Omega \to \mathbb{R}^d$ such that

$$\operatorname{div} \sigma(u) = -\varphi_0 \quad \text{in } \Omega, \tag{2.1}$$

$$\sigma(u) = \mathcal{A}\varepsilon(u) \quad \text{in } \Omega, \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\nu = \varphi_2 \quad \text{on } \Gamma_2, \tag{2.4}$$

$$\sigma_\nu = -p(u_\nu) \quad \text{on } \Gamma_3, \tag{2.5}$$

$$\sigma_\tau = 0 \quad \text{on } \Gamma_3. \tag{2.6}$$

Here (2.1) represents the equilibrium equation such that $\sigma = \sigma(u)$ denotes the stress tensor. Equation (2.2) is the elastic constitutive law where $\mathcal{A}$ is the fourth order tensor of elasticity coefficients; (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. The condition (2.5) represents the contact with normal compliance in which σ_ν denotes the normal stress and p is a prescribed nonnegative function. The condition (2.6) represents the frictionless contact. Recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and S^d are given by

$$u \cdot v = u_i v_i, \qquad \|v\| = (v \cdot v)^{1/2} \qquad \text{for all } u, v \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\| = (\tau \cdot \tau)^{1/2} \qquad \text{for all } \sigma, \tau \in S^d,$$

where S^d is the space of second order symmetric tensors on $\mathbb{R}^d$, $d = 2, 3$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H = (L^2(\Omega))^d, \qquad H_1 = (H^1(\Omega))^d, \qquad Q = \{\sigma = (\sigma_{ij})\colon \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\}.$$

Note that H and Q are real Hilbert spaces endowed with the respective canonical inner products:

$$(u, v)_H = \int_\Omega u_i v_i \,\mathrm{d}x, \qquad (\sigma, \tau)_Q = \int_\Omega \sigma_{ij}\tau_{ij} \,\mathrm{d}x.$$

The linearized strain tensor is defined as

$$\varepsilon(v) = (\varepsilon_{ij}(v)) \qquad \text{for all } v \in H_1, \text{ where } \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}),$$

$\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of σ. For every $v \in H_1$, we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and the tangential components of v on the boundary Γ given by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$. Also, for a regular function (say C^1) $\sigma \in Q$, we define its normal and tangential components by $\sigma_\nu = (\sigma\nu) \cdot \nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$ and we recall that the following Green's formula holds:

$$(\sigma, \varepsilon(v))_Q + (\operatorname{div}\sigma, v)_H = \int_\Gamma \sigma\nu \cdot v \,\mathrm{d}a \qquad \text{for all } v \in H_1,$$

where $\mathrm{d}a$ is the surface measure element. Let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1 \colon v = 0 \ \text{ on } \ \Gamma_1\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, the following Korn's inequality holds,

$$\|\varepsilon(v)\|_Q \geq c_\Omega \|v\|_{H_1} \qquad \text{for all } v \in V, \tag{2.7}$$

where the constant $c_\Omega > 0$ depends only on Ω and Γ_1. We endow V with the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_Q \qquad \text{for all } u, v \in V,$$

and $\|\cdot\|_V$ is the associated norm. It follows from Korn's inequality (2.7) that the norms $\|\cdot\|_H$ and $\|\cdot\|_V$ are equivalent on V. Then $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover by Sobolev's trace theorem, there exists $\mathrm{d}_\Omega > 0$ which depends only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \le \mathrm{d}_\Omega \|v\|_V \qquad \text{for all } v \in V. \tag{2.8}$$

We assume that the body forces and surface tractions have the regularity

$$\varphi_0 \in H, \qquad \varphi_2 \in (L^2(\Gamma_2))^d. \tag{2.9}$$

Also we define the functional $j \colon V \times V \to \mathbb{R}$ by

$$j(u, v) = \int_{\Gamma_3} p(u_\nu)|v_\nu| \, \mathrm{d}a \qquad \text{for all } u, v \in V,$$

where we assume that the normal compliance function $p \colon \mathbb{R} \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) there exists } L_p > 0 \text{ such that:} \\ \qquad |p(r_1) - p(r_2)| \le L_p |r_1 - r_2| \;\; \text{for all } r_1, r_2 \in \mathbb{R}; \\ \text{(b) } p(r) = 0 \;\; \text{for all } r \le 0. \end{cases} \tag{2.10}$$

Next, in the study of Problem (P_1) we assume that the linear elasticity operator $\mathcal{A}$ verifies

$$\begin{cases} \text{(a) } \mathcal{A} = (\mathcal{A}_{ijkl}) \colon \Omega \times S^d \to S^d; \\ \text{(b) there exists } m > 0 \text{ such that} \\ \qquad \mathcal{A}\varepsilon \cdot \varepsilon \ge m\|\varepsilon\|^2 \;\; \text{for all } \varepsilon \in S^d \;\; \text{a.e. in } \Omega; \\ \text{(c) } \mathcal{A} \in Q_\infty. \end{cases} \tag{2.11}$$

We recall that Q_∞ is the space of fourth-order tensor fields defined as

$$Q_\infty = \{\mathcal{E} = (\mathcal{E}_{ijkl}) \colon \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega) \;\; \text{for all } \; i, j, k, l \in \{1, \ldots, d\}\},$$

and Q_∞ is a real Banach space endowed with the norm

$$\|\mathcal{E}\|_{Q_\infty} = \max_{1 \le i,j,k,l \le d} \|\mathcal{E}_{jikl}\|_{L^\infty(\Omega)}.$$

Finally, we need to assume that

$$\mathrm{d}_\Omega^2 L_p < m. \tag{2.12}$$

Now by assuming the solution to be sufficiently regular, we obtain by using Green's formula that Problem (P_1) has the following variational formulation.

Problem (P_2)**.** Find a displacement field $u \in V$ such that

$$(2.13)\qquad \begin{aligned}&(Au, v-u)_V + j(u,v) - j(u,u)\\ &\qquad \geq (\varphi_0, v-u)_H + (\varphi_2, v-u)_{(L^2(\Gamma_2))^d} \qquad \text{for all } v \in V.\end{aligned}$$

Herein, the operator $A\colon V \to V$ is defined as

$$(Au, v)_V = (\mathcal{A}\varepsilon(u), \varepsilon(v))_Q \qquad \text{for all } \ u, v \in V.$$

Theorem 2.1. *Assume that* (2.9), (2.10), (2.11) *and* (2.12) *hold. Then, there exists a unique solution of Problem* (P_2)*.*

Proof: By (2.11), the operator A is Lipschitz continuous and strongly monotone; using (2.10), the functional j is proper, convex and lower semicontinuous. Moreover by (2.9), it follows from [26] that the inequality (2.13) has a unique solution under the condition (2.12). □

3. The optimal control problem

We now suppose that $\varphi_0 \in H$ is fixed and consider the following state variational problem.

Problem (PS1)**.** For a given $\varphi_2 \in (L^2(\Gamma_2))^d$ (called control), find $u \in V$ such that

$$(3.1)\qquad \begin{aligned}&(Au, v-u)_V + j(u,v) - j(u,u)\\ &\qquad \geq (\varphi_0, v-u)_H + (\varphi_2, v-u)_{(L^2(\Gamma_2))^d} \qquad \text{for all } v \in V.\end{aligned}$$

Following the existence and uniqueness of Problem (P_2), we deduce that for every control $\varphi_2 \in (L^2(\Gamma_2))^d$, the state variational problem (PS1) has a unique solution $u \in V$, $u = u(\varphi)$ if (2.12) is verified. Now, by acting the control on the boundary Γ_2, we like to get that the resulting stress be close to a given target σ_d. We assume that $\sigma_d = \mathcal{A}\varepsilon(u_d)$ where $u_d \in V$ and recall that $\sigma(u) = \mathcal{A}\varepsilon(u)$; then, it follows by (2.11) (c) that there exists a constant $M > 0$ such that $\|\sigma(u) - \sigma_d\|_Q \leq M\|u - u_d\|_V$.

Thus, we see that if $\|u - u_d\|_V$ is sufficiently small then $\sigma(u)$ is close to σ_d. Let now the cost functional

$$\mathcal{L}\colon (L^2(\Gamma_2))^d \times V \to \mathbb{R},$$

be given by

$$(3.2)\qquad \mathcal{L}(\varphi, v) = \frac{\alpha}{2}\|v - u_d\|_V^2 + \frac{\beta}{2}\|\varphi\|^2_{(L^2(\Gamma_2))^d},$$

where $\alpha, \beta > 0$.

Next, we define the set U_{ad} as

$$U_{ad} = \{(u, \varphi_2) \in V \times (L^2(\Gamma_2))^d \colon (3.1) \text{ is satisfied}\},$$

and we consider the following optimal control problem.

Problem (POC1)**.** Find $(u^*, \varphi^*) \in U_{ad}$ such that

$$\mathcal{L}(\varphi^*, u^*) = \min_{(u,\varphi)\in U_{ad}} \mathcal{L}(\varphi, u).$$

Theorem 3.1. *Assume* (2.7), (2.8), (2.10) (b) *and* (2.11) (b). *Then Problem* (POC1) *has at least one solution.*

Proof: Take $v = 0_V$ in (3.1), using (2.7), (2.8) and (2.11) (c), we deduce that the solution u of Problem (PS1) is bounded in V as

$$\|u\|_V \leq \frac{\|\varphi_0\|_H + c\|\varphi_2\|_{(L^2(\Gamma_2))^d}}{m},$$

where $c > 0$. This inequality implies that

$$0 \leq \inf_{(u,\varphi_2)\in U_{ad}} \{\mathcal{L}(\varphi_2, u)\} < \infty.$$

Then, there exists a minimizing sequence $(u^n, \varphi_2^n) \subset U_{ad}$ such that

$$\lim_{n\to\infty} \mathcal{L}(\varphi_2^n, u^n) = \inf_{(u,\varphi_2)\in U_{ad}} \{\mathcal{L}(\varphi_2, u)\}.$$

The sequence (u^n, φ_2^n) is bounded in $V \times (L^2(\Gamma_2))^d$, so there exists an element

$$(u^*, \varphi_2^*) \in V \times (L^2(\Gamma_2))^d,$$

such that passing to a subsequence, still denoted by (u^n, φ_2^n), we deduce as $n \to \infty$,

$$\left\{ \begin{array}{ll} \text{(i) } u^n \to u^* & \text{weakly in } \ V, \\ \text{(ii) } \varphi^n \to \varphi^* & \text{weakly in } \ (L^2(\Gamma_2))^d. \end{array} \right. \tag{3.3}$$

Now to end the proof we need to prove that

$$u^n \to u^* \qquad \text{strongly in } \ V \text{ as } n \to \infty. \tag{3.4}$$

Indeed, as $(u^n, \varphi_2^n) \in U_{ad}$, then u^n verifies the inequality:

$$\begin{aligned} &(Au^n, v - u^n)_V + \ j(u^n, v) - j(u^n, u^n) \\ &\qquad \geq (\varphi_0, v - u^n)_H + (\varphi_2, v - u^n)_{(L^2(\Gamma_2))^d} \qquad \text{for all } \ v \in V. \end{aligned} \tag{3.5}$$

Using (2.11) (b) and (3.5), we arrive at

$$\begin{aligned} m\|u^n - u^*\|_V^2 &\leq (Au^n - Au^*, u^n - u^*)_V \\ &\leq (Au^n, u^n - u^*)_V - (Au^*, u^n - u^*)_V \\ &\leq (Au^*, u^* - u^n)_V + j(u^n, u^*) - j(u^n, u^n) \\ &\quad + (\varphi_0, u^n - u^*)_H + (\varphi_2^n, u^n - u^*)_{(L^2(\Gamma_2))^d}. \end{aligned} \tag{3.6}$$

Then, as we have

$$|j(u^n,u^*)-j(u^n,u^n)|\le \int_{\Gamma_3} p(u^n_\nu)|u^*_\nu-u^n_\nu|\ \mathrm{d}a$$

and moreover using (2.8), (2.10) and (3.3) (i), it follows that

$$\lim_{n\to\infty}|j(u^n,u^*)-j(u^n,u^n)|=0.$$

On the other hand, since (φ_2^n) is bounded in $(L^2(\Gamma_2))^d$, we obtain

$$\lim_{n\to\infty}(\varphi_2^n,u^n-u^*)_{(L^2(\Gamma_2))^d}=0.$$

Consequently,

$$\lim_{n\to\infty}\big(\ j(u^n,u^*)-j(u^n,u^n)+(\varphi_0,u^n-u^*)_H+(\varphi_2^n,u^n-u^*)_{(L^2(\Gamma_2))^d}\big)=0.$$

Using now (3.6), we get (3.4). Moreover, passing to the limit as $n\to\infty$ in (3.5), one obtains that $(u^*,\varphi^*)\in U_{ad}$ and it is a solution of Problem (POC1). □

Remark 3.2. For any minimizer (u^*,φ^*) of Problem (POC1), φ^* is a minimizer of the functional $J\colon (L^2(\Gamma_2))^d\to\mathbb{R}$ defined as,

$$J(\varphi)=\frac{\alpha}{2}\|u(\varphi)-u_d\|_V^2+\frac{\beta}{2}\|\varphi\|^2_{(L^2(\Gamma_2))^d}.$$

4. The regularized optimal control problem

Let $\delta>0$ and consider the functional $j_\delta\colon V\times V\to\mathbb{R}$ defined as

$$j_\delta(u,v)=\int_{\Gamma_3} p_\delta(u_\nu)\big(\sqrt{v_\nu^2+\delta^2}-\delta\big)\ \mathrm{d}a,$$

where p_δ satisfies the hypotheses below.

$$(4.1)\qquad \begin{cases} \text{(a) } p_\delta\in C^1(\mathbb{R},\mathbb{R}_+);\\ \text{(b) } p_\delta(r)=0 \ \text{ for all } r\le 0;\\ \text{(c) there exists } M_1,\ L_1>0 \text{ such that}\\ \qquad |p_\delta(r)|\le M_1, |p'_\delta(r)|\le L_1 \ \text{ for all } r\in\mathbb{R};\\ \text{(d) there exists } G_p\colon\mathbb{R}\to\mathbb{R}_+ \text{ such that}\\ \qquad |p_\delta(r)-p(r)|\le G_p(\delta) \ \text{ for all } r\in\mathbb{R}\\ \qquad \text{and } \lim\limits_{\delta\to 0}G_p(\delta)=0. \end{cases}$$

Let $u,v\in V$. There exists an element $\nabla_2 j_\delta(u,v)\in V$ such that

$$\lim_{h\to 0}\frac{j_\delta(u,v+hw)-j_\delta(u,v)}{h}=(\nabla_2 j_\delta(u,v),w)_V \qquad \text{for all } w\in V,$$

the existence of the Gâteaux gradient $\nabla_2 j_\delta(u,v) \in V$ is ensured by the Riesz representation theorem since the application

$$w \in V \to \int_{\Gamma_3} p(u_\nu)\frac{v_\nu w_\nu}{\sqrt{v_\nu^2+\delta^2}}\,\mathrm{d}a$$

is a linear and continuous functional. Thus, giving $u, v \in V$ there exists a unique element in V denoted by $\nabla_2 j_\delta(u,v)$ such that

$$(\nabla_2 j_\delta(u,v), w)_V = \int_{\Gamma_3} p(u_\nu)\frac{v_\nu w_\nu}{\sqrt{v_\nu^2+\delta^2}}\,\mathrm{d}a \qquad \text{for all } w \in V. \tag{4.2}$$

Now let $\delta > 0$ and $\varphi_2 \in (L^2(\Gamma_2))^d$, we state the regularized problem below.

Problem (P_δ)**.** Find the displacement field $u^\delta \in V$ such that

$$\begin{aligned}&(Au^\delta, v-u^\delta)_V + j_\delta(u^\delta, v) - j_\delta(u^\delta,u^\delta)\\ &\qquad \geq (\varphi_0, v-u^\delta)_H + (\varphi_2, v-u^\delta)_{(L^2(\Gamma_2))^d} \qquad \text{for all } v \in V.\end{aligned}$$

Theorem 4.1. *Assume that* (2.9), (2.10), *and* (2.11). *Then, there exists a unique solution of Problem* (P_δ) *if* $L_1 d_\Omega^2 < m$.

PROOF: With the same arguments used in the proof of Theorem 2.1, Problem (P_δ) admits a unique solution, see [16]. ☐

Now, for $\delta > 0$ and a fixed $\varphi_0 \in H$, we introduce the following regularized state problem.

Problem (PS2)**.** For a given $\varphi_2 \in (L^2(\Gamma_2))^d$ (called control), find $u^\delta \in V$ such that

$$\begin{aligned}&(Au^\delta, v-u^\delta)_V + j_\delta(u^\delta, v) - j_\delta(u^\delta,u^\delta)\\ &\qquad \geq (\varphi_0, v-u^\delta)_H + (\varphi_2, v-u^\delta)_{(L^2(\Gamma_2))^d} \qquad \text{for all } v \in V.\end{aligned} \tag{4.3}$$

By Theorem 4.1, Problem (PS2) has a unique solution $u \in V$, $u^\delta = u(\varphi_2)$. In addition, we have

$$\|u^\delta\|_V \leq \frac{1}{m}(\|\varphi_0\|_H + c\|\varphi_2\|_{(L^2(\Gamma_2))^d}).$$

Furthermore, we define the set U_{ad}^δ as

$$U_{ad}^\delta = \{u, \varphi_2 \in V \times (L^2(\Gamma_2))^d \colon (4.3) \text{ is satisfied}\}.$$

Using the functional $\mathcal{L}$, given by (3.2), we introduce the regularized optimal control problem defined below as

Problem (POC2). Find $[\bar{u}, \overline{\varphi}_2] \in U^\delta_{ad}$ such that

$$\mathcal{L}(\overline{\varphi}_2, \bar{u}) = \min_{[u,\varphi_2] \in U^\delta_{ad}} \{\mathcal{L}(\varphi_2, u)\}.$$

With arguments similar to those used in Theorem 3.1, the following result can be proved.

Theorem 4.2. *Assume* (2.7), (2.8), (2.10) (b) *and* (2.11) (b). *Then Problem* (POC2) *has at least one solution.*

Proof: A solution of Problem (POC2) is called a *regularized optimal pair* and the second component $\overline{\varphi}_2$ is called *regularized optimal control.* Like for Problem (POC1), we have that $\overline{\varphi}_2$ is a minimizer of the functional

$$J\colon (L^2(\Gamma_2))^d \to \mathbb{R}, \qquad J(\varphi_2) = \frac{\alpha}{2}\|u^\delta(\varphi_2) - u_d\|^2_V + \frac{\beta}{2}\|\varphi_2\|^2_{(L^2(\Gamma_2))^d}, \tag{4.4}$$

where $u^\delta(\varphi_2)$ is the solution of the state problem (PS2). Again, the functional J is not convex in general. □

5. Optimality conditions for the regularized problem

In this section, we use the following standard result that can be found in [15]. The sketch of its proof is provided in [1]. This result will be the key in obtaining our optimal condition. In addition, in this section we denote u_δ and p_δ instead of u and p, respectively.

Lemma 5.1. *Let B be a Banach space, X and Y two reflexive Banach spaces. Let also be given two C^1 functions*

$$G\colon \mathcal{B}\times X \to Y, \qquad \mathcal{L}\colon \mathcal{B}\times X \to \mathbb{R}.$$

We suppose that for all $\eta \in B$:

i) *there exists a unique $\tilde{u}(\eta)$ such that $G(\eta, \tilde{u}(\eta)) = 0$,*
ii) *$\partial_2 G(\eta, \tilde{u}(\eta))$ is an isomorphism from X onto Y.*

Then, $J(\eta) = L(\eta, \tilde{u}(\eta))$ is differentiable and for all $\zeta \in B$,

$$\frac{\mathrm{d}J}{\mathrm{d}\eta}(\eta) = \partial_1\mathcal{L}(\eta, \tilde{u}(\eta))\zeta - \langle g(\eta), \partial_1 G(\eta, \tilde{u}(\eta))\rangle_{Y'\times Y}, \tag{5.1}$$

where $g(\eta) \in Y'$ is the adjoint state, unique solution of

$$[\partial_2 G(\eta, \tilde{u}(\eta))]^* g(\eta) = \partial_2\mathcal{L}(\eta, \tilde{u}(\eta)) \in X'. \tag{5.2}$$

Now before we start with the proof of the theorem below, we need to prove the following lemma.

Lemma 5.2. *Let $u \in V$ be the unique solution of Problem* (PS2), *then for any $\phi \in (L^2(\Gamma_2))^d$, there exists a unique $z \in V$ such that*

$$Au + \nabla_2 j_\delta(u, v) = z + y(\phi). \tag{5.3}$$

Proof: We use Riesz's representation theorem to define for each $\phi \in (L^2(\Gamma_2))^d$, the element $y(\phi) \in V$ by

$$(y(\phi), v)_V = \int_{\Gamma_3} \phi \cdot v \, \mathrm{d}a \qquad \text{for all } v \in V. \tag{5.4}$$

Furthermore, there exists a unique $z \in V$ such that

$$(z, v)_V = \int_{\Gamma_3} \varphi_0 \cdot v \, \mathrm{d}x \qquad \text{for all } v \in V. \tag{5.5}$$

Let $u \in V$ be the unique solution of Problem (PS2) and let us define as in [18],

$$\partial_2 j_\delta(u, u) = \{\zeta \in V \colon j_\delta(u, v) - j_\delta(u, u) \geq (\zeta, v - u)_V \ \text{ for all } v \in V\}.$$

Using (5.5), the inequality (4.3) is equivalent with the following inclusion

$$z + y(\phi) - Au \in \partial_2 j_\delta(u, u).$$

Since $j_\delta(.,.)$ is convex and Gâteaux differentiable in the second argument

$$\partial_2 j_\delta(u, u) = \{\nabla_2 j_\delta(u, u)\}. \tag{5.6}$$

Thus with (5.6), the inequality (4.3) is equivalent with the nonlinear equation (5.3). □

Theorem 5.3. *(Optimality condition): Any optimal control $\overline{\varphi}_2$ of the state Problem* (PS2) *verifies*

$$\overline{\varphi}_2 = -\frac{1}{\beta}\gamma(g(\overline{\varphi}_2)), \tag{5.7}$$

where γ is the trace operator for vector valued functions and $g(\overline{\varphi}_2)$ is the unique solution of the variational equation,

$$\alpha(u(\overline{\varphi}_2) - u_d, w)_V = ((g(\overline{\varphi}_2), Fw + D_2^2 j_\delta(u(\overline{\varphi}_2)), u(\overline{\varphi}_2))w)_V \qquad \textit{for all } \ w \in V,$$

$u(\overline{\varphi}_2)$ being the solution of Problem (PS2) *with $\varphi_2 = \overline{\varphi}_2$ and for all $v \in V$, writing u instead of $u(\overline{\varphi}_2)$,*

$$\begin{aligned}(D_2^2 j_\delta(u(\overline{\varphi}_2), u(\overline{\varphi}_2))w)_V = &\int_{\Gamma_3} p'(u_\nu)\frac{u_\nu v_\nu w_\nu}{\sqrt{u_\nu^2 + \delta^2}} \, \mathrm{d}a \\ &+ \int_{\Gamma_3} p(u_\nu)\frac{v_\nu w_\nu \delta^2}{(u_\nu^2 + \delta^2)^{3/2}} \, \mathrm{d}a.\end{aligned} \tag{5.8}$$

PROOF: For every $(u, \varphi_2) \in V \times (L^2(\Gamma_2))^d$, we have

$$\partial_1 \mathcal{L}(\varphi_2, u)\colon (L^2(\Gamma_2))^d \to \mathbb{R},\ \ \partial_1 \mathcal{L}(\varphi_2, u)\zeta = \beta(\varphi_2, \zeta)_{L^2(\Gamma_2)^d} \ \text{ for all } \zeta \in (L^2(\Gamma_2))^d,$$

and

$$\partial_2 \mathcal{L}(\varphi_2, u)\colon V \to \mathbb{R}, \qquad \partial_2 \mathcal{L}(\varphi_2, u)v = \alpha(u - u_d, v)_V \qquad \text{for all } v \in V.$$

Now, using (5.3) and define $G\colon (L^2(\Gamma_2))^d \times V \to V$,

$$G(\varphi_2, u) = Au + \nabla_2 j_\delta(u, u) - z - y(\varphi_2) \qquad \text{for all } \ (u, \varphi_2) \in V \times (L^2(\Gamma_2))^d.$$

For every $(u, \varphi_2) \in V \times (L^2(\Gamma_2))^d$ we have

$$\partial_1 G(\varphi_2, u)\colon (L^2(\Gamma_2))^d \to V, \qquad \partial_1 G(\varphi_2, u)\zeta = -y(\zeta) \qquad \text{for all } \zeta \in (L^2(\Gamma_2))^d.$$

To compute $\partial_2 G(u, \varphi_2)$, let $v, w \in V$ and $h > 0$. Then we have

$$\begin{aligned}\frac{(G(\varphi_2, u + hv) - G(\varphi_2, u), w)_V}{h} &= \frac{(A(u + hv) - Au, w)_V}{h}\\ &\quad + \frac{(\nabla_2 j_\delta(u + hv, u + hv) - \nabla_2 j_\delta(u, u), w)_V}{h}.\end{aligned}$$

By taking into account (4.2), we get

$$\begin{aligned}&\frac{(\nabla_2 j_\delta(u + hv, u + hv) - \nabla_2 j_\delta(u, u), w)_V}{h}\\
&\qquad = \frac{1}{h}\int_{\Gamma_3} p(u_\nu + hv_\nu)\frac{(u_\nu + hv_\nu)w_\nu}{\sqrt{(u_\nu + hv_\nu)^2 + \delta^2}}\\
&\qquad\quad - \frac{1}{h}\int_{\Gamma_3} p(u_\nu)\frac{u_\nu w_\nu}{\sqrt{u_\nu^2 + \delta^2}}\ \mathrm{d}a\\
&\qquad = \int_{\Gamma_3} \frac{p(u_\nu + hv_\nu) - p(u_\nu)}{h}\,\frac{(u_\nu + hv_\nu)w_\nu}{\sqrt{(u_\nu + hv_\nu)^2 + \delta^2}}\ \mathrm{d}a\\
&\qquad\quad + \int_{\Gamma_3} p(u_\nu)\Big[\frac{(u_\nu + hv_\nu)w_\nu}{\sqrt{(u_\nu + hv_\nu)^2 + \delta^2}} - \frac{u_\nu w_\nu}{\sqrt{u_\nu^2 + \delta^2}}\Big]\ \mathrm{d}a.\end{aligned}$$

Then, we deduce that

$$\begin{aligned}\lim_{h\to 0}\frac{(\nabla_2 j_\delta(u + hv, u + hv) - \nabla_2 j_\delta(u, u), w)_V}{h} &= \int_{\Gamma_3} p'(u_\nu)\frac{u_\nu v_\nu w_\nu}{\sqrt{u_\nu^2 + \delta^2}}\ \mathrm{d}a\\
&\quad + \int_{\Gamma_3} p(u_\nu)\frac{v_\nu w_\nu \delta^2}{(u_\nu^2 + \delta^2)^{3/2}}\ \mathrm{d}a.\end{aligned}$$

Let us denote by $D_2^2 j_\delta(u, u)v$ the unique element of V such that for all $w \in V$,

$$(D_2^2 j_\delta(u, u)v, w)_V = \int_{\Gamma_3} p'(u_\nu)\frac{u_\nu v_\nu w_\nu}{\sqrt{u_\nu^2 + \delta^2}}\ \mathrm{d}a + \int_{\Gamma_3} p(u_\nu)\frac{v_\nu w_\nu \delta^2}{(u_\nu^2 + \delta^2)^{3/2}}\ \mathrm{d}a,$$

so

$$\partial_2 G(\varphi_2, u)\colon V \to V, \qquad \partial_2 G(\varphi_2, u)v = Av + D_2^2 j_\delta(u,u)v \qquad \text{for all } v \in V.$$

Let $\varphi_2 \in (L^2(\Gamma_2))^d$, and $u(\varphi_2) \in V$ be the corresponding solution of the regularized state Problem (PS2)

$$\partial_2 G(\varphi_2, u(\varphi_2))v = Av + D_2^2 j_\delta(u(\varphi_2), u(\varphi_2))v \qquad \text{for all } v \in V.$$

Then, we shall prove that $\partial_2 G(\varphi_2, u(\varphi_2))\colon V \to V$ is an isomorphism. In fact, we have for all $\zeta \in V$ there exists a unique element $v^* \in V$ such that,

$$(Av^*, w)_V + (D_2^2 j_\delta(u(\varphi_2), u(\varphi_2))v^*, w)_V = (\zeta, w)_V \qquad \text{for all } w \in V. \tag{5.9}$$

Let us define a bilinear form $a\colon V \times V \to \mathbb{R}$ by

$$a(v,w) = (Av, w)_V + (D_2^2 j_\delta(u(\varphi_2), u(\varphi_2)v, w)_V,$$

then using the properties of the operator A and of the function p, (2.11) and (5.9), we deduce that a is continuous and V-elliptic. Thus, by Lax–Milgram theorem, we get that (5.9) has a unique solution. Now, let $g(\varphi_2)$ be the unique solution of the following equation

$$[\partial_2 G(\varphi_2, u(\varphi_2))]^* g(\varphi) = \partial_2 \mathcal{L}(\varphi_2, u(\varphi_2)).$$

Since

$$\begin{aligned}\partial_2 \mathcal{L}(\varphi_2, u(\varphi_2))w &= ([\partial_2 G(\varphi_2, u(\varphi_2))]^* g(\varphi_2), w)_V \\ &= (g(\varphi_2), \partial_2 G(\varphi_2, u(\varphi_2))w_V,\end{aligned}$$

$g(\varphi_2) \in V$ is in fact the unique solution of the following variational equation:

$$\alpha(u(\varphi) - u_d, w)_V = ((g(\varphi_2), Aw + D_2^2 j_\delta u(\varphi_2)), u(\varphi_2)w)_V. \tag{5.10}$$

Now, from Lemma 5.1, we deduce that for all $q \in (L^2(\Gamma_2))^d$,

$$\begin{aligned}\frac{\mathrm{d}J}{\mathrm{d}\varphi}(\varphi_2)q &= \partial_1 \mathcal{L}(\varphi_2, u(\varphi_2))q - (g(\varphi_2), \partial_1 G(\varphi_2, u(\varphi_2)q))_V \\ &= \beta(\varphi_2, q)_{(L^2(\Gamma_2))^d} + (g(\varphi_2), y(q))_V,\end{aligned}$$

where $y(q)$ is defined by (5.4). Since $\overline{\varphi}_2$ is a minimizer of J, we obtain that

$$\frac{\mathrm{d}J}{\mathrm{d}\varphi}(\overline{\varphi}_2)q = 0 \qquad \text{for all } q \in (L^2(\Gamma_2))^d,$$

and thus, we get the following optimality condition

$$\beta(\overline{\varphi}_2, q)_{L^2(\Gamma_2)^d} + (g(\overline{\varphi}_2), y(q))_V = 0 \qquad \text{for all } q \in (L^2(\Gamma_2))^d, \tag{5.11}$$

where $g(\overline{\varphi}_2)$ is the unique solution of (5.10) with $\varphi = \overline{\varphi}_2$. Using (5.4) and (5.11), we get (5.7) and the proof of the theorem ends. □

6. Convergence results

In the first part of this section, we prove that the unique solution of the regularized state Problem (PS2) converges to the unique solution of the state Problem (PS1). More precisely, the following theorem takes place.

Theorem 6.1. *Let $\delta > 0$, $\varphi_0 \in H$ and $\varphi_2 \in (L^2(\Gamma_2))^d$ be given. If $u^\delta, u \in V$ are respectively the solutions of Problems* (PS2) *and* (PS1), *then*

$$u^\delta \to u \qquad \text{strongly in } V \text{ as } \delta \to 0. \tag{6.1}$$

PROOF: We take $v = u^\delta$ in (3.1) for $u = u^\delta$, $v = u$ in (4.3). Then, by adding the two inequalities obtained, one gets

$$\begin{aligned} m\|u^\delta - u\|_V^2 &\le (Au^\delta - Au, u^\delta - u)_V \\ &\le j(u, u^\delta) - j(u, u) + j(u^\delta, u) - j(u^\delta, u^\delta) \\ &\quad + j_\delta(u^\delta, u) - j(u^\delta, u) + j(u^\delta, u^\delta) - j_\delta(u^\delta, u^\delta). \end{aligned}$$

By (2.10) and (2.11) we have

$$j(u, u^\delta) - j(u, u) + j(u^\delta, u) - j(u^\delta, u^\delta) \le d_\Omega^2 L_p \|u^\delta - u\|_V^2,$$

then we get

$$(m - L_p d_\Omega^2)\|u^\delta - u\|_V^2 \le j_\delta(u^\delta, u) - j(u^\delta, u) + j(u^\delta, u^\delta) - j_\delta(u^\delta, u^\delta). \tag{6.2}$$

So, with (2.10), (4.1) and definitions of j and j_δ, after some computations, it follows that

$$j_\delta(u^\delta, u) - j(u^\delta, u) + j(u^\delta, u^\delta) - j_\delta(u^\delta, u^\delta) \to 0 \qquad \text{as } \delta \to 0.$$

Finally, by using (2.12) and (6.2) we get (6.1). □

Now we need to prove the following convergence results.

Theorem 6.2. *Let $[\bar{u}^\delta, \overline{\varphi}^\delta]$ be a solution of Problem* (POC2). *Then, there exists a solution of Problem* (POC1), *$[\bar{u}, \overline{\varphi}]$ such that, for $\delta \to 0$,*

$$\begin{aligned} \bar{u}^\delta &\to \bar{u} \qquad \text{strongly in } V, \\ \overline{\varphi}^\delta &\to \overline{\varphi} \qquad \text{weakly in } (L^2(\Gamma_2))^d. \end{aligned}$$

PROOF: Let $u_0^\delta \in V$ be the unique solution of Problem (PS2) with $\varphi_2 = 0_{(L^2(\Gamma_2))^d}$. We have

$$\mathcal{L}(0_{(L^2(\Gamma_2))^d}, u_0^\delta) = \frac{1}{2}\|u_0^\delta - u_d\|_V^2 \le (\|u_0^\delta\|_V^2 + \|u_d\|_V^2)$$

and since

$$\|u_0^\delta\|_V \le \frac{1}{m}\|\varphi_0\|_H,$$

we deduce that there exists $c > 0$ such that

$$\mathcal{L}(\overline{\varphi}^\delta, \bar{u}^\delta) \leq \mathcal{L}(0_{(L^2(\Gamma_2))^d}, u_0^\delta) \leq c(\|\varphi_0\|_H^2 + \|u_d\|_V^2).$$

Therefore, $(\bar{u}^\delta, \overline{\varphi}^\delta)_\delta$ is a bounded sequence in $V \times (L^2(\Gamma_2))^d$. Consequently, there exists $[\bar{u}, \overline{\varphi}] \in V \times (L^2(\Gamma_2))^d$ such that as $\delta \to 0$,

$$\bar{u}^\delta \to \bar{u} \quad \text{weakly in } V, \qquad \overline{\varphi}^\delta \to \overline{\varphi} \quad \text{weakly in } (L^2(\Gamma_2))^d.$$

Moreover, we have

$$\begin{aligned} m\|\bar{u}^\delta - \bar{u}\|_V^2 &\leq (A\bar{u} - A\bar{u}^\delta, \bar{u} - \bar{u}^\delta)_V \\ &\leq (A\bar{u}, \bar{u} - \bar{u}^\delta)_V + j_\delta(\bar{u}^\delta, \bar{u}) - j_\delta(\bar{u}^\delta, \bar{u}^\delta) \\ &\quad + (\varphi_0, \bar{u} - \bar{u}^\delta)_H + (\overline{\varphi}^\delta, \overline{u} - \bar{u}^\delta)_{(L^2(\Gamma_2))^d}. \end{aligned} \tag{6.3}$$

On the other hand as $\bar{u}^\delta \to \bar{u}$ weakly in V implies that $\bar{u}^\delta \to \bar{u}$ strongly in $(L^2(\Gamma_2))^d$, then $j_\delta(\bar{u}^\delta, \bar{u}) - j_\delta(\bar{u}^\delta, \bar{u}^\delta) \to 0$ as $\delta \to 0$. Hence we deduce that

$$\lim_{\delta\to 0}(A\bar{u}, \bar{u} - \bar{u}^\delta)_V + j_\delta(\bar{u}^\delta, \bar{u}) - j_\delta(\bar{u}^\delta, \bar{u}^\delta) + (\varphi_0, \bar{u} - \bar{u}^\delta)_H + (\overline{\varphi}^\delta, \bar{u} - \bar{u}^\delta)_{(L^2(\Gamma_2))^d} = 0.$$

Using this relation and passing to the limit as $\delta \to 0$ on both sides of the inequality (6.3), one obtains $\bar{u}^\delta \rightharpoonup \bar{u}$ strongly in V as $\delta \to 0$. Now, we must prove that $[\bar{u}, \overline{\varphi}] \in U_{ad}$. Indeed, using (6.3), it follows, after some computations that when $\delta \to 0$, the following limits hold:

$$\begin{gathered} (A\bar{u}^\delta, v - \bar{u}^\delta)_V \to (A\bar{u}, v - \bar{u})_V, \\ j_\delta(v, \bar{u}^\delta) - j_\delta(\bar{u}^\delta, \bar{u}^\delta) \to j(v, \bar{u}) - j(\bar{u}, \bar{u}), \\ (\varphi_0, v - \bar{u}^\delta)_H + (\overline{\varphi}^\delta, v - \bar{u}^\delta)_{(L^2(\Gamma_2))^d} \to (\varphi_0, v - \bar{u})_H + (\overline{\varphi}, v - \bar{u})_{(L^2(\Gamma_2))^d}. \end{gathered}$$

Therefore, passing to the limit as $\delta \to 0$ in (4.3), we deduce that $(\bar{u}, \overline{\varphi})$ satisfies (3.1) and $[\bar{u}, \overline{\varphi}] \in U_{ad}$.

Let (u^*, φ^*) be a solution of Problem (POC1) and let us consider the sequence $(u^\delta)_\delta$ such that for each $\delta > 0$, u^δ is the unique solution of Problem (PS2) with the data $\varphi_0 \in H$ and $\varphi^* \in (L^2(\Gamma_2))^d$. Obviously, for every $\delta > 0, (u^\delta, \varphi^*) \in U_{ad}^\delta$. Using Theorem 6.1, since the functional $\mathcal{L}$ is convex and continuous, we have

$$\mathcal{L}(\varphi^*, u^*) \leq \liminf_{\delta\to 0} \mathcal{L}(\overline{\varphi}^\delta, \bar{u}^\delta), \tag{6.4}$$

$$(u^\delta, \varphi^*) \to (u^*, \varphi^*) \text{ strongly in } V \times (L^2(\Gamma_2))^d \text{ as } \delta \to 0. \tag{6.5}$$

We also have, as $(\bar{u}^\delta, \overline{\varphi}^\delta)$ is a solution of Problem (POC2):

$$\limsup_{\delta\to 0} \mathcal{L}(\overline{\varphi}^\delta, \bar{u}^\delta) \leq \limsup_{\delta\to 0} \mathcal{L}(\overline{\varphi}^\delta, u^\delta). \tag{6.6}$$

Now, using (6.3), we have

$$\limsup_{\delta\to 0} \mathcal{L}(\overline{\varphi}^{\delta}, u^{\delta}) = \mathcal{L}(\varphi^*, u^*), \tag{6.7}$$

and as (φ^*, u^*) is a solution of Problem (POC1),

$$\mathcal{L}(\varphi^*, u^*) \le \mathcal{L}(\overline{\varphi}, \bar{u}). \tag{6.8}$$

Thus, from (6.4)–(6.8), we deduce that

$$\mathcal{L}(\overline{\varphi}, \bar{u}) = \mathcal{L}(\varphi^*, u^*),$$

and then we obtain

$$\lim_{\delta\to 0} \mathcal{L}(\overline{\varphi}^{\delta}, u^{\delta}) = \mathcal{L}(\varphi^*, u^*).$$

□

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