Homogeneity and rigidity in Erdős spaces

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To the memory of Bohuslav Balcar

Abstract. The classical Erdős spaces are obtained as the subspaces of real separable Hilbert space consisting of the points with all coordinates rational or all coordinates irrational, respectively.

One can create variations by specifying in which set each coordinate is allowed to vary. We investigate the homogeneity of the resulting subspaces. Our two main results are: in case all coordinates are allowed to vary in the same set the subspace need not be homogeneous, and by specifying different sets for different coordinates it is possible to create a rigid subspace.

Keywords: Erdős space; homogeneity; rigidity; sphere

Classification: 54F99, 46A45, 54B05, 54D65, 54E50, 54F50

Introduction

We let l_2 denote real separable Hilbert space, that is

$$l_2 = \Bigl\{ \boldsymbol{x} \in \mathbb{R}^\omega \colon \sum_{i\in\omega} x_i^2 < \infty \Bigr\}.$$

In this paper we consider (topological) subspaces of l_2 that are obtained by taking a sequence $\boldsymbol{X} = \langle X_i \colon i \in \omega \rangle$ of subsets of $\mathbb{R}$ and then defining

$$E(\boldsymbol{X}) = \{\boldsymbol{x} \in l_2 \colon \forall i \;\; x_i \in X_i\}.$$

If all X_i are equal to one fixed set X we simply write $E(X)$. Since $E(\mathbb{R})$ is just l_2 itself we henceforth tacitly assume that $X \neq \mathbb{R}$ when we deal with a single set X.

These subspaces are generally known as Erdős spaces because Erdős showed that $E(S)$ and $E(\mathbb{Q})$ are natural examples of totally disconnected spaces of dimension one that are also homeomorphic to their own squares, see [4]. Here S denotes the convergent sequence $\{2^{-n} \colon n \in \omega\} \cup \{0\}$ and $\mathbb{Q}$ denotes the set of rational numbers. These two spaces have been the object of intense study, Chapter 2 of [1] summarizes much of the earlier history and contains references to, among others, a proof that $E(S)$ and $E(\mathbb{P})$ are homeomorphic, where $\mathbb{P}$ denotes the set of irrational numbers.

DOI 10.14712/1213-7243.2015.265

The purpose of this paper is to see what can be said of the spaces $E(X)$ and $E(\boldsymbol{X})$ in terms of homogeneity and rigidity. The standard examples $E(\mathbb{Q})$ and $E(\mathbb{P})$ are homogeneous, they are even (homeomorphic to) topological groups. Note that this shows that $E(S)$ is homogeneous, even though S is not of course.

The result of L. B. Lawrence from [6], that states that an infinite power of a zero-dimensional subspace of $\mathbb{R}$ is always homogeneous, might lead one to think that $E(X)$ is always homogeneous too. There are two important differences though: X^ω is a much larger subset of $\mathbb{R}^\omega$ than $E(X)$, and the topology of $E(X)$ is finer than the product topology. Our first result, in Section 1, shows that these differences present real obstacles: we construct a subset X of $\mathbb{R}$ such that all autohomeomorphisms of $E(X)$ are norm-preserving.

This of course raises the question whether this can be sharpened to: every autohomeomorphism of $E(X)$ must be norm-preserving and all spheres centered at the origin are homogeneous. We will comment on this after the construction.

In the case of a single set one can say for certain that $E(X)$ is not rigid: any permutation of ω induces an autohomeomorphism of $E(X)$. These are not the only 'easy' autohomeomorphisms of $E(X)$. Assume there is a real number r not in X such that both $X \cap (r, \infty)$ and $X \cap (-\infty, r)$ are nonempty and consider the clopen subset $C = \{\boldsymbol{x} \in E(X) \colon x_0, x_1 > r\}$ of $E(X)$. One can define $f\colon E(X) \to E(X)$ to be the identity outside C and have $f(\boldsymbol{x}) = (x_1, x_0, x_2, \ldots)$ if $\boldsymbol{x} \in C$. If, as is quite often the case, X has a dense complement in $\mathbb{R}$ then one can create many 'easy' autohomeomorphisms in this way and we are forced to conclude that the notion of a 'trivial' autohomeomorphism of $E(X)$ may be hard to pin down.

To obtain a truly rigid space of the form $E(\boldsymbol{X})$ one must have all sets X_i distinct for otherwise exchanging two coordinates would result in a nontrivial autohomeomorphism. In Section 2 we exhibit a sequence $\boldsymbol{X}$ for which $E(\boldsymbol{X})$ is rigid.

Our constructions use Sierpiński's method of killing homeomorphisms from [7], which in turn is based on Lavrentieff's theorem from [5]. The latter theorem states that a homeomorphism between two subsets, A and B, of completely metrizable spaces can be extended to a homeomorphism between G_δ-subsets, A^* and B^*, in those spaces that contain A and B, respectively (see also [3, Theorem 4.3.20]). It is well-known that a separable metric space, like l_2, contains continuum many G_δ-subsets and that each such set admits continuum many continuous functions into l_2. As will be seen below this will allow us to kill all unwanted homeomorphisms in a recursive construction of length $\mathfrak{c}$.

We shall conclude this note with some questions and suggestions for further research.

1. A non-homogeneous Erdős space

We shall show that there is a subset X of $\mathbb{R}$ for which $E(X)$ is not homogeneous. In fact our X will be such that the autohomeomorphisms of $E(X)$ must be norm-preserving. As observed in the introduction we cannot go all the way and make

$E(X)$ rigid: every permutation of ω induces a unitary operator on l_2 that maps $E(X)$ to itself. Thus, the autohomeomorphism group of $E(X)$ contains, at least, the symmetry group S_ω.

We shall construct a dense subset X of $\mathbb{R}$ in a recursion of length $\mathfrak{c}$. The set $E(X)$ will then be dense in l_2. If $f\colon E(X) \to E(X)$ is an autohomeomorphism then we can apply Lavrentieff's theorem to find a G_δ-set A that contains $E(X)$ and an autohomeomorphism $\bar{f}$ of A that extends f. By continuity the map $\bar{f}$ is norm-preserving if and only if f is. This tells us how we can ensure that $E(X)$ has norm-preserving autohomeomorphisms only: make sure that whenever A is a dense G_δ-subset of l_2 that contains $E(X)$ and $f\colon A \to A$ is an autohomeomorphism that is *not* norm-preserving then $E(X)$ is *not* invariant under f.

To make our construction run a bit smoother we note that it suffices to ensure that every autohomeomorphism $f\colon E(X) \to E(X)$ does not increase norms anywhere, that is, it satisfies $\|\boldsymbol{x}\| \geq \|f(\boldsymbol{x})\|$ for all $\boldsymbol{x}$. For if f is an autohomeomorphism then so is its inverse f^{-1} and from $(\forall \boldsymbol{x})\ (\|\boldsymbol{x}\| \geq \|f^{-1}(\boldsymbol{x})\|)$ we then deduce $(\forall \boldsymbol{x})\ (\|f(\boldsymbol{x})\| \geq \|\boldsymbol{x}\|)$.

We enumerate the set of pairs $\langle A, f\rangle$, where A is a dense G_δ-subset of l_2 and f is an autohomeomorphism of A that increases the norm somewhere as $\langle\langle A_\alpha, f_\alpha\rangle : \alpha < \mathfrak{c}\rangle$.

By transfinite recursion we build increasing sequences $\langle X_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle Y_\alpha : \alpha < \mathfrak{c}\rangle$ of subsets of $\mathbb{R}$ such that for all α

(1) $|X_\alpha \cup Y_\alpha| < \mathfrak{c}$,
(2) $X_\alpha \cap Y_\alpha = \emptyset$, and
(3) if $E(X_\alpha) \subseteq A_\alpha$ then there is a point $\boldsymbol{x}_\alpha$ in A_α such that $X_{\alpha+1}$ consists of X_α and the coordinates of $\boldsymbol{x}_\alpha$, and $Y_{\alpha+1}$ consists of Y_α and at least one coordinate of $f_\alpha(\boldsymbol{x}_\alpha)$.

To see that this suffices let $X = \bigcup_{\alpha<\mathfrak{c}} X_\alpha$ and assume f is an autohomeomorphism of $E(X)$ that increases the norm of at least one point. Apply Lavrentieff's theorem to extend f to an autohomeomorphism $\bar{f}$ of a G_δ-set A that contains $E(X)$. Then A is dense and $\bar{f}$ increases the norm of at least one point so there is an α such that $\langle A, \bar{f}\rangle = \langle A_\alpha, f_\alpha\rangle$. But now consider the point $\boldsymbol{x}_\alpha$. It belongs to $E(X_{\alpha+1})$ and hence to $E(X)$; on the other hand one of the coordinates of $f_\alpha(\boldsymbol{x}_\alpha)$ belongs to $Y_{\alpha+1}$ and it follows that $f_\alpha(\boldsymbol{x}_\alpha) \notin E(X)$ as $Y_{\alpha+1} \cap X = \emptyset$. This shows that $\bar{f}$ does not extend f, as $f(\boldsymbol{x}_\alpha)$ must be in $E(X)$, which is a contradiction.

To start the construction let $X_0 = \mathbb{Q}$, to ensure density of $E(X)$, and $Y_0 = \emptyset$.

At limit stages we take unions, so it remains to show what to do at successor stages.

To avoid having to carry the index α around all the time we formulate the successor step as the following lemma, in which Z plays the role of the union $X_\alpha \cup Y_\alpha$.

Lemma 1.1. *Let A be a dense G_δ-subset of l_2 and let $f\colon A \to A$ be an autohomeomorphism that increases the norm of at least one point. Furthermore let Z be a subset of $\mathbb{R}$ of cardinality less than $\mathfrak{c}$. Then there is a point $\boldsymbol{x}$ in A such that*

(1) *none of the coordinates of $\boldsymbol{x}$ and $f(\boldsymbol{x})$ are in Z, and*
(2) *at least one coordinate of $f(\boldsymbol{x})$ is not among the coordinates of $\boldsymbol{x}$ itself.*

Proof: We take $\boldsymbol{a} \in A$ such that $\|\boldsymbol{a}\| < \|f(\boldsymbol{a})\|$.

First we show that we can assume, without loss of generality, that $\boldsymbol{a}$ has two additional properties: 1) all coordinates of $f(\boldsymbol{a})$ are nonzero, and 2) all coordinates of $f(\boldsymbol{a})$ are distinct.

This follows from the fact that the following sets are closed and nowhere dense in l_2:

(1) $\{\boldsymbol{x} \in l_2 \colon x_i = 0\}$ for every i, and
(2) $\{\boldsymbol{x} \in l_2 \colon x_i \neq x_j\}$ whenever $i < j$.

By continuity and because A is a dense G_δ-subset of l_2 we may choose $\boldsymbol{a}$ so that $f(\boldsymbol{a})$ is not in any one of these sets.

We claim that there is an $i \in \omega$ such that $f(\boldsymbol{a})_i \neq a_j$ for all j. If not then we can choose for each i the smallest k_i such that $f(\boldsymbol{a})_i = a_{k_i}$. Because all coordinates of $f(\boldsymbol{a})$ are distinct the map $i \mapsto k_i$ must be injective. But then

$$\sum_{i=0}^{\infty} f(\boldsymbol{a})_i^2 = \sum_{i=0}^{\infty} a_{k_i}^2 \leq \sum_{j=0}^{\infty} a_j^2,$$

which contradicts our assumption that $\|f(\boldsymbol{a})\| > \|\boldsymbol{a}\|$.

Fix an i as above. Since $\lim_j a_j = 0$, and $f(\boldsymbol{a})_i \neq 0$ and $f(\boldsymbol{a})_i \neq a_j$ for all j, we can take $\varepsilon > 0$ such that $|f(\boldsymbol{a})_i - a_j| \geq 3\varepsilon$ for all j.

By continuity we can take $\delta > 0$ such that $\delta \leq \varepsilon$ and such that $\|\boldsymbol{x} - \boldsymbol{a}\| < \delta$ implies $\|f(\boldsymbol{x}) - f(\boldsymbol{a})\| < \varepsilon$.

By the triangle inequality we have $|f(\boldsymbol{x})_i - x_j| \geq \varepsilon$ for all j when $\|\boldsymbol{x} - \boldsymbol{a}\| < \delta$.

Now we apply Lemma 4.2 from [2]. The conditions of this lemma are that we have a separable completely metrizable space, for this we take $M = B(\boldsymbol{a}, \delta) \cap A$. Next we need a family of countably many continuous functions to one space, for this we take the coordinate maps $\pi_j \colon \boldsymbol{x} \mapsto x_j$ and their compositions with f, that is $\varrho_j \colon \boldsymbol{x} \mapsto f(\boldsymbol{x})_j$; the codomain is the real line $\mathbb{R}$. Finally, we need to know that whenever $C \subseteq \mathbb{R}$ is countable the complement of

$$\bigcup_{j\in\omega} (\pi_j^{\leftarrow}[C] \cup \varrho_j^{\leftarrow}[C])$$

in M is not countable. This is true because the preimages of points under the π_j and the ϱ_j are nowhere dense, so that the complement is a dense G_δ-subset of M.

The conclusion then is that there is a (copy of the) Cantor set K inside $B(\boldsymbol{a}, \delta) \cap A$ such that all maps π_j and ϱ_j are injective.

Because $|K| = \mathfrak{c}$ this then yields many points $\boldsymbol{x} \in K$ such that $x_j, f(\boldsymbol{x})_j \notin Z$ for all j. All these points are as required. $\square$

Remark 1.2. One would like to make this example as sharp as possible, for example by making all spheres centered at the origin homogeneous. This seems harder than one might think at first, as some straightforward modifications of the construction in this section will aft gang agley.

One might try to add some unitary operators as autohomeomorphisms of $E(X)$. To keep the sets X_α and Y_α small these should introduce as few new coordinates as possible. But even a simple transformation of the first two coordinates as given by $u_0 = \frac{3}{5}x_0 + \frac{4}{5}x_1$ and $u_1 = -\frac{4}{5}x_0 + \frac{3}{5}x_1$ is potentially quite dangerous. For if $x_0 = x_1$ then $u_0 = \frac{9}{5}x_0$ and $u_1 = -\frac{1}{5}x_0$. This shows that if $E(X)$ is to be invariant under just this operator then the set X itself must be invariant under scaling by $\frac{9}{5}$ and $-\frac{1}{5}$. This would introduce norm-changing autohomeomorphisms of $E(X)$.

Another possibility would be to make $E(X)$ invariant under standard reflections: if $\|\boldsymbol{a}\| = \|\boldsymbol{b}\|$ then

$$R(\boldsymbol{x}) = \boldsymbol{x} - 2\frac{\boldsymbol{x}\cdot(\boldsymbol{a}-\boldsymbol{b})}{(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})}(\boldsymbol{a}-\boldsymbol{b})$$

defines the reflection in the perpendicular bisecting hyperplane of $\boldsymbol{a}$ and $\boldsymbol{b}$. Unfortunately this would mean that as soon as X_α is infinite and dense there would be $\mathfrak{c}$ many of these maps, and hence $\mathfrak{c}$ many coordinates to avoid. This would make it quite difficult to keep the sets X and Y above disjoint.

Remark 1.3. It is relatively easy to create situations where some spheres are not homogeneous. Simply take a set X that has 0 as an element and as an accumulation point, and an isolated point x, e.g., the convergent sequence S mentioned in the introduction. In $E(X)$ the sphere $H = \{\boldsymbol{y} \in E(X)\colon \|\boldsymbol{y}\| = |x|\}$ is not homogeneous.

Indeed, the point $\boldsymbol{x} = \langle x, 0, 0, \ldots\rangle$ is isolated in H. To see this take $\varepsilon > 0$ such that $\{x\} = X \cap (x-\varepsilon, x+\varepsilon)$ and consider any $\boldsymbol{y} \in H \setminus \{\boldsymbol{x}\}$. Then $y_0 \neq x$ because $\sum_i y_i^2 = x^2$, hence $|y_0 - x| \geq \varepsilon$ and also $\|\boldsymbol{y} - \boldsymbol{x}\| \geq \varepsilon$.

On the other hand, using a nontrivial convergent sequence in X with limit 0 it is an elementary exercise to construct a nontrivial convergent sequence in H.

2. A rigid example

In this section we construct a rigid Erdős space. As noted before, in this case we need a sequence $\boldsymbol{X} = \langle X_i\colon i \in \omega\rangle$ of subsets of $\mathbb{R}$ simply because we need to disallow permutations of coordinates as autohomeomorphisms.

The construction is similar to, but easier than, that in Section 1. We list the set of pairs $\langle A, f\rangle$, where A is a dense G_δ-subset of l_2 and $f\colon A \to A$ is a homeomorphism that is not the identity, as $\langle\langle A_\alpha, f_\alpha\rangle\colon \alpha < \mathfrak{c}\rangle$.

We now build countably many increasing sequences $\langle X_{i,\alpha}\colon \alpha < \mathfrak{c}\rangle$ of subsets of $\mathbb{R}$, one for each i, and countably many auxiliary sequences $\langle Y_{i,\alpha}\colon \alpha < \mathfrak{c}\rangle$ such that $Y_{i,\alpha} \cap X_{i,\alpha} = \emptyset$ for all i and all α.

We start with a sequence $\langle X_{i,0}\colon i \in \omega\rangle$ of pairwise disjoint countable dense subsets of $\mathbb{R}$ and $Y_{i,0} = \emptyset$ for all i.

At limit stages we take unions and at a successor stage we consider $\boldsymbol{X}_\alpha = \langle X_{i,\alpha} \colon i \in \omega\rangle$, the corresponding Erdős space $E(\boldsymbol{X}_\alpha)$, and the pair $\langle A_\alpha, f_\alpha\rangle$. In case $E(\boldsymbol{X}_\alpha) \subseteq A_\alpha$ we take a point $\boldsymbol{a} \in A_\alpha$ such that $f_\alpha(\boldsymbol{a}) \neq \boldsymbol{a}$ and we fix a coordinate j such that $a_j \neq f_\alpha(\boldsymbol{a})_j$, let $\varepsilon = |a_j - f_\alpha(\boldsymbol{a})_j|/2$, and $\delta > 0$ such that $\|\boldsymbol{x} - \boldsymbol{a}\| < \delta$ implies $\|f_\alpha(\boldsymbol{x}) - f_\alpha(\boldsymbol{a})\| < \varepsilon$.

As in the previous section we apply [2, Lemma 4.2] to find a Cantor set in $B(\boldsymbol{a}, \delta) \cap A_\alpha$ on which all projections π_i and the composition $\pi_j \circ f_\alpha$ are injective. This shows us that we can assume that $\boldsymbol{a}$ is such that $a_i \notin Y_{i,\alpha}$ for all i and $f_\alpha(\boldsymbol{a})_j \notin X_{j,\alpha}$. We then put $X_{i,\alpha+1} = X_{i,\alpha} \cup \{a_i\}$ for all i, and $Y_{i,\alpha+1} = Y_{i,\alpha}$ for all $i \neq j$, and $Y_{j,\alpha+1} = Y_{j,\alpha} \cup \{f(\boldsymbol{a})_j\}$.

In the end we let $X_i = \bigcup_{\alpha<\mathfrak{c}} X_{i,\alpha}$.

As in the previous section if f is an autohomeomorphism of $E(\boldsymbol{X})$ that is not the identity then there is an α such that $E(\boldsymbol{X}) \subseteq A_\alpha$ and f_α extends f. However, for the point $\boldsymbol{a}$ chosen at that stage we have $\boldsymbol{a} \in E(\boldsymbol{X})$ and $f_\alpha(\boldsymbol{a}) \notin E(\boldsymbol{X})$.

3. Some questions

In this last section we formulate two questions that we deem of particular interest in the context of homogeneity and rigidity in Erdős spaces.

Question 3.1. Given a subset X of $\mathbb{R}$ with a dense complement, what are the 'trivial' autohomeomorphisms of $E(X)$?

This is, to some extent, a subjective question, but a first approximation of 'trivial' could be "describable without using special properties of X other than its having a dense complement". Since at least one $E(X)$ has norm-preserving autohomeomorphisms only we know that 'trivial' should imply that property.

Question 3.2. Is there a set X such that $E(X)$ has norm-preserving autohomeomorphisms only and such that all spheres centered at the origin are homogeneous?

Note that one can split the last condition into two possibilities: one can ask whether the spheres can be made homogeneous as spaces in their own right or whether one can use autohomeomorphisms of $E(X)$ to establish their homogeneity. The failed attempts described in Remark 1.2 were of the latter kind.

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(*Received* February 12, 2018, *revised* May 17, 2018)