Asymmetric tie-points and almost clopen subsets of $\mathbb{N}^*$

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To the memory of Bohuslav Balcar

Abstract. A tie-point of compact space is analogous to a cut-point: the complement of the point falls apart into two relatively clopen non-compact subsets. We review some of the many consistency results that have depended on the construction of tie-points of $\mathbb{N}^*$. One especially important application, due to Veličković, was to the existence of nontrivial involutions on $\mathbb{N}^*$. A tie-point of $\mathbb{N}^*$ has been called symmetric if it is the unique fixed point of an involution. We define the notion of an almost clopen set to be the closure of one of the proper relatively clopen subsets of the complement of a tie-point. We explore asymmetries of almost clopen subsets of $\mathbb{N}^*$ in the sense of how may an almost clopen set differ from its natural complementary almost clopen set.

Keywords: ultrafilter; cardinal invariants of continuum

Classification: 54D80, 03E15

1. Introduction

In this introductory section we review some background to motivate our interest in further study of tie-points and almost clopen sets. The Stone–Čech compactification of the integers $\mathbb{N}$, is denoted as $\beta\mathbb{N}$ and, as a set, is equal to $\mathbb{N}$ together with all the free ultrafilters on $\mathbb{N}$. The remainder $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ can be topologized as a subspace of the Stone space of the power set of $\mathbb{N}$ as a Boolean algebra and, in particular, for a subset a of $\mathbb{N}$, the set a^* of all free ultrafilters with a as an element, is a basic clopen subset of $\mathbb{N}^*$.

A point x of a space X is a butterfly point (or b-point, see [23]) if there are sets $D, E \subset X \setminus \{x\}$ such that $\{x\} = \overline{D} \cap \overline{E}$. In [4], the authors introduced the tie-point terminology.

Definition 1.1. A point x is a tie-point of a space X if there are closed sets A, B of X such that $X = A \cup B$, $\{x\} = A \cap B$ and x is a limit point of each of A and B. We picture (and denote) this as $X = A \underset{x}{\bowtie} B$ where A, B are the closed sets which have a unique common accumulation point x and say that x is a tie-point as witnessed by A, B.

DOI 10.14712/1213-7243.2015.268

The research of the first author was supported by the NSF grant No. NSF-DMS 1501506. The research of the second author was supported by the United States-Israel Binational Science Foundation (BSF Grant No. 2010405), and by the NSF grant No. NSF-DMS 1101597.

In this note the focus is on the local properties of x with respect to each of the closed sets A and B such that $A \underset{x}{\bowtie} B$ in the case when A, B witness that x is a tie-point. For this reason we introduce the notion of an almost clopen subset of $\mathbb{N}^*$.

Definition 1.2. A set $A \subset \mathbb{N}^*$ is almost clopen if A is the closure of an open subset of $\mathbb{N}^*$ and has a unique boundary point, which we denote x_A.

Proposition 1.3. *If A is an almost clopen subset of $\mathbb{N}^*$, then $B = \{x_A\} \cup (\mathbb{N}^* \setminus A)$ is almost clopen and $x_B = x_A$. In addition x_A is a tie-point as witnessed by A, B.*

Definition 1.4 ([4])**.** A tie-point x is a symmetric tie-point of $\mathbb{N}^*$ if there is a pair A, B witnessing that x is a tie-point and if there is a homeomorphism $h\colon A \to B$ satisfying that $h(x) = x$.

If A is almost clopen, then we refer to $B = \{x_A\} \cup (\mathbb{N}^* \setminus A)$ as the almost clopen complement of A. A more set-theoretically inclined reader would surely prefer a straightforward translation of almost clopen to properties of ideals of subsets of $\mathbb{N}$ and the usual mod finite ordering $\subset^*$.

Definition 1.5. If A is any subset of $\mathbb{N}^*$, then $\mathcal{I}_A$ is defined as the set $\{a \subset \mathbb{N}\colon a^* \subset A\}$.

For any family $\mathcal{A}$ of subsets of $\mathbb{N}$ (or ω), we define $\mathcal{A}^\perp$ to be the orthogonal ideal $\{b \subset \mathbb{N}\colon \forall\, a \in A \;\; b \cap a =^* \emptyset\}$. We state this next result for easy reference.

Proposition 1.6. *If $\mathcal{I}$ is an ideal that has no $\subset^*$-maximal element, then the ideal generated by $\mathcal{I} \cup \mathcal{I}^\perp$ is a proper ideal.*

Lemma 1.7. *If A is an almost clopen subset of $\mathbb{N}^*$ with almost clopen complement B, then $\mathcal{I}_A \cap \mathcal{I}_B$ is the Fréchet ideal $[\mathbb{N}]^{<\aleph_0}$, $\mathcal{I}_B = \mathcal{I}_A^\perp$, and x_A is the unique ultrafilter that is disjoint from $\mathcal{I}_A \cup \mathcal{I}_B$.*

Almost clopen sets (and tie-points) first arose implicitly in the work [9] of N. J. Fine and L. Gillman in the investigation of extending continuous functions on dense subsets of $\mathbb{N}^*$. A subset Y of a space X is C^*-embedded if every bounded continuous real-valued function on Y can be continuously extended to all of X. The character of a point $x \in \mathbb{N}^*$ is the minimal cardinality of a filter base for x as an ultrafilter on $\mathbb{N}$.

Proposition 1.8 ([9])**.** *If x is a tie-point of $\mathbb{N}^*$, then $\mathbb{N}^* \setminus \{x\}$ is not C^*-embedded in $\mathbb{N}^*$. Every point of character $\aleph_1$ is a tie-point of $\mathbb{N}^*$.*

It was shown in [3] to be consistent with Zermelo–Fraenkel set theory (ZFC) that $\mathbb{N}^* \setminus \{x\}$ is C^*-embedded for all $x \in \mathbb{N}^*$. It was also shown by J. E. Baumgartner in [1] that this result holds in models of the Proper forcing axiom (PFA).

Proposition 1.9 ([3], [1])**.** *PFA implies that $\mathbb{N}^* \setminus \{x\}$ is C^*-embedded in $\mathbb{N}^*$ for all $x \in \mathbb{N}^*$*

Corollary 1.10. *PFA implies that there are no almost clopen sets and no tie-points in* $\mathbb{N}^*$.

Almost clopen sets arise in the study of minimal extensions of Boolean algebras (see [15]) and in the application of this method of construction for building a variety of counterexamples (e.g. [16], [12], [22], [6]). The next application of almost clopen subsets of $\mathbb{N}^*$ was to the study of nontrivial automorphisms of $\mathscr{P}(\mathbb{N})/\operatorname{fin}$, or nontrivial autohomeomorphisms of $\mathbb{N}^*$. M. Katětov in [14] proved that the set of fixed points of an autohomeomorphism of $\beta\mathbb{N}$ will be a clopen set. It is immediate from N. J. Fine and L. Gillman's work [9] that every P-point of character of $\aleph_1$ is a fixed point of a nontrivial autohomeomorphism of $\mathbb{N}^*$.

Definition 1.11. A point x of $\mathbb{N}^*$ is a P-point if the ultrafilter x is countably complete mod finite. For a cardinal κ, an ultrafilter x on $\mathbb{N}$ is a simple P_κ-point if x has a base well-ordered by mod finite inclusion of order type κ.

Proposition 1.12 ([9]). *If A is an almost clopen subset of $\mathbb{N}^*$ and x_A is a simple $P_{\aleph_1}$-point of $\mathbb{N}^*$, then*

(1) *A is homeomorphic to $\mathbb{N}^*$;*
(2) *x_A is a symmetric tie-point;*
(3) *there is an autohomeomorphism f on N^* such that $\{x\}$ is the only fixed point of f.*

As we have seen above, PFA implies that there are no almost clopen subsets of $\mathbb{N}^*$, and of course, PFA also implies that all autohomeomorphisms of $\mathbb{N}^*$ are trivial, see [24]. However Veličković utilized a similar simple P-point trick (motivating our definition of symmetric tie-point) in order to prove that this is not a consequence of Martin's axiom (MA).

Proposition 1.13 ([27]). *It is consistent with MA and $\mathfrak{c} = \aleph_2$ that there is an almost clopen set A of $\mathbb{N}^*$ such that x_A is a simple $P_{\aleph_2}$-point and,*

(1) *x_A is a symmetric tie-point,*
(2) *there is an autohomeomorphism f on $\mathbb{N}^*$ such that $\{x\}$ is the only fixed point of f.*

Veličković's result and approach was further generalized in [25] and [26]. It is very interesting to know if an almost clopen subset of $\mathbb{N}^*$ is itself homeomorphic to $\mathbb{N}^*$, see [8], [13]. This question also arose in the authors' work on two-to-one images of $\mathbb{N}^*$, see [5]. Veličković's method was slightly modified in [5] to produce a complementary pair of almost clopen sets so that neither is homeomorphic to $\mathbb{N}^*$, but it is not known if there is a symmetric tie-point $A \underset{x}{\bowtie} B$ where A is not a copy of $\mathbb{N}^*$.

Our final mention of recent interest in almost clopen subsets of $\mathbb{N}^*$ is in connection to the question, see [19], [7], of whether the Banach space l_∞/c_0 is necessarily primary. It was noted by P. Koszmider in [20, page 577] that a special almost clopen subset of $\mathbb{N}^*$ could possibly resolve the problem. For a compact space K, we let $C(K)$ denote the Banach space of continuous real-valued functions on K

with the supremum norm. It is well-known that $C(\mathbb{N}^*)$ is isomorphic (as a Banach space) to l_∞/c_0. Naturally if a space A is homeomorphic to $\mathbb{N}^*$, then $C(A)$ is isomorphic to $C(\mathbb{N}^*)$.

Proposition 1.14 ([20, page 577]). *Suppose that A is an almost clopen subset of $\mathbb{N}^*$ and that B is its almost clopen complement. If $C(\mathbb{N}^*)$ is not homeomorphic to either of $C(A)$ or $C(B)$, then l_∞/c_0 is not primary.*

2. Asymmetric tie-points

In many of the applications mentioned in the introductory section, the tie-points utilized were symmetric tie-points. In other applications for example the primariness of l_∞/c_0, it may be useful to find examples where the witnessing sets A, B for a tie-point are quite different. There are any number of local topological properties that x may enjoy as a point in A that it may not share as a point in B. We make the following definition in connection with simple P_κ-points.

Definition 2.1. Let κ be a regular cardinal. An almost clopen set A is simple of type κ if $\mathcal{I}_A$ has a $\subset^*$-cofinal $\subset^*$-increasing chain $\{a_\alpha\colon \alpha\in\kappa\}$ of type κ.

If $\{a_\alpha\colon \alpha\in\kappa\}$ is strictly $\subset^*$-increasing and $\subset^*$-cofinal in $\mathcal{I}_A$ for an almost clopen set A, then the family $\mathcal{A}_A = \{a_{\alpha+1}\setminus a_\alpha\colon \alpha\in\kappa\}$ cannot be *reaped.* A family $\mathcal{A}\subset[\mathbb{N}]^{\aleph_0}$ is reaped by a set $c\subset\mathbb{N}$ if $|a\setminus c| = |a\cap c|$ for all $a\in\mathcal{A}$. We verify that $\mathcal{A}_A$ cannot be reaped. Let $c\subset\mathbb{N}$ and, by symmetry, assume that c is not an element of the ultrafilter x_A. Therefore $c^*\cap A$ is a compact subset of $A\setminus\{x\}$ and so is contained in a^*_α for some $\alpha\in\kappa$. Thus $(c\setminus a_\alpha)^*$ is disjoint from A and so $c\cap(a_{\alpha+1}\setminus a_\alpha)$ is finite. This proves that $\mathcal{A}_A$ is not reaped by c.

The reaping number $\mathfrak{r}$ is the minimum cardinal of a family that cannot be reaped, see [11] and [18]. For any infinite set $a\subset\mathbb{N}$, let $\mathrm{next}(a,\cdot)$ be the function in $\mathbb{N}^\mathbb{N}$ defined by $\mathrm{next}(a,k) = \min(a\setminus\{1,\dots,k\})$. As usual, for $f,g\in\mathbb{N}^\mathbb{N}$ we say that $f<^* g$ if $\{k\colon g(k)\le f(k)\}$ is finite.

Proposition 2.2 ([11]). *If $\mathcal{A}\subset[\mathbb{N}]^{\aleph_0}$ and if there is some $g\in\mathbb{N}^\mathbb{N}$ such that $\mathrm{next}(a,\cdot)<^* g$ for all $a\in\mathcal{A}$, then $\mathcal{A}$ can be reaped. In particular, $\mathfrak{b}\le\mathfrak{r}$.*

If $\{a_\alpha\colon \alpha\in\kappa\}$ is strictly $\subset^*$-increasing and $\subset^*$-cofinal in $\mathcal{I}_A$ for an almost clopen set A, then the family $\{a_{\alpha+1}\setminus a_\alpha\colon \alpha\in\kappa\}$ is an example of a *converging* family of infinite sets.

Definition 2.3. Let $\mathcal{A}$ be a family of infinite subsets of $\mathbb{N}$. We say that $\mathcal{A}$ converges if there is an ultrafilter x on $\mathbb{N}$ such that for each $U\in x$, the set $\{a\in\mathcal{A}\colon a\setminus U\neq^*\emptyset\}$ has cardinality less than that of $\mathcal{A}$.

We say that $\mathcal{A}$ is hereditarily unreapable if each reapable subfamily of $\mathcal{A}$ has cardinality less than that of $\mathcal{A}$.

Lemma 2.4. *If for a cardinal κ, $\{a_\alpha\colon \alpha\in\kappa\}$ is a strictly $\subset^*$-increasing family of subsets of $\mathbb{N}$, and if $\{a_{\alpha+1}\setminus a_\alpha\colon \alpha\in\kappa\}$ converges, then the closure of $\bigcup\{a^*_\alpha\colon \alpha\in\kappa\}$ is a simple almost clopen set of type κ.*

PROOF: Suppose that $\{a_{\alpha+1} \setminus a_\alpha \colon \alpha \in \kappa\}$ converges to $x \in \mathbb{N}^*$ and let $A = \{x\} \cup \bigcup\{a^*_\alpha \colon \alpha \in \kappa\}$. It suffices to prove that A is closed. Choose any $y \in \mathbb{N}^* \setminus A$ and let a be an element of $x \setminus y$. Choose $\beta \in \kappa$ so that $(a_{\alpha+1} \setminus a_\alpha) \subset^* a$ for all $\beta \leq \alpha \in \kappa$. By induction on $\alpha < \kappa$, it follows that $a_\alpha \setminus a_\beta \subset^* a$. Therefore A is contained in the clopen set $(a_\beta \cup a)^* \subset \mathbb{N}^* \setminus \{y\}$. □

An ultrafilter x of $\mathbb{N}^*$ is said to be an almost P_κ-point (or pseudo-P_κ-point) if each set of fewer than κ many members of x have a pseudo-intersection (an infinite set mod finite contained in each of them). Certainly a converging family is hereditarily unreapable and converges to a point that is an almost P_κ-point where κ is the cardinality of the family. Clearly the cardinality of any hereditarily unreapable family will have cofinality not larger than the splitting number $\mathfrak{s}$. Therefore if, for example, $\mathfrak{s} = \aleph_1$ and $\mathfrak{r} = \mathfrak{c} = \aleph_2$, there will be no hereditarily unreapable family. If $\mathfrak{s} = \mathfrak{c}$, then there is a hereditarily unreapable family of cardinality $\mathfrak{s}$. In the Mathias model of $\mathfrak{s} = \mathfrak{c} = \mathfrak{b} = \aleph_2$, there is no converging unreapable family because there is no almost $P_{\aleph_2}$-point. In the Goldstern–Shelah model of $\mathfrak{r} = \mathfrak{s} = \aleph_1 < \mathfrak{u}$, see [11], there is (easily checked) no converging family of cardinality $\mathfrak{r}$.

If there is a simple almost clopen set of type κ, are there restrictions on the behavior of its almost clopen complement and can there be simple almost clopen sets of different types (including the complement)? These are the types of questions that stimulated this study. The most compelling of these has been answered.

Theorem 2.5. *If A is a simple almost clopen set of type κ and if the complementary almost clopen set B is simple, then it also has type κ.*

We defer the proof of this, and the next two theorems, until the next section. Similarly, there is restriction on what the type of a simple almost clopen set can be that is shared by simple P_κ-points (as shown, for simple P_κ-points, by Nyikos (unpublished), see [2]).

Theorem 2.6. *If A is a simple almost clopen set of type κ, then κ is one of $\{\mathfrak{b}, \mathfrak{d}\}$.*

Now that we understand the limits on the behavior of a complementary pair of simple almost clopen sets, we look to the properties of the complement B when it is not assumed to be simple. The topological properties of character and tightness of x_B in B are natural cardinal invariants to examine. These correspond to natural properties of $\mathcal{I}_B$ as well. An indexed subset $\{x_\beta \colon \beta < \lambda\}$ of a space X is said to be a free sequence if the closure of each initial segment is disjoint from the closure of its complementary final segment. A λ-sequence $\{x_\beta \colon \beta < \lambda\}$ is converging if there is a point x such that every neighborhood of x contains a final segment of $\{x_\beta \colon \beta < \lambda\}$. A subset D of $\mathbb{N}^*$ is said to be *strongly* discrete, see [10] and [21], if there is a family of pairwise disjoint clopen subsets of $\mathbb{N}^*$ each containing a single point of D.

Theorem 2.7. *If $\kappa < \lambda$ are regular cardinals with $\mathfrak{c} \leq \lambda$, then there is a countable chain condition (ccc) forcing extension in which there is a simple almost clopen set*

A of type κ such that the almost clopen complement B contains a free λ-sequence $\{x_\beta \colon \beta < \lambda\}$ that converges to x_A.

We finish this section by formulating some open problems about almost clopen sets and possible asymmetries.

Question 2.8. Can there exist simple almost clopen sets of different types?

If A is a simple almost clopen set of type κ, then, as a point in A, x_A is a P-point with a linear neighborhood base of clopen sets. Therefore the next two questions seem natural. See also the remark about Nyikos' result, see [2], about simple P_κ-points preceding Theorem 2.6.

Question 2.9. If there is a simple almost clopen set of type κ is there a point of $\mathbb{N}^*$ of character κ? Is there a simple P_κ-point?

Question 2.10. Is a simple almost clopen set of type $\aleph_1$ necessarily homeomorphic to $\mathbb{N}^*$?

Question 2.11. If A is a simple almost clopen set of type κ, is there a simple almost clopen set B' contained in the almost clopen complement B of A such that $x_A = x_{B'}$? Is there a family of κ-many members of $\mathcal{I}_B$ that converges to x_A?

3. Proofs

Our analysis of simple almost clopen sets depends on the connection between the type of the clopen set and the ultrafilter ordering of functions from $\mathbb{N}$ to $\mathbb{N}$. For an ultrafilter x on $\mathbb{N}$ the ordering $<_x$ is defined on $\mathbb{N}^\mathbb{N}$ by the condition that $f <_x g$ if $\{n \in \mathbb{N} \colon f(n) < g(n)\} \in x$. Since x is an ultrafilter, a set $F \subset \mathbb{N}^\mathbb{N}$ is cofinal in $(\mathbb{N}^\mathbb{N}, <_x)$ if it is not bounded. Of course a subset of $\mathbb{N}^\mathbb{N}$ that is unbounded with respect to the $<_x$-ordering is also unbounded with respect to the mod finite ordering $<^*$.

Fix a $<^*$-unbounded family $\{f_\xi \colon \xi < \mathfrak{b}\} \subset \mathbb{N}^\mathbb{N}$ such that each f_ξ is strictly increasing and such that $f_\eta <^* f_\xi$ for all $\eta < \xi < \mathfrak{b}$. The following well-known fact will be useful.

Proposition 3.1. *For each infinite $b \subset \mathbb{N}$ and each unbounded $\Gamma \subset \mathfrak{b}$, the family $\{f_\xi \restriction b \colon \xi \in \Gamma\}$ is $<^*$-unbounded in $\mathbb{N}^b$.*

Proof: For each $\eta < \mathfrak{b}$, there is a $\xi \in \Gamma \setminus \eta$ such that $f_\eta <^* f_\xi$, hence $\{f_\xi \colon \xi \in \Gamma\}$ is $<^*$-unbounded. If $g \in \mathbb{N}^b$, then $g \circ \mathrm{next}(b, \cdot) \in \mathbb{N}^\mathbb{N}$. So there is a $\xi \in \Gamma$ such that $f_\xi \not<^* g \circ \mathrm{next}(b, \cdot)$. Since f_ξ is strictly increasing, $f_\xi \restriction b \not<^* g$. □

Lemma 3.2. *If a family $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ converges to an ultrafilter x and if $\{f_\xi \colon \xi \in \mathfrak{b}\}$ is bounded $\mod <_x$, then $\mathcal{A}$ has cardinality $\mathfrak{b}$.*

Proof: Choose $g \in \mathbb{N}^\mathbb{N}$ so that $f_\xi <_x g$ for all $\xi < \mathfrak{b}$. Since $\mathcal{A}$ cannot be reaped, Proposition 2.2 implies that $\mathfrak{b} \le |\mathcal{A}|$. For each ξ, let $U_\xi = \{n \in \mathbb{N} \colon f_\xi(n) < g(n)\} \in x$. If $\mathfrak{b} < |\mathcal{A}|$, then there is a $b \in \mathcal{A}$ such that $b \subset^* U_\xi$ for all $\xi < \mathfrak{b}$

(i.e. x is an almost $P_{\mathfrak{b}^+}$-point). However we would then have that $f_\xi \restriction b <^* g \restriction b$ for all $\xi < \mathfrak{b}$, and by Proposition 3.1, there is no such set b. This completes the proof. $\square$

Concerning this next result, we do not know if it is consistent to have a family in $[\mathbb{N}]^{\aleph_0}$ of singular cardinality that is converging.

Lemma 3.3. *If a family $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ of regular cardinality converges to an ultrafilter x and if $\{f_\xi \colon \xi \in \mathfrak{b}\}$ is unbounded $\mod <_x$, then $\mathcal{A}$ has cardinality equal to $\mathfrak{d}$.*

Proof: Since we are assuming that $\{f_\xi \colon \xi \in \mathfrak{b}\}$ is $<_x$-unbounded, it is actually $<_x$-cofinal. Also, we check that the family $\{f_\xi \circ \mathrm{next}(a,\cdot) \colon \xi < \mathfrak{b},\ a \in \mathcal{A}\}$ is a $<^*$-dominating family. Take any strictly increasing $g \in \mathbb{N}^\mathbb{N}$ and choose $\xi < \mathfrak{b}$ such that $U = \{n \colon g(n) < f_\xi(n)\} \in x$. Since $\mathcal{A}$ converges to x, there is an $a \in \mathcal{A}$ such that $a \subset^* U$. Since g is strictly increasing, it is clear that $g < f_\xi \circ \mathrm{next}(U,\cdot) <^* f_\xi \circ \mathrm{next}(a,\cdot)$. Again, since $\mathcal{A}$ cannot be reaped, we have $\mathfrak{b} \leq |\mathcal{A}|$ and this implies that $\mathfrak{d} \leq |\mathcal{A}|$. Assume that $\{g_\beta \colon \beta < \mathfrak{d}\} \subset \mathbb{N}^\mathbb{N}$ is a $<^*$-dominating family. For each $a \in \mathcal{A}$, there is a $\beta_a < \mathfrak{d}$ such that $\mathrm{next}(a,\cdot) <^* g_{\beta_a}$. Now since $\mathcal{A}$ is hereditarily unreapable, Proposition 2.2 implies that the mapping $a \mapsto \beta_a$ is $<|\mathcal{A}|$-to-1. Since we assume that $|\mathcal{A}|$ is regular, this implies that $|\mathcal{A}| \leq \mathfrak{d}$. $\square$

Corollary 3.4. *If A is a simple almost clopen set of type κ, then $\kappa = \mathfrak{b}$ if $\{f_\xi \colon \xi < \mathfrak{b}\}$ is $<_{x_A}$-bounded. Otherwise $\kappa = \mathfrak{d}$.*

Proof: Let $\{a_\alpha \colon \alpha \in \kappa\}$ be the family contained in $\mathcal{I}_A$ witnessing that A has type κ. Set $\mathcal{A}$ equal to the family $\{a_{\alpha+1} \setminus a_\alpha \colon \alpha \in \kappa\}$ which converges to x_A. If $\{f_\xi \colon \xi < \mathfrak{b}\}$ is $<_{x_A}$-bounded, then by Lemma 3.2, $\kappa = \mathfrak{b}$. Otherwise, since κ is a regular cardinal, we have by Lemma 3.3, $\kappa = \mathfrak{d}$. $\square$

Proof of Theorem 2.5: Assume that A and its complementary almost clopen set B are both simple and let $x = x_A$. If $\{f_\xi \colon \xi < \mathfrak{b}\}$ is $<_x$-bounded then, by Corollary 3.4 they both have type $\mathfrak{b}$; otherwise they both have type $\mathfrak{d}$. $\square$

Proof of Theorem 2.6: Immediate from Corollary 3.4. $\square$

We can improve Theorem 2.5. First we recall that a family $\mathcal{A} \subset [\mathbb{N}]^{\aleph_0}$ is a *splitting* family if for all infinite $b \subset \mathbb{N}$, there is an $a \in \mathcal{A}$ such that $|b \cap a| = |b \setminus a|$. We say that b is split by a. The splitting number $\mathfrak{s}$ is the least cardinality of a splitting family and $\mathfrak{s} \leq \mathfrak{d}$, see [11].

Proposition 3.5. *There is no almost $P_{\mathfrak{s}^+}$-point in $\mathbb{N}^*$.*

Proof: Let $\mathcal{A}$ be a splitting family of cardinality $\mathfrak{s}$. We may assume that $\mathcal{A}$ is closed under complements. Let x be any point of $\mathbb{N}^*$. It is easily seen that any pseudointersection of $x \cap \mathcal{A}$ is not split by any member of $\mathcal{A}$. Since $\mathcal{A}$ is splitting, $x \cap \mathcal{A}$ has no pseudointersection, and so x is not an almost $P_{\mathfrak{s}^+}$-point. $\square$

Now we improve Theorem 2.5.

Theorem 3.6. *If A is a simple almost clopen set of type κ then x_A is not an almost P_{κ^+}-point.*

Proof: We first note that by Proposition 3.5 we must have that $\kappa < \mathfrak{d}$. Therefore, by Lemma 3.3, $\{f_\xi \colon \xi < \mathfrak{b}\}$ is $<_{x_A}$-bounded. Choose any $g \in \mathbb{N}^{\mathbb{N}}$ so that $f_\xi <_{x_A} g$ for all $\xi < \mathfrak{b}$. For each ξ, let $U_\xi = \{n \in \mathbb{N} \colon f_\xi(n) < g(n)\}$. By Proposition 3.1, we have that the collection $\{U_\xi \colon \xi < \mathfrak{b}\} \subset x$ has no pseudointersection. By Theorem 2.5, $\mathfrak{b} \leq \kappa$ and this proves the theorem. $\square$

Proof of Theorem 2.7: We first prove the easier special case when $\kappa = \aleph_1$. An α-length finite support iteration sequence of posets, denoted $(\langle \mathbb{P}_\beta \colon \beta \leq \alpha \rangle, \langle \dot{Q}_\beta \colon \beta < \alpha \rangle)$, will mean that $\langle \mathbb{P}_\beta \colon \beta \leq \alpha \rangle$ is an increasing chain of posets, $\dot{Q}_\beta$ is a $\mathbb{P}_\beta$-name of a poset for each $\beta < \alpha$, and members p of $\mathbb{P}_\alpha$ will be functions with domain a finite subset, $\operatorname{supp}(p)$, of α satisfying that $p \restriction \beta \in \mathbb{P}_\beta$ forces that $p(\beta) \in \dot{Q}_\beta$. As usual, $p_2 < p_1$ providing $p_2 \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p_2(\beta) < p_1(\beta)$" for all $\beta \in \operatorname{supp}(p_1)$. Since $\mathbb{P}_0$ is the trivial poset, we will allow ourselves to simply specify a poset Q_0 in such an iteration sequence rather than the $\mathbb{P}_0$-name of that poset.

Definition 3.7. Let $\mathcal{A} = \{a_\beta \colon \beta < \alpha\}$ be a $\subset^*$-increasing chain of subsets of ω, and let $\mathcal{I}$ be an ideal contained in $\mathcal{A}^\perp$. We define the poset $Q = Q(\mathcal{A}; \mathcal{I})$ where $q \in Q$ if $q = (F_q, \sigma_q, b_q)$ where

(1) $F_q \in [\omega]^{<\aleph_0}$;
(2) $b_q \in \mathcal{I}$ is disjoint from F_q;
(3) $\sigma_q \colon H_q \to \omega$ and $H_q \in [\alpha]^{<\aleph_0}$;
(4) for each $\beta \in H_q$, $a_\beta \setminus \sigma_q(\beta)$ is disjoint from b_q.

For $r, q \in Q$ we define $r < q$ providing $F_r \supset F_q$, $\sigma_r \supset \sigma_q$, and $b_r \supset b_q$.

Definition 3.8. If Q is $Q(\mathcal{A}; \mathcal{I})$ for some $\subset^*$-increasing chain of subsets of ω and ideal $\mathcal{I} \subset \mathcal{A}^\perp$, then the Q-generic set $\dot{a}_Q$ is defined as the natural name $\{(\check{n}, q) \colon q \in Q \text{ and } n \in F_q\}$, i.e. for each Q-generic filter G, $\operatorname{val}_G(\dot{a}_Q)$ is equal to the union of the family $\{F_q \colon q \in G\}$.

Lemma 3.9. *Let $\mathcal{A} = \{a_\beta \colon \beta < \alpha\}$ and $\mathcal{I}$ be as given in the definition of $Q(\mathcal{A}; \mathcal{I})$ and let $Q = Q(\mathcal{A}; \mathcal{I})$. Then $1 \Vdash_Q$ "$\dot{a}_Q \in \mathcal{I}^\perp$", and for all $q \in Q(\mathcal{A}; \mathcal{I})$ and $\beta \in \operatorname{dom}(\sigma_q)$, $q \Vdash_Q$ "$a_\beta \setminus \sigma_q(\beta) \subset \dot{a}_Q$".*

Proof: Let $q \in Q$, $\beta \in \sigma_q$ and $b \in \mathcal{I}$. Since $q_b = (F_q, \sigma_q, b_q \cup (b \setminus F_q))$ forces that $\dot{a}_Q$ is disjoint from b, this proves that $1 \Vdash_Q$ "$\dot{a}_Q \in \mathcal{I}^\perp$". Choose any $k \in a_\beta \setminus \sigma_q(\beta)$ and let $r \leq q$. By condition (4), $k \notin b_r$. It is easily checked that $r' = (F_r \cup \{k\}, \sigma_r, b_r) < r$ and that $r' \Vdash_Q$ "$k \in \dot{a}_Q$". $\square$

For constructing our models, we will need instances of $Q(\mathcal{A}; \mathcal{I})$ that are ccc. This first result is sufficient to produce simple almost clopen sets of type $\aleph_1$.

Lemma 3.10. *Let $\mathcal{A} = \{a_\beta \colon \beta < \alpha\}$ be a $\subset^*$-increasing chain of subsets of ω and let $\mathcal{I}$ be an ideal contained in $\mathcal{A}^\perp$. If α is countable or if $\mathcal{I} = [\omega]^{<\aleph_0}$, then $Q(\mathcal{A}; \mathcal{I})$ is σ-centered.*

Proof: Let b, $F \in [\omega]^{<\aleph_0}$ and $H \in [\alpha]^{<\aleph_0}$ and $\sigma \colon H_q \to \omega$. Then each of the sets $\{q \in Q(\mathcal{A};\mathcal{I})\colon F_q = F \text{ and } \sigma_q = \sigma\}$ and $\{q \in Q(\mathcal{A};\mathcal{I})\colon F_q = F \text{ and } b_q = b\}$ is centered. $\square$

Lemma 3.11. *If λ is a regular cardinal with $\mathfrak{c} \le \lambda$, then there is a ccc forcing extension in which there is a simple almost clopen set A of type $\aleph_1$ such that there is a strongly discrete free λ-sequence converging to x_A.*

Proof: There are ccc posets of cardinality λ that add a strictly $\subset^*$-increasing sequence $\{b_\zeta\colon \zeta < \lambda\}$ of infinite subsets of ω (e.g. [17, II Example 22]). Alternatively, by Definition 3.7 and Lemma 3.10, we could let Q_0 be a λ-length finite support sequence of posets of the form $Q(\{b_\beta\colon \beta < \zeta\}; [\omega]^{<\omega})$ and recursively let b_ζ be the resulting $\dot{a}_Q$ as in Definition 3.8. In each of these models we obtain that $\mathfrak{c} = \lambda$.

For convenience we now work in such a ccc forcing extension in which $\lambda = \mathfrak{c}$ and we construct a finite support ccc iteration sequence of cardinality λ and length ω_1 that will add a strictly $\subset^*$-increasing sequence $\{a_\alpha\colon \alpha \in \omega_1\}$ of infinite subsets of ω so that the closure A of $\bigcup\{a_\alpha^*\colon \alpha \in \omega_1\}$ is almost clopen. Suppose that we do this in such a way that $\{b_\zeta\colon \zeta < \lambda\}$ is contained in $\{a_\alpha\colon \alpha \in \omega_1\}^\perp$ and for all $U \in x_A$ and all $\zeta < \lambda$ there is an $\eta < \lambda$ such that $U \cap (b_\eta \setminus b_\zeta)$ is infinite. We check that there is then a strongly discrete free λ-sequence converging to x_A. Let $\{U_\zeta\colon \zeta < \lambda\}$ enumerate the members of x_A. Recursively define a strictly increasing function g from λ into λ satisfying that $U_\zeta \cap (b_{g(\delta)} \setminus b_\delta)$ is infinite for all $\zeta < \delta \in \lambda$. Choose any cub $C \subset \lambda$ satisfying that the family $\{U_\xi\colon \xi < \delta\}$ is closed under finite intersections. For each $\delta \in C$, the family $\{U_\zeta \cap (b_{g(\delta)} \setminus b_\delta)\colon \zeta < \delta\}$ is closed under finite intersections, so we may choose an ultrafilter x_δ extending it. Pass to a cub subset $C_1 \subset C$ satisfying that $g(\eta) < \delta$ for all $\eta < \delta$ in C_1. Since the family $\{b_\zeta\colon \zeta < \lambda\}$ is mod finite increasing and $b_\delta \setminus b_\eta \in x_\eta$ for all $\eta < \delta$ in C_1, it follows that $\{x_\delta\colon \delta \in C_1\}$ is strongly discrete and free. Similarly, the sequence converges to x_A since $U_\zeta \in x_\delta$ for all $\zeta < \delta \in C_1$.

Now we construct the iteration sequence to define the $\subset^*$-increasing chain $\{a_\alpha\colon \alpha \in \omega_1\}$ that will be cofinal in $\mathcal{I}_A$. We will use iterands of the form $\dot{Q}_\alpha = Q(\{\dot{a}_\beta\colon \beta < \alpha\}; \dot{\mathcal{I}}_\alpha)$ for $0 < \alpha < \omega_1$, and will recursively let $\dot{a}_\alpha$ be the standard $\mathbb{P}_{\alpha+1}$-name for $a_{\dot{Q}_\alpha}$ (as in Definition 3.8). Clearly the only choices we have for the construction are the definition of a_0 and, by recursion, the definition of $\dot{\mathcal{I}}_\alpha$. By recursion on $\gamma < \alpha < \omega_1$ we ensure that $\mathbb{P}_\alpha$ forces each of the following:

(1) $\dot{\mathcal{I}}_{\gamma+1} \subset \dot{\mathcal{I}}_\alpha \subset \{\dot{a}_\beta\colon \beta < \alpha\}^\perp$;
(2) $\{b_\zeta\colon \zeta < \lambda\}$ is a $\subset^*$-unbounded subset of $\dot{\mathcal{I}}_{\gamma+1}$;
(3) if $\alpha = \gamma + 1$, then $\dot{\mathcal{I}}_\alpha \cup \{\dot{a}_\beta\colon \beta < \alpha\}$ generates a proper maximal ideal.

To start the recursion, using Lemma 1.6, we can choose a proper maximal ideal $\mathcal{I}_1$ extending the ideal generated by $\{b_\zeta\colon \zeta < \lambda\} \cup \{b_\zeta\colon \zeta < \lambda\}^\perp$. There is no $\subset^*$-bound for $\{b_\zeta\colon \zeta < \lambda\}$ in $\mathcal{I}_1$ since the complement of any such bound is in $\mathcal{I}_1$. Very likely $\{b_\zeta\colon \zeta < \lambda\}^\perp$ is simply $[\omega]^{<\omega}$, so we let a_0 be exceptional and equal the emptyset. We now have our definition (working in the extension by Q_0)

of $Q_1 = Q(\{a_0\};\mathcal{I}_1)$ and the generic set $\dot a_1 = \dot a_{Q_1}$ is forced to be almost disjoint from every member of $\mathcal{I}_1$ (it is a pseudointersection of the ultrafilter dual to $\mathcal{I}_1$). Now assume that $\alpha < \omega_1$ and that we have defined $\dot{\mathcal{I}}_\beta$ for all $\beta < \alpha$. For the definition of $\dot{\mathcal{I}}_\alpha$ we break into two cases.

If α is a limit ordinal, then we define $\dot{\mathcal{I}}_\alpha$ to be the $\mathbb{P}_\alpha$-name of the ideal $\{\dot a_\beta\colon \beta < \alpha\}^\perp$. By induction we have for $\gamma < \alpha$ that $\Vdash_{\mathbb{P}_\alpha}$ "$\dot{\mathcal{I}}_\gamma \subset \{\dot a_\beta\colon \beta < \alpha\}^\perp = \dot{\mathcal{I}}_\alpha$" as required in (1). Condition (2) holds by induction and (3) is vacuous.

In the case that $\alpha = \gamma + 1$, we note that, by the genericity of $\dot a_\gamma$ and the induction hypothesis (1), $\mathbb{P}_\alpha$ forces that the family $\{\dot a_\gamma\} \cup \bigcup\{\dot{\mathcal{I}}_{\eta+1}\colon \eta < \gamma\}$ generates a proper ideal $\dot{\mathcal{J}}_\alpha$. Furthermore, hypotheses (2) (and the definition of $\dot a_\gamma$) ensures that $\mathbb{P}_\alpha$ forces that $\{b_\zeta\colon \zeta < \lambda\}$ is $\subset^*$-unbounded in $\dot{\mathcal{J}}_\alpha$. Again using Lemma 1.6, we can choose $\dot{\mathcal{J}}'_\alpha$ to be the $\mathbb{P}_\alpha$-name of any proper maximal ideal that contains $\dot{\mathcal{J}}_\alpha \cup (\dot{\mathcal{J}}_\alpha)^\perp$. Just as in the first inductive step, $\mathbb{P}_\alpha$ again forces that $\{b_\zeta\colon \zeta < \lambda\}$ is $\subset^*$-unbounded in $\dot{\mathcal{J}}'_\alpha$. The definition of $\dot{\mathcal{I}}_\alpha$ is the $\mathbb{P}_\alpha$-name of $\{b \in \dot{\mathcal{J}}'_\alpha\colon b \cap \dot a_\gamma =^* \emptyset\}$. The set $\dot a_\alpha \setminus \dot a_\gamma$ will be forced by $\mathbb{P}_{\alpha+1}$ to be an infinite pseudointersection of the ultrafilter that is dual to $\dot{\mathcal{J}}'_\alpha$. For convenience, let $\dot y_\alpha$ denote the $\mathbb{P}_\alpha$-name of this ultrafilter.

This completes the definition of the poset $\mathbb{P}_{\omega_1}$. Now we establish some additional properties. It will be convenient to observe that for each $\alpha = \gamma + 1 < \omega_1$ the $\mathbb{P}_\alpha$-name for $\{\omega \setminus (b \cup \dot a_\gamma)\colon b \in \dot{\mathcal{I}}_\alpha\}$ is forced to be a base for $\dot y_\alpha$. Let $\dot A$ denote a $\mathbb{P}_{\omega_1}$-name of the closure in ω^* of $\bigcup\{\dot a^*_\alpha\colon \alpha \in \omega_1\}$.

Claim 1. *For each $\beta < \alpha < \omega_1$, $\mathbb{P}_{\alpha+1}$ forces that $\dot a_\alpha \setminus \dot a_\beta$ is a pseudointersection of the filter $\dot y_{\beta+1}$.*

Proof: We prove the claim by induction on $\alpha \geq \beta + 1$. For $\alpha = \beta + 1$, $\dot a_\alpha$ is almost disjoint from each member of $\dot{\mathcal{I}}_\alpha$, and so $\dot a_\alpha \setminus \dot a_\beta$ is almost disjoint from every member of $\dot{\mathcal{J}}'_\alpha$. Thus $\dot a_\alpha \setminus \dot a_\beta$ is forced to be mod finite contained in every member of the dual filter, namely $\dot y_{\beta+1}$. Similarly, for $\alpha > \beta + 1$, $\dot a_\alpha$ is forced to be almost disjoint from each member of $\dot{\mathcal{I}}_\alpha$. This means that $\dot a_\alpha$ is almost disjoint from each member of $\dot{\mathcal{I}}_{\beta+1}$, and so $\dot a_\alpha \setminus \dot a_{\beta+1}$ is also almost disjoint from every member of $\dot{\mathcal{J}}'_{\beta+1}$. □

Claim 2. *The family $\{\dot y_{\beta+1}\colon \beta < \omega_1\}$ is a family of $\mathbb{P}_{\omega_1}$-names and the union is forced to generate an ultrafilter $\dot x_{\dot A}$ that is indeed the unique boundary point of $\dot A$.*

Proof: Since, for each $\beta < \alpha < \omega_1$, $\dot{\mathcal{I}}_{\beta+1}$ is contained in $\dot{\mathcal{I}}_{\alpha+1}$ and $\dot a_\beta$ is forced to be a mod finite subset of $\dot a_\alpha$, the filter dual to the ideal generated by $\{\dot a_\beta\} \cup \dot{\mathcal{I}}_{\beta+1}$ is contained in the filter dual to the ideal generated by $\{\dot a_\alpha\} \cup \dot{\mathcal{I}}_{\alpha+1}$. This implies that P_{ω_1} forces that $\bigcup\{\dot y_{\beta+1}\colon \beta < \omega_1\}$ is a filter. Furthermore, since $\mathbb{P}_{\omega_1}$ is ccc, every $\mathbb{P}_{\omega_1}$-name of a subset of ω is forced to be equal to a $\mathbb{P}_\beta$-name for some $\beta < \omega_1$. Since $\mathbb{P}_{\beta+1}$ forces that $\dot y_{\beta+1}$ is an ultrafilter, this shows that $\mathbb{P}_{\omega_1}$ forces that $\dot x_{\dot A}$ is an ultrafilter. Finally, it follows from the previous claim and Lemma 2.4 that $\dot x_{\dot A}$ is the unique boundary point of $\dot A$. □

We now verify the final property required of the construction.

Claim 3. *Let $\mathbb{P}_{\omega_1}$ are forces that for each $U \in \dot{x}_{\dot{A}}$ and each $\zeta < \lambda$, there is an $\eta < \lambda$ such that $U \cap (b_\eta \setminus b_\zeta)$ is infinite.*

PROOF: Let $\dot{U}$ be any $\mathbb{P}_{\omega_1}$-name and assume that $p \in \mathbb{P}_{\omega_1}$ forces that $\dot{U} \in \dot{x}_{\dot{A}}$. Since P_{ω_1} is ccc and $p \Vdash_{\mathbb{P}_{\omega_1}}$ "$\dot{U} \subset \omega$", we may suppose there is an $\alpha < \omega_1$ such that $p \in \mathbb{P}_{\alpha+1}$ and $\dot{U}$ is a $\mathbb{P}_{\alpha+1}$-name. By Claim 2, $p \Vdash_{\mathbb{P}_{\alpha+1}}$ "$\dot{U} \in \dot{y}_{\alpha+1}$". Let $\dot{W}$ be the $\mathbb{P}_{\alpha+1}$-name of $\omega \setminus (\dot{U} \cup \dot{a}_\alpha)$, and note that $p \Vdash_{\mathbb{P}_{\alpha+1}}$ "$\dot{W} \in \dot{\mathcal{I}}_{\alpha+1}$". Therefore, by induction hypothesis (2), $p \Vdash_{\mathbb{P}_{\alpha+1}}$ "$(\exists\, \eta < \lambda)\ b_\eta \setminus (b_\zeta \cup \dot{W}) \neq^* \emptyset$". Since $\mathbb{P}_{\alpha+1}$ is ccc and λ has uncountable cofinality, we may choose an $\eta < \lambda$ such that $p \Vdash_{\mathbb{P}_{\alpha+1}}$ "$b_\eta \setminus (b_\zeta \cup \dot{W}) \neq^* \emptyset$". Finally we have that $p \Vdash_{\mathbb{P}_{\alpha+1}}$ "$(b_\eta \setminus b_\zeta) \cap \dot{U} \neq^* \emptyset$" since $p \Vdash_{\mathbb{P}_{\alpha+1}}$ "$\dot{a}_\alpha \cap b_\eta =^* \emptyset$". $\square$

This completes the proof of Lemma 3.9. $\square$

Now we return to the complete proof of Theorem 2.7.

Definition 3.12. We say that $\mathcal{A}$ is a pre-ccc sequence if $\mathcal{A} = \{a_\beta \colon \beta < \alpha\} \subset [\omega]^{\aleph_0}$ is $\subset^*$-increasing and for each sequence $\{b_\xi \colon \xi \in \omega_1\} \subset \mathcal{A}^\perp$, and sequence $\{\gamma_\xi : \xi \in \omega_1\} \subset \alpha$ there are $\xi < \eta$ such that either $a_{\gamma_\xi} \cap b_\xi \neq \emptyset$ or $(a_{\gamma_\xi} \cup a_{\gamma_\eta}) \cap (b_\xi \cup b_\eta) = \emptyset$.

Lemma 3.13. *If $\mathcal{A}$ is a pre-ccc sequence, then $Q(\mathcal{A};\mathcal{I})$ is ccc for any ideal $\mathcal{I} \subset \mathcal{A}^\perp$.*

PROOF: Let $\mathcal{A} = \{a_\beta \colon \beta < \alpha\} \subset [\omega]^{\aleph_0}$ be $\subset^*$-increasing. Let $\{q_\xi \colon \xi \in \omega_1\} \subset Q = Q(\mathcal{A};\mathcal{I})$. By passing to a subcollection we can suppose there is a single $F \in [\omega]^{<\aleph_0}$ such that $F_{q_\xi} = F$ for all ξ. For each ξ, let $b_\xi = b_{q_\xi}$, $\sigma_\xi = \sigma_{q_\xi}$, and $H_\xi = \mathrm{dom}(\sigma_\xi)$. By passing to an uncountable subcollection, we can assume that the family $\{H_\xi \colon \xi \in \omega_1\}$ is a Δ-system with root H. Similarly, we can assume that for all ξ, η, $\sigma_\xi \restriction H = \sigma_\eta \restriction H$. For each ξ, let γ_ξ be the maximum element of H_ξ.

Next, we choose an integer $\overline{m}$ sufficiently large so that there is again an uncountable $I \subset \omega_1$ and a subset $\bar{b}$ of $\overline{m}$ such that for all $\xi \in I$ and all $\beta \in H_\xi$ $\sigma_\xi(\beta) < \overline{m}$, $a_\beta \setminus \overline{m} \subset a_{\gamma_\xi}$, and $b_\xi \cap \overline{m} = \bar{b}$. Now we apply the pre-ccc property for the family $\{\gamma_\xi \colon \xi \in I\}$ and the sequence $\{b_\xi \setminus \overline{m} \colon \xi \in I\}$. Since a_{γ_ξ} is disjoint from $b_\xi \setminus \overline{m}$ for all $\xi \in I$, there must be $\xi < \eta$ from I so that $(a_{\gamma_\xi} \cup a_{\gamma_\eta})$ is disjoint from $((b_\xi \cup b_\eta) \setminus \overline{m})$.

We claim that $r = (F, \sigma_\xi \cup \sigma_\eta, b_\xi \cup b_\eta)$ is in Q and is an extension of each of q_ξ and q_η. It suffices to prove that for $\beta \in H_\xi$, $a_\beta \setminus \sigma_\xi(\beta)$ is disjoint from b_η, and similarly, that $a_\beta \setminus \sigma_\eta(\beta)$ is disjoint from b_ξ for $\beta \in H_\eta$. Since $b_\xi \cap \overline{m} = b_\eta \cap \overline{m} = \bar{b}$, it suffices to consider $a_\beta \setminus \overline{m}$ in each case. For $\beta \in H_\xi$, we have $a_\beta \setminus \overline{m} \subset a_{\gamma_\xi}$ and a_{γ_ξ} is disjoint from $b_\eta \setminus \overline{m}$. For $\beta \in H_\eta$, we similarly have that $a_\beta \setminus \overline{m} \subset a_{\gamma_\eta}$ and a_{γ_η} is disjoint from $b_\xi \setminus \overline{m}$. $\square$

For the remainder of the section, $\kappa < \lambda$ are regular cardinals with $\mathfrak{c} \leq \lambda$ (as in Theorem 2.7).

Definition 3.14. $\mathfrak{A}$ is the class of triples $(\mathfrak{P}, \mathcal{A}, \mathfrak{I})$ such that, there is an ordinal $0 < \alpha \leq \kappa$, and the following holds for each $\gamma < \beta < \alpha$:

(1) $\mathfrak{P} = (\langle \mathbb{P}_\beta \colon \beta \leq \alpha\rangle, \langle \dot{Q}_\beta \colon \beta < \alpha\rangle)$ is a finite support iteration sequence of ccc posets, and has cardinality λ;
(2) $\mathcal{A}$ is an α-sequence $\{\dot{a}_\beta \colon \beta < \alpha\}$ and $\dot{a}_0 = \emptyset$;
(3) $\mathfrak{I}$ is an α-sequence $\{\dot{\mathcal{I}}_\beta \colon \beta < \alpha\}$;
(4) $\dot{a}_\beta$ is a $\mathbb{P}_{\beta+1}$-name, and $\Vdash_{\mathbb{P}_{\beta+1}}$ "$\dot{a}_\gamma \subset^* \dot{a}_\beta \subset \omega$";
(5) $\mathbb{P}_1$ forces that $\dot{\mathcal{I}}_1$ is an ideal on ω that has a cofinal $\subset^*$-increasing chain of type λ;
(6) $\dot{\mathcal{I}}_\beta$ is a $\mathbb{P}_\beta$-name, and $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathcal{I}}_\beta \subset \{\dot{a}_\xi \colon \xi < \beta\}^\perp$ is an ideal ";
(7) $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathcal{I}}_{\gamma+1} \subset \dot{\mathcal{I}}_\beta$", and $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathcal{I}}_1$ is $\subset^*$ -unbounded in $\dot{\mathcal{I}}_{\gamma+1}$";
(8) $\Vdash_{\mathbb{P}_\beta}$ "$\dot{Q}_\beta = Q(\{\dot{a}_\xi \colon \xi < \beta\}; \dot{\mathcal{I}}_\beta)$";
(9) $\mathbb{P}_{\gamma+1}$ forces that $\{\dot{a}_\gamma\} \cup \dot{\mathcal{I}}_{\gamma+1}$ generates a maximal ideal;
(10) $\dot{a}_\beta$ is the $\mathbb{P}_{\beta+1}$-name for $\dot{a}_{Q(\{\dot{a}_\xi \colon \xi<\beta\};\dot{\mathcal{I}}_\beta)}$;
(11) if $\mathrm{cf}(\beta) = \omega$, then $\Vdash_{\mathbb{P}_\beta}$ "$\dot{\mathcal{I}}_\beta = \bigcup\{\dot{\mathcal{I}}_{\xi+1} \colon \xi < \beta\}$".

We omit the routine modifications of the proof of Lemma 3.11 needed to prove the following.

Lemma 3.15. *If $(\mathfrak{P}, \mathcal{A}, \mathcal{J}) \in \mathfrak{A}$ and $\mathfrak{P} = (\langle \mathbb{P}_\beta \colon \beta \leq \kappa\rangle, \langle \dot{Q}_\beta \colon \beta < \kappa\rangle)$, and $\mathcal{A} = \{\dot{a}_\beta \colon \beta < \kappa\}$, then $\mathbb{P}_\kappa$ is a ccc poset that forces that the closure A of $\bigcup\{\dot{a}^*_\beta \colon \beta < \kappa\}$ is a simple almost clopen set of type κ and that there is a strongly discrete free λ-sequence converging to x_A.*

To finish the proof of Theorem 2.7 we have to prove there is an element of $\mathfrak{A}$ of length κ. Towards doing so, we let $<_{\mathfrak{A}}$ denote the obvious coordinatewise ordering on the elements of $\mathfrak{A}$. Using Zorn's lemma, we let $\mathcal{C} \subset \mathfrak{A}$ be a maximal $<_{\mathfrak{A}}$-chain. It is easily checked that the maximality of $\mathcal{C}$ implies that it has a maximal element which we denote

$$(\mathfrak{P}, \mathcal{A}, \mathcal{J}) = (\langle \mathbb{P}_\beta \colon \beta \leq \alpha\rangle, \langle \dot{Q}_\beta \colon \beta < \alpha\rangle), \{\dot{a}_\beta \colon \beta < \alpha\}, \{\dot{\mathcal{I}}_\beta \colon \beta < \alpha\})$$

and we prove that $\alpha = \kappa$.

Lemma 3.16. *If $\alpha < \kappa$ then $(\mathfrak{P}, \mathcal{A}, \mathcal{J})$ is not maximal.*

Proof: In order to extend $(\mathfrak{P}, \mathcal{A}, \mathcal{J})$ we have to define a $\mathbb{P}_\alpha$-name $\dot{\mathcal{I}}_\alpha$ and to then let $\dot{Q}_\alpha$ be the $\mathbb{P}_\alpha$-name of $Q(\mathcal{A}; \dot{\mathcal{I}}_\alpha)$. We then set $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{Q}_\alpha$ and let $\dot{a}_\alpha$ be the $\mathbb{P}_{\alpha+1}$-name for $\dot{a}_{Q(\mathcal{A};\dot{\mathcal{I}}_\alpha)}$. We must show that $(\mathfrak{P}^\frown\{(\langle \mathbb{P}_{\alpha+1}\rangle, \langle \dot{Q}_\alpha\rangle)\}, \mathcal{A}^\frown\{\dot{a}_\alpha\}, \mathcal{J}^\frown\{\dot{\mathcal{I}}_\alpha\})$ is in $\mathfrak{A}$ to complete the proof. We defer to the end of the proof that our choice of $\dot{\mathcal{I}}_\alpha$ will result in the fact that $\mathbb{P}_\alpha$ forces that $Q(\mathcal{A}; \dot{\mathcal{I}}_\alpha)$ is ccc.

First we assume that α is a limit ordinal and explain the construction of $\dot{\mathcal{I}}_\alpha$. If α has uncountable cofinality, we let $\dot{\mathcal{I}}_\alpha$ simply be the $\mathbb{P}_\alpha$-name for $\mathcal{A}^\perp$. If α has countable cofinality, then define $\dot{\mathcal{I}}_\alpha = \bigcup\{\dot{\mathcal{I}}_{\xi+1} \colon \xi < \alpha\}$ as required in item (11). These definitions ensure that items (2)–(11) in the definition of $\mathfrak{A}$ are fulfilled.

In the case that α is a successor, let $\alpha = \delta + 1$. We proceed exactly as in the successor case in the proof of Lemma 3.11. By property (7) of $\mathfrak{A}$, $\mathbb{P}_\alpha$ forces that $\dot{\mathcal{I}}_1$ is unbounded in $\dot{\mathcal{I}}_{\gamma+1}$ for all $\gamma < \delta$. In this case, $\mathcal{A}$ has a maximal element $\dot{a}_\alpha$ and we follow the steps in Lemma 3.11 to define a $\mathbb{P}_\alpha$-name of an ideal $\dot{\mathcal{I}}_\alpha$ that is forced to contain for each $\gamma < \delta$, $\dot{\mathcal{I}}_{\gamma+1}$ as an $\subset^*$-unbounded subset, and in such a way that every member is almost disjoint from $\dot{a}_\alpha$ and the ideal generated by $\{\dot{a}_\alpha\} \cup \dot{\mathcal{I}}_\alpha$ is a maximal ideal. This ensures that properties (2)–(11) of $\mathfrak{A}$ will hold in this successor case.

Now to finish the proof, we have to prove that $\mathbb{P}_\alpha$ forces that $Q(\mathcal{A}; \dot{\mathcal{I}}_\alpha)$ is ccc. By Lemma 3.13, it suffices to prove that $\mathbb{P}_\alpha$ forces that $\mathcal{A}$ is a pre-ccc sequence. To prove this we consider, as in the definition of pre-ccc, $\mathbb{P}_\alpha$-names $\{\dot{\gamma}_\xi : \xi \in \omega_1\}$ and $\{\dot{b}_\xi : \xi \in \omega_1\}$ such that there is some $p_0 \in \mathbb{P}_\alpha$ forcing that for each $\xi < \omega_1$, $\dot{\gamma}_\xi \in \alpha$, $\dot{b}_\xi \in \dot{\mathcal{A}}^\perp$ and $\dot{a}_{\gamma_\xi} \cap \dot{b}_\xi$ is empty.

Assume first that, by possibly passing to an extension of p_0 and to an uncountable subsequence, there is some $\beta < \alpha$ such that p_0 forces that $\dot{\gamma}_\xi < \beta$ for all $\xi < \omega_1$. For each $0 < \xi < \omega_1$ choose $p_\xi < p_0$, an integer m_ξ and a $\gamma_\xi < \beta$ so that $a_{\gamma_\xi} \setminus a_\beta \subset m_\xi$ and $p_\xi \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\gamma}_\xi = \gamma_\xi$ and $a_\beta \cap \dot{b}_\xi \subset m_\xi$". We may also assume that there is some $d_\xi \subset m_\xi$ such that $p_\xi \Vdash_{\mathbb{P}_\alpha}$ "$\dot{b}_\xi \cap m_\xi = d_\xi$". There is a pair m, d so that $S = \{\xi : m_\xi = m \text{ and } d_\xi = d\}$ is uncountable. Since $\mathbb{P}_\alpha$ is ccc, we may choose $\xi < \eta$ from S so that p_ξ and p_η are compatible. Let $q \in \mathbb{P}_\alpha$ be any common extension of p_ξ and p_η. Note that q forces that $\dot{a}_{\dot{\gamma}_\xi} \cup \dot{a}_{\dot{\gamma}_\eta}$ is contained in $a_\beta \cup m$ and that $(\dot{b}_\xi \cup \dot{b}_\eta) \setminus m$ is disjoint from a_β. Since q also forces that $\dot{b}_\xi \cap m = \dot{b}_\eta \cap m$, it should now be clear that q forces that $\dot{a}_{\dot{\gamma}_\xi} \cup \dot{a}_{\dot{\gamma}_\eta}$ is disjoint from $\dot{b}_\xi \cup \dot{b}_\eta$. This proves that p_0 does not force this to be a violation of the pre-ccc property.

Now we may suppose that p_0 forces that $\{\dot{\gamma}_\xi : \xi \in \omega_1\}$ is strictly increasing and cofinal in α. We may assume that p_0 decides the value, γ_0, of $\dot{\gamma}_0$. For each $\xi < \omega_1$, choose any $p_\xi < p_0$ that decides a value, γ_ξ, of $\dot{\gamma}_\xi$ and that $\dot{b}_\xi$ is a $\mathbb{P}_\beta$-name for some $\beta \in \operatorname{supp}(p_\xi)$. Let g be a continuous strictly increasing function from ω_1 into α with cofinal range. Since $\mathbb{P}_\alpha$ is ccc we have, for each $\delta \in \omega_1$, the set $\{\xi : \gamma_\xi < g(\delta)\}$ is countable. Therefore there is a cub $C \subset \omega_1$ such that for all $\delta \in C$, $g(\delta) \le \gamma_\delta$ and $g(\delta)$ is a limit ordinal with countable cofinality. We may also arrange that for each $\delta \in C$ and $\xi < \delta$, $\operatorname{supp}(p_\xi) \subset g(\delta)$.

For each $\delta \in C$, we may extend p_δ so as to ensure that each of $g(\delta)$ and γ_δ are in $\operatorname{supp}(p_\delta)$ and such that there is a $\beta \in \operatorname{supp}(p_\delta)$ such that $\dot{b}_\delta$ is a $\mathbb{P}_\beta$-name. We also extend each p_δ so that we can arrange a list of special properties (referred to as "determined" in many similar constructions). Specifically, for each $\beta \in \operatorname{supp}(p_\delta)$

(1) there are $F^\delta_\beta \in [\omega]^{<\aleph_0}$, $H^\delta_\beta \in [\beta]^{<\aleph_0}$, $\sigma^\delta_\beta : H^\delta_\beta \to \omega$, and a $\mathbb{P}_\beta$-name $\dot{b}^\delta_\beta$ such that $p_\delta \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$p_\delta(\beta) = (F^\delta_\beta, \sigma^\delta_\beta, \dot{b}^\delta_\beta)$";
(2) if $g(\delta) < \gamma_\delta$, then $g(\delta) \in H^\delta_{\gamma_\delta}$;
(3) $H^\delta_\beta \subset \operatorname{supp}(p_\delta)$;

(4) if β is a limit with countable cofinality, then there is a $\mu^\delta_\beta < \beta$ such that $\dot b^\delta_\beta$ is a $\mathbb{P}_{\mu^\delta_\beta}$-name and $\mathrm{supp}(p_\delta) \cap \beta \subset \mu^\delta_\beta$;

(5) if $\iota < \beta$ is in $\mathrm{supp}(p_\delta)$, then $H^\delta_\iota \supset H^\delta_\beta \cap \iota$.

Let us note that, by our assumption on C, we have defined $\mu^\delta_{g(\delta)} < g(\delta)$, as in (4) for all $\delta \in C$. By the pressing down lemma, there is a stationary set $S \subset C$ and a $\mu < \alpha$ such that $\mu^\delta_{g(\delta)} < \mu$ for all $\delta \in S$ and so $g(\delta)$ is the minimum element of $\mathrm{supp}(p_\delta) \setminus \mu$. Since $\mathbb{P}_{\mu+1}$ is ccc, it follows by a standard argument that there is a $\bar p \in \mathbb{P}_{\mu+1}$ with the property that for all cub $\widetilde C \subset C$, the set $\{p_\xi \restriction \mu \colon \xi \in S \cap \widetilde C\}$ is pre-dense below $\bar p$. Pick such a $\bar p$ and let $\bar p \in G_{\mu+1}$ be $\mathbb{P}_{\mu+1}$-generic filter. Now we work in $V[G_{\mu+1}]$. In effect, $\bar p$ has forced that $\{\xi \in S \colon p_\xi \restriction \mu \in G_{\mu+1}\}$ is stationary. By passing to a stationary subset S_1 of this set, we may also arrange that the values of the pair $\{F^\xi_{g(\xi)}, F^\xi_{\gamma_\xi}\}$ is the same for all $\xi \in S_1$. For all $\beta \in \mu+1$, we let a_β denote the valuation of $\dot a_\beta$ by $G_{\mu+1}$. By further shrinking S_1 we may suppose there is an $\overline{m} \in \omega$ and a $\bar b \subset \overline{m}$, satisfying that for all $\delta \in S_1$

(1) for all $\beta \in \mathrm{supp}(p_\delta)$, $F^\delta_\beta \subset \overline{m}$, and for all $\iota \in H^\delta_\beta$, $\sigma^\delta_\beta(\iota) < \overline{m}$;

(2) for all $\beta \in \mathrm{supp}(p_\delta) \cap \mu$, $a_\iota \setminus a_\mu \subset \overline{m}$;

(3) $\bar b = \overline{m} \cap b^\delta_{g(\delta)}$, where $b^\delta_{g(\delta)}$ is the valuation of $\dot b^\delta_{g(\delta)}$ by $G_{\mu+1}$;

(4) $b^\delta_{g(\delta)} \cap a_\mu \subset \overline{m}$.

Fix any $\xi < \eta$ from S_1. Define q_ξ so that $\mathrm{supp}(q_\xi) = \mathrm{supp}(p_\xi) \setminus \mu + 1$, and for $\beta \in \mathrm{supp}(q_\xi)$,

$$q_\xi(\beta) = \begin{cases} (F^\xi_{g(\xi)}, \sigma^\xi_{g(\xi)} \cup \{(\mu, \overline{m})\}, \dot b^\xi_{g(\xi)} \cup \dot b^\eta_{g(\eta)}) & \text{if } \beta = g(\xi), \\ (F^\xi_\beta, \sigma^\xi_\beta, \dot b^\xi_\beta \cup b^\eta_{g(\eta)} \setminus \overline{m}) & \text{if } g(\xi) < \beta . \end{cases}$$

We prove by induction on $\beta \in \mathrm{supp}(q_\xi)$, that there is a condition $r^\xi_\beta \in G_{\mu+1}$ such that $r^\xi_\beta \cup (q_\xi \restriction \beta+1) \le p_\xi \restriction \beta+1$. Evidently, for the case $\beta = g(\xi)$, $F^\xi_{g(\xi)}$ and $\overline{m} \cap a_\iota$ are disjoint from $\bar b$ and so there is some condition in $G_{\mu+1}$ that forces that, they are disjoint from $\dot b^\eta_{g(\eta)}$. Similarly, for $\iota \in H^\xi_{g(\xi)}$, $a_\iota \setminus \overline{m} \subset a_\mu$, and since $a_\mu \setminus \overline{m} \cap (b^\xi_{g(\xi)} \cup b^\eta_{g(\eta)})$ is empty, there is a condition r in $G_{\mu+1}$ that forces that $q_\xi(g(\xi)) \in \dot Q_{g(\xi)}$ and that $q_\xi(g(\xi)) < p_\xi(g(\xi))$. In addition, $r \cup q_\xi \restriction g(\xi)+1$ forces that $\dot a_{g(\xi)}$ is disjoint from $b^\eta_{g(\eta)} \setminus \overline{m}$. Now, suppose that $g(\xi) < \beta \in \mathrm{supp}(p_\xi)$, and that $r \cup q \restriction \beta$ is a condition in $\mathbb{P}_\beta$ that is below $p_\xi \restriction \beta$. We recall that $H^\xi_\beta \subset \mathrm{supp}(p_\xi)$, and so it follows that $r \cup q_\xi \restriction \beta$ forces that $\dot a_\iota$ is disjoint from $b^\eta_{g(\eta)} \setminus \overline{m}$ for all $\iota \in H^\xi_\beta$. This is the only thing that needs verifying when checking that $r \cup q_\xi \restriction \beta+1 < p_\xi \restriction \beta+1$.

Now that we have that $r \cup q_\xi$ forces that $\dot a_{\gamma_\xi}$ is disjoint from $b^\eta_{g(\eta)} \setminus \overline{m}$, we can add $\{(\gamma_\xi, \overline{m})\}$ to $\sigma^\eta_{g(\eta)}$ and still have a condition. Similarly, for all $\iota \in \mathrm{supp}(p_\eta) \cap \mu$, $a_\iota \setminus \overline{m}$ is contained in a_μ, and $r \cup q_\xi$ forces that $a_\mu \setminus \overline{m}$, being a subset of $\dot a_{\gamma_\xi}$, is disjoint from $\dot b_\xi$. This implies that $r \cup q_\xi$ forces that $(F^\eta_{g(\eta)}, \sigma^\eta_{g(\eta)} \cup \{(\gamma_\xi, \overline{m})\}, b^\eta_{g(\eta)} \cup$

$(\dot{b}_\xi \setminus \overline{m}))$ is a condition in $\dot{Q}_{g(\eta)}$ and is less than $p_\eta(g(\eta))$. Now we define a condition q_η so that $\mathrm{supp}(q_\eta) = \mathrm{supp}(p_\eta) \setminus \mu + 1$, and for $\beta \in \mathrm{supp}(q_\eta)$,

$$q_\eta(\beta) = \begin{cases} (F^\eta_{g(\eta)}, \sigma^\eta_{g(\eta)} \cup \{(\gamma_\xi, \overline{m})\}, \dot{b}^\eta_{g(\eta)} \cup \dot{b}_{\gamma_\xi}) & \text{if } \beta = g(\xi), \\ (F^\eta_\beta, \sigma^\eta_\beta, \dot{b}^\eta_\beta \cup (\dot{b}_{\gamma_\xi} \setminus \overline{m})) & \text{if } g(\eta) < \beta \ . \end{cases}$$

It again follows, by induction on $\beta \in \mathrm{supp}(q_\eta)$, that $r \cup q_\xi$ forces that $r \cup q_\xi \cup (q_\eta \restriction \beta + 1)$ is a condition in $\mathbb{P}_{\beta+1}$ and is below $p_\eta \restriction \beta + 1$. Finally, we observe that $r \cup q_\xi \cup q_\eta$ forces that $\dot{a}_{\gamma_\xi} \subset \dot{a}_{\gamma_\eta}$ because it forces that $\dot{a}_{\gamma_\xi} \cap \overline{m} = \dot{a}_{\gamma_\eta} \cap \overline{m}$ and that $\dot{a}_{\gamma_\xi} \setminus \overline{m} \subset \dot{a}_{g(\eta)} \setminus \overline{m} \subset \dot{a}_{\gamma_\eta}$. Similarly $r \cup q_\xi \cup q_\eta$ forces that $\dot{a}_{\gamma_\eta}$ is disjoint from $\dot{b}_\xi$ because $\dot{b}_\xi \cap \overline{m} = \bar{b}$ and $\dot{a}_{\gamma_\eta}$ is disjoint from $\dot{b}_\xi \setminus \overline{m}$. This completes the proof that $\mathcal{A}$ is pre-ccc and the proof of the Lemma 3.16. □

This finishes the proof of the main theorem, Theorem 2.7. □

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(*Received* December 7, 2017*, revised* August 7, 2018)