More remarks on the intersection ideal $\mathcal{M} \cap \mathcal{N}$

Tomasz Weiss

Abstract. We prove in ZFC that every $\mathcal{M}\cap\mathcal{N}$ additive set is $\mathcal{N}$ additive, thus we solve Problem 20 from paper [Weiss T., *A note on the intersection ideal* $\mathcal{M}\cap\mathcal{N}$, Comment. Math. Univ. Carolin. **54** (2013), no. 3, 437–445] in the negative.

Keywords: intersection ideal $\mathcal{M} \cap \mathcal{N}$; null additive set; meager additive set

Classification: 03E05, 03E17

Introduction. In this paper, we continue our considerations (see [6]) of sets belonging to the intersection ideal $\mathcal{M} \cap \mathcal{N}$ in terms of their translations.

Suppose that "+" is the standard modulo 2 coordinatewise addition in 2^ω, and I, J are σ-ideals of subsets of 2^ω with $I \subseteq J$.

Definition 1. We say that $X \subseteq 2^\omega$ is I additive, or $X \in I^*$, if and only if $X + A = \{x + a \colon x \in X,\ a \in A\} \in I$ for any set $A \in I$, and $X \in (I,J)^*$ if and only if for every set $A \in I$, $X + A \in J$.

The σ-ideal of meager subsets of 2^ω is denoted by $\mathcal{M}$, $\mathcal{N}$ is the σ-ideal of measure zero subsets of 2^ω, and $\mathcal{E}$ denotes the σ-ideal generated by F_σ measure zero subsets of 2^ω. It is well-known that $\mathcal{E}$ is strictly contained in the intersection ideal $\mathcal{M} \cap \mathcal{N}$. The following diagram of inclusions holds, where "$\to$" stands for the inclusion and crossed arrow "$\not\leftarrow$" means that the reverse inclusion cannot be proved in ZFC (Zermelo–Fraenkel set theory). See Proposition 19 in [6].

$$\begin{tikzcd}
 & & & & (\mathcal{E},\mathcal{M})^* \arrow[r, shift left, "/" marking] \arrow[dl, shift left, "/" marking] \arrow[dd, shift left, "/" marking] & SFC \arrow[l, shift left] \\
\mathcal{N}^* \arrow[r, shift left] & (\mathcal{M}\cap\mathcal{N})^* \arrow[l, shift left, "?"] \arrow[r, shift left] & \mathcal{E}^* = \mathcal{M}^* \arrow[l, shift left, "/" marking] \arrow[r, shift left] & (\mathcal{E},\mathcal{M}\cap\mathcal{N})^* \arrow[l, shift left, "/" marking] \arrow[ur, shift left] \arrow[dr, shift left] & & \\
 & & & & (\mathcal{E},\mathcal{N})^* = SMZ \arrow[ul, shift left, "/" marking] \arrow[uu, shift left, "/" marking] &
\end{tikzcd}$$

Recall that $SMZ = \{X \subseteq 2^\omega \colon \text{ for every } A \in \mathcal{M},\ X + A \neq 2^\omega\}$, and $SFC = \{X \subseteq 2^\omega \colon \text{ for every } B \in \mathcal{N},\ X + B \neq 2^\omega\}$.

Question 2 (Problem 20 in [6])**.** Is it consistent with ZFC that the class $(\mathcal{M}\cap\mathcal{N})^*$ contains sets that are not in $\mathcal{N}^*$?

DOI 10.14712/1213-7243.2015.252

Main theorems. We begin with the answer to Question 2 which is surprisingly negative.

Theorem 3. $(\mathcal{M}\cap\mathcal{N})^* \subsetneq \mathcal{N}^*$.

To prove this theorem we apply the following sequence of lemmas. The first one is Lemma 0 in [5].

Lemma 4. *Let $m \geq n + 2^n k$, $k, m, n \in \omega$. Then there exists $T \subseteq 2^m$ with measure $\mu(T) = 2^{-k}$ such that for all $\langle \sigma_i, \tau_i\rangle \in 2^n \times 2^{[n,m)}$, $i \in I$, with σ_i distinct the sets $T + \langle \sigma_i, \tau_i\rangle$ are stochastically independent.*

Lemma 5 (Theorem 23 in [6]). $(\mathcal{M}\cap\mathcal{N}, \mathcal{M})^* \subseteq \mathcal{E}^* = \mathcal{M}^*$.

Proof of Theorem 3: We combine the procedures of (♠) in [5], Theorem 2.7.18 in [2] and Lemma 5 above.

Suppose that $X \in (\mathcal{M}\cap\mathcal{N})^*$, and an increasing $f \in \omega^\omega$ is such that $f(n+1) \geq f(n) + n$ for every $n \in \omega$. By Lemma 5 the set X is meager additive and by the Bartoszyński–Judah–Shelah characterization (see Theorem 2.7.17 from [2]), there are an increasing $g \in \omega^\omega$ and $y \in 2^\omega$, so that

$$X \subseteq \{x \in 2^\omega \colon \exists m\ \forall n \geq m\ \exists k\ \ (g(n) \leq f(k) < f(k+1) \leq g(n+1) \quad \text{and} \\ x \restriction [f(k), f(k+1)) = y \restriction [f(k), f(k+1)))\}.$$

Assume without loss of generality that g is sufficiently fast increasing and put $a_n = g(2n)$, $b_n = g(2n+1)$ for $n \in \omega$. From now on, each number $b_i - a_i$ and $a_{i+1} - b_i$ will play the role of n and $m - n$, respectively, from Lemma 4. Each set T_i with $\mu(T_i) = 1/2^i$ and used in the expression below plays the role of a set T which appears in Lemma 4. Let $A = \bigcap_{m\in\omega}\bigcup_{n\geq m} A_n$, where for $n \in \omega$,

$$A_n = \{x \in 2^\omega \colon x \restriction [a_n, a_{n+1}) \in T_n\}.$$

Since $\mu(A_n) = 1/2^n$ for $n \in \omega$, we have that $\mu(A) = 0$. Suppose that $h \in 2^\omega$ is such that

$$A' = A \cap \{x \in 2^\omega \colon \exists m\ \forall n \geq m\ \ x \restriction [a_n, a_{n+1}) \neq h \restriction [a_n, a_{n+1})\}$$

is nonempty. Notice that the second set in the above formula is meager (see Theorem 2.2.4 in [2]), thus $A' \in \mathcal{M}\cap\mathcal{N}$, and by the assumption $X + A' \in \mathcal{N}$.

Let $G \subseteq 2^\omega$, $\mu(G) < 1$, be an open set such that $X + A' \subseteq G$, and suppose that for every $\tau \in 2^{<\omega}$, $[\tau]$ is the basic clopen set $\{x \in 2^\omega \colon \tau \subseteq x\}$. Since we can delete from $2^\omega \setminus G$ every set $[\tau]$ which satisfies $\mu([\tau]\setminus G) = 0$, we may assume that for each basic clopen set $[\tau]$, $[\tau] \not\subseteq G$, we have that $\mu([\tau]\setminus G) > 0$. By De Morgan law

$$\bigcap_{x\in X} ((x + (2^\omega \setminus A)) \cup (x + B)) \supseteq [\tau] \setminus G,$$

where

$$B = \{x \in 2^\omega \colon \forall m\ \exists n \geq m\ \ x \restriction [a_n, a_{n+1}) = h \restriction [a_n, a_{n+1})\}.$$

It is easy to see that

$$\bigcap_{x\in X} ((x + (2^\omega \setminus A)) \cup (x + B)) \subseteq \bigcap_{x\in X} (x + (2^\omega \setminus A)) \cup \bigcup_{x\in X} (x + B).$$

We show that the union at the end of the above expression is a null set.

Fact 6. *$X + B$ is of measure zero.*

Proof: Notice that $X + B \subseteq \bigcap_{m\in\omega} \bigcup_{n\geq m} C_n$, where for $n \in \omega$,

$$\begin{aligned} C_n = \{x \in 2^\omega : \exists k\ (g(n) \leq f(k) < f(k+1) \leq g(n+1) \quad &\text{and} \\ x \restriction [f(k), f(k+1)) = y \restriction [f(k), f(k+1)))\} \\ + \{x \in 2^\omega : x \restriction [g(n), g(n+1)) = h \restriction [g(n), g(n+1))\} \\ \subseteq \bigcup_{k:\ g(n)\leq f(k)<f(k+1)\leq g(n+1)} \{x \in 2^\omega : x \restriction [f(k), f(k+1)) = y \restriction [f(k), f(k+1))\} \\ + \{x \in 2^\omega : x \restriction [f(k), f(k+1)) = h \restriction [f(k), f(k+1))\}. \end{aligned}$$

Clearly, $\sum_{n\in\omega} \mu(C_n) < \infty$. This finishes the proof of Fact 6. $\square$

By Fact 6 for each basic clopen $[\tau]$, $[\tau] \not\subseteq G$, there is $a_\tau \subseteq [\tau] \setminus G$ such that $\mu(a_\tau) > 0$, and

$$a_\tau \subseteq \bigcap_{x\in X} (x + (2^\omega \setminus A)).$$

This implies that for every such a_τ we have that

$$\left(\bigcup_{x\in X} (x + A)\right) \cap a_\tau = \emptyset.$$

We now follow the main argument and the notation from (♠) in [5]. By earlier remarks we have that for every $x \in X$ and every basic clopen set $[\tau]$, $[\tau] \not\subseteq G$,

$$\left(\bigcap_{m\in\omega} \bigcup_{n\geq m} (x + A_n)\right) \cap a_\tau = \emptyset.$$

By applying the Baire category theorem in $2^\omega \setminus G$ for each $x \in X$ one can find $m_x \in \omega$ and a basic clopen τ_x, $[\tau_x] \not\subseteq G$ such that

$$\left(\bigcup_{n\geq m_x} (x + A_n)\right) \cap a_{\tau_x} = \emptyset, \qquad \text{or equivalently} \qquad a_{\tau_x} \subseteq \bigcap_{n\geq m_x} (x + (2^\omega \setminus A_n)).$$

Define for $n \in \omega$ and $[\tau] \not\subseteq G$

$$K_n^\tau = \{x \restriction [a_n, b_n) : x \in X,\ \text{and } (x + A_n) \cap a_\tau = \emptyset\}.$$

It is clear that for every $x \in X$, $x \restriction [a_n, b_n) \in K_n^{\tau_x}$, where $n \geq m_x$.

Let $\{x^\tau_{k,n}\colon k < |K^\tau_n|\}$ be a list of all x's such that $x \restriction [a_n, b_n)$ are distinct and give the entire set K^τ_n. We have

$$a_\tau \subseteq \bigcap_{n\in\omega}\left(2^\omega \setminus \bigcup_{k<|K^\tau_n|}(x^\tau_{k,n} + A_n)\right)$$

thus by the stochastic independence condition from Lemma 4 above this implies that

$$\prod_{n\in\omega}\Big(1-\frac{1}{2^n}\Big)^{|K^\tau_n|} > 0.$$

Hence

$$\sum_{n\in\omega}\frac{|K^\tau_n|}{2^n} < \infty.$$

For each τ, $[\tau] \nsubseteq G$, let $n(\tau)$ be such that $|K^\tau_n| \le 2^n$ for $n \ge n(\tau)$. Let $\{\tau_n\}$ be a list of all τ's which satisfy $[\tau] \nsubseteq G$. Define for every $n \in \omega$

$$D_n = \bigcup_{m<n}\{K^{\tau_m}_n : \tau_m \text{ is such that } n(\tau_m) \le n\}.$$

Clearly, $|D_n| \le n2^n$ for $n \in \omega$. This shows that there exists a sequence $\{D_n\}_{n\in\omega}$ with $D_n \subseteq 2^{[a_n,b_n)}$ and $|D_n| \le n2^n$ for $n \in \omega$ such that for each $x \in X$ and almost every $n \in \omega$

$$x \restriction [a_n, b_n) \in D_n.$$

Notice that by using simultaneously the same procedure for intervals of the form $[b_n, b_{n+1})$ we show that there is a sequence $\{D'_n\}_{n\in\omega}$ with $D'_n \subseteq 2^{[b_n,a_{n+1})}$ and $|D'_n| \le (n+1)2^{n+1}$ for $n \in \omega$ so that for each $x \in X$ and almost every $n \in \omega$

$$x \restriction [b_n, a_{n+1}) \in D'_n.$$

To obtain this sequence we can choose the function $g \in \omega^\omega$ at the beginning of the proof of Theorem 3 sufficiently fast increasing, so that each interval $[b_n, a_{n+1})$ is "large enough" in comparison to $[a_n, b_n)$ (each number $a_{n+1} - b_n$ and $b_{n+1} - a_{n+1}$ will play the role of n and $m - n$, respectively, from Lemma 4) and then we can define the sets $\widetilde{T}_n$, $\widetilde{T}_n \subseteq 2^{[b_n,b_{n+1})}$ for $n \in \omega$, and $\widetilde{A}$, $\widetilde{A}'$ analogously to the sets from the first part of the proof of Theorem 3. By Theorem 2.7.18.4 in [2] this proves that $X \in \mathcal{N}^*$. □

According to the referees' suggestions we consider two classes $(\mathcal{M}\cap\mathcal{N},\mathcal{N})^*$ and $(\mathcal{M}\cap\mathcal{N},\mathcal{M})^*$ which have not been explored before.

Proposition 7. $(\mathcal{M}\cap\mathcal{N},\mathcal{M})^* \not\to (\mathcal{M}\cap\mathcal{N},\mathcal{N})^*$.

Proof: See Theorem 22 in [6]. □

Question 8. $(\mathcal{M}\cap\mathcal{N},\mathcal{N})^* \to (\mathcal{M}\cap\mathcal{N},\mathcal{M})^*$?

In [6], the author asks the following question (see Problem 21 in [6]).

Question 9. Is there a model of ZFC in which every element of the class $(\mathcal{E}, \mathcal{M})^*$ is at most countable?

Question 10 (B. Tsaban, personal communication)**.** Does ZFC imply that there is an uncountable $X \subseteq 2^\omega$ such that $X + F \neq 2^\omega$ for every $F \in \mathcal{E}$?

Below we show that the positive answer to B. Tsaban's question proves that there is in ZFC a particularly small uncountable set, that is an uncountable $X \in (\mathcal{E}, \mathcal{M})^*$. This solves Question 9 in the negative. By Theorem 2 in [1] the following holds: if $\mathfrak{b} = \aleph_1$, then there is $X \subseteq 2^\omega$, $|X| = \aleph_1$, and X is meager additive. In Theorem 3.6 from [4], the authors prove that under $\mathfrak{b} = \aleph_1$, there is an uncountable $X \subseteq 2^\omega$, $|X| = \aleph_1$, with a stronger property than meager additivity. For the other case (i.e. $\mathfrak{b} > \aleph_1$) we use the following proposition.

Proposition 11. *If $X \subseteq 2^\omega$, $|X| < \mathfrak{b}$, is such that $X + F \neq 2^\omega$ for every $F \in \mathcal{E}$, then $X + F$ is meager for every $F \in \mathcal{E}$.*

Proof: Suppose that $X + F \neq 2^\omega$ for a fixed $F \in \mathcal{E}$. We may assume without loss of generality that $F + \mathbf{Q} = F$, where $\mathbf{Q} = \{x \in 2^\omega \colon \exists m \, \forall n \geq m \;\; x(n) = 0\}$. Thus there is $z_0 \in 2^\omega$ such that

$$(z_0 + \mathbf{Q}) \cap (X + F) = \emptyset.$$

Hence

$$(z_0 + \mathbf{Q}) \cap \left(\bigcup_{x \in X} (x + F) \right) = \emptyset.$$

Since $z_0 + \mathbf{Q}$ is dense, and $|X| < \mathfrak{b}$, we can follow directly the implication (5) $\Rightarrow$ (1) from Lemma 2.2.6 in [2] and the arguments from Lemma 2.2.7 and after Lemma 2.2.8 both in [2] to show that $2^\omega \setminus \left(\bigcup_{x \in X} (x + F) \right)$ contains a dense G_δ set. □

Notice that the only property of a set $F \in \mathcal{E}$ that we use in the proof of the above proposition is the assumption that it is an F_σ meager set. Thus we essentially proved the following.

Corollary 12. *If $X \in SMZ$ and $|X| < \mathfrak{b}$, then $X \in \mathcal{M}^*$.*

Proof: Clear. □

An example of a meager set $X \in SMZ$, $|X| = \mathfrak{b}$, which is not meager additive is given in Theorem 10 from [6].

It was also pointed out by the referees that by earlier remarks and Proposition 11 a positive answer to Question 9 provides a negative answer to Question 10 which in turn implies the result Con(ZFC + Borel conjecture + dual Borel conjecture) of the paper [3].

Acknowledgment. The author thanks the referees for many helpful suggestions.

References

[1] Bartoszyński T., *Remarks on small sets of reals*, Proc. Amer. Math. Soc **131** (2003), no. 2, 625–630.

[2] Bartoszyński T., Judah H., *Set Theory. On the Structure of the Real Line*, A K Peters, Wellesley, 1995.

[3] Goldstern M., Kellner J., Shelah S., Wohofsky W., *Borel conjecture and dual Borel conjecture*, Trans. Amer. Math. Soc. **366** (2014), no. 1, 245–307.

[4] Orenshtein T., Tsaban B., *Linear σ-additivity and some applications*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3621–3637.

[5] Pawlikowski J., *A characterization of strong measure zero sets*, Israel J. Math. **93** (1996), 171–183.

[6] Weiss T., *A note on the intersection ideal $\mathcal{M} \cap \mathcal{N}$*, Comment. Math. Univ. Carolin. **54** (2013), no. 3, 437–445.

T. Weiss:
Institute of Mathematics, College of Science, Cardinal Stefan Wyszyński University, Dewajtis 5, 01-815 Warsaw, Poland

E-mail: tomaszweiss@o2.pl

(*Received* July 6, 2016*, revised* June 5, 2018)