Convergence and submeasures in Boolean algebras

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In memory of Bohuslav Balcar

Abstract. A Boolean algebra carries a strictly positive exhaustive submeasure if and only if it has a sequential topology that is uniformly Fréchet.

Keywords: Boolean algebra; exhaustive submeasure; sequential topology; uniformly Fréchet topology

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1. Introduction

In 1937, John von Neumann proposed the problem of characterizing the complete Boolean algebras that carry a strictly positive σ-additive measure (The Scottish Book [17], Problem 163). Such algebras satisfy the *countable chain condition* (ccc), that is, every subset of mutually disjoint elements is countable. In addition, von Neumann isolated a weak distributive law and asked whether these two properties are sufficient for the existence of measure.

The first major contribution to von Neumann's problem was given by D. Maharam in [16] who identified a part of the problem as a topological problem. She showed that a complete Boolean algebra carries a strictly positive continuous submeasure if and only if the sequential topology given by the algebraic convergence is metrizable. This gives a natural decomposition of the von Neumann problem into two parts:

I. Does every complete weakly distributive ccc Boolean algebra carry a strictly positive continuous submeasure?

II. Does the existence of a strictly positive continuous submeasure imply the existence of a strictly positive σ-additive measure?

Part II has become the well known and well-studied Control measure problem, see [12], in functional analysis and remained open until Talagrand's solution in [22] in 2008.

As for Part I, D. Maharam showed that a negative answer would follow from the existence of a Suslin algebra. Its existence is equivalent to the negation of Suslin's hypothesis in [20] from 1920 but was not known until 1967 (cf. [23], [9]). The existence of a Suslin algebra is not provable; it is only consistent with the

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axioms of set theory [19]. The work of B. Balcar, T. Jech and T. Pazák [3] in 2003 established that the positive answer to Part I of Maharam's problem is also consistent.

The work on Maharam's problem I followed in Maharam's footsteps and was carried out mostly by Bohuslav Balcar and his collaborators (cf. [1], [2], [3], [4], [5]), with additional contributions of S. Todorčević, see [24], [25]. The metrizability assumption of Maharam was successively weakened to "ccc and Hausdorff", see [1], "ccc, weakly distributive and G_δ", see [3], "weakly distributive and σ-finite cc", see [25], and "uniformly weakly distributive", see [4].

In this paper we generalize the topological characterization to finitely additive measures on Boolean algebras, when neither the algebra nor the measure are complete. As the Control measure problem has been solved by M. Talagrand, and there exists an exhaustive submeasure that is not equivalent to a measure, the goal is to characterize Boolean algebras that carry an exhaustive submeasure. We give a necessary and sufficient condition in Theorem 3.6.

Our investigations were inspired by the work of Bohuslav Balcar who advocated the use of sequential topology and who introduced me to Maharam's problems. This paper is dedicated to his memory.

2. Submeasures on Boolean algebras

A Boolean algebra is a set B with Boolean operations $a \vee b$ (join), $a \wedge b$ (meet), $-a$ (complement) and $a \triangle b$ (symmetric difference), the partial order $a \leq b$, and the smallest and greatest element, $\mathbf{0}$ and $\mathbf{1}$. To avoid trivialities we assume that B is infinite and atomless. We let $B^+ = B - \{\mathbf{0}\}$.

An *antichain* in B is a set $A \subset B$ whose any two elements a, b are *disjoint*, i.e. $a \wedge b = \mathbf{0}$. B satisfies the *countable chain condition* (ccc) if it has no uncountable antichains.

Definition 2.1. A *submeasure* on a Boolean algebra B is a real valued function m on B such that

(i) $m(\mathbf{0}) = 0$ and $m(\mathbf{1}) = 1$,
(ii) if $a \leq b$ then $m(a) \leq m(b)$,
(iii) $m(a \vee b) \leq m(a) + m(b)$.

A submeasure m is *strictly positive* if

(iv) $m(a) = 0$ only if $a = \mathbf{0}$.

A submeasure m is a *measure* if

(v) $m(a \vee b) = m(a) + m(b)$ whenever a and b are disjoint.

Following M. Talagrand in [21], a submeasure m is *exhaustive* if for every infinite antichain $\{a_n\}_n$, $\lim_n m(a_n) = 0$; m is *uniformly exhaustive* if for every $\varepsilon > 0$ there exists a K such that every antichain has at most K elements with $m(a) \geq \varepsilon$.

Note that some condition like "exhaustive" is needed to exclude trivial examples of submeasures like $m(a) = 1$ for all $a \neq \mathbf{0}$. Every measure is uniformly exhaustive, and uniformly exhaustive implies exhaustive.

If m is a strictly positive exhaustive submeasure, let for each $n \in \mathbb{N}$

$$C_n = \left\{a \in B \colon m(a) \geq \frac{1}{n}\right\}.$$

Then $C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots$ and $\bigcup_n C_n = B^+$, and for each n, every antichain in C_n is finite. If m is uniformly exhaustive then there exist K_n, $n \in \mathbb{N}$, such that for each n, every antichain in C_n has size at most K_n. We say that B is *σ-finite* cc or *σ-bounded* cc, respectively.

In [13], N. J. Kalton and J. W. Roberts proved that if m is a uniformly exhaustive submeasure then it is equivalent to a measure. This reduced the Control measure problem to the question whether there exists an exhaustive submeasure that is not uniformly exhaustive. Such a submeasure was constructed by M. Talagrand in [22].

Let m be a strictly positive exhaustive submeasure. Letting

$$\varrho(a, b) = m(a \triangle b),$$

we obtain a distance function that makes B a metric space. The metric ϱ defines a topology on B that is invariant under Boolean operations; in particular if U is an open neighborhood of $\mathbf{0}$ then $a \triangle U$ is an open neighborhood of a.

We shall isolate a property of topological Boolean algebras that is necessary and sufficient for the existence of an exhaustive submeasure.

3. Convergence ideals on Boolean algebras

We consider an abstract theory of convergent sequences in topological Boolean algebras. This is inspired by an earlier work of B. Balcar, e.g. [1] and [2].

Definition 3.1. Let B be a Boolean algebra. A *convergence ideal* on B is a set I of infinite sequences $\{a_n\}_n$ in B with the following properties:

(i) $\bigwedge_n a_n = \mathbf{0}$, i.e. there is no $a > \mathbf{0}$ such that $a \leq a_n$ for all n.
(ii) If $s \in I$ and if t is an infinite subsequence of s then $t \in I$.
(iii) If $\{a_n\}_n \in I$ and $b_n \leq a_n$ for all n then $\{b_n\}_n \in I$.
(iv) If $\{a_n\}_n \in I$ and $\{b_n\}_n \in I$ then $\{a_n \vee b_n\}_n \in I$.
(v) If $\{a_n\}_n$ is an infinite antichain then $\{a_n\}_n \in I$.

Given a convergence ideal I we write $\lim_n a_n = \mathbf{0}$ instead of $\{a_n\}_n \in I$ and say that $\{a_n\}$ converges to $\mathbf{0}$. We extend this to convergence to any a by

$$\lim a_n = a \qquad \text{whenever } \lim(a_n \triangle a) = \mathbf{0}.$$

This notion of convergence has the obvious properties like

- each a_n converges to at most one limit,

- $\circ$ each constant sequence converges to the constant,
- $\circ$ $\lim(a_n \vee b_n) = \lim a_n \vee \lim b_n$, etc.

Given a convergence ideal we define a topology on B as follows: a set A is *closed* if it contains the limits of all sequences in A. We call such a topology *sequential*.

Definition 3.2. A sequential topology on B is *Fréchet* if for every A, the closure of A is the set of all limits of sequences in A.

It should be clear that a sequential topology is Fréchet if and only if it has this property: For every A, if for every $k \in \mathbb{N}$, $\{a_n^k\}_n$ is a sequence in A with $\lim_n a_n^k = b_k$, and if $\lim_k b_k = c$, then there exists a sequence $\{c_k\}_k$ in A such that $\lim_k c_k = c$.

Lemma 3.3. *A sequential topology on B is Fréchet if and only if the convergence ideal has the diagonal property: if $\lim_n x_n^k = \mathbf{0}$ for all k, then there exists a function $F\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_k x_{F(k)}^k = \mathbf{0}$.*

Proof: Assume that I has the diagonal property and let a_n^k, b_k and c be such that $\lim_n a_n^k = b_k$ and $\lim_k b_k = c$. We shall find a function $F\colon \mathbb{N} \to \mathbb{N}$ such that $\lim_k a_{F(k)}^k = c$.

Let $x_n^k = a_n^k \mathbin{\triangle} b_k$; then $\lim_n x_n^k = \mathbf{0}$ for each k, and so there exists a function F such that $\lim_k x_{F(k)}^k = \mathbf{0}$. Now because

$$a_{F(k)}^k \mathbin{\triangle} c \le (a_{F(k)}^k \mathbin{\triangle} b_k) \vee (b_k \mathbin{\triangle} c),$$

we have

$$\lim_k (a_{F(k)}^k \mathbin{\triangle} c) \le \lim_k (a_{F(k)}^k \mathbin{\triangle} b_k) \vee \lim_k (b_k \mathbin{\triangle} c) = \mathbf{0}$$

and so $\lim_k a_{F(k)}^k = c$.

For the converse, assume that the sequential topology is Fréchet and let x_n^k be such that $\lim_n x_n^k = \mathbf{0}$ for each k. Let $A = \{a_k\}_k$ be an infinite antichain, and let for each n

$$y_n^k = x_n^0 \vee x_n^1 \vee \cdots \vee x_n^k \qquad \text{and} \qquad z_n^k = y_n^k \vee a_k.$$

For each k we have $\lim_n y_n^k = \mathbf{0}$ and so $\lim_n z_n^k = a_k$.

Let $Z = \{z_n^k\colon k, n \in \mathbb{N}\}$. Each a_k is in the closure of Z, and so is $\mathbf{0}$ since $\mathbf{0} = \lim_k a_k$. Because the topology is Fréchet, there is a sequence in Z with limit $\mathbf{0}$; hence there is an infinite set $E \subset \mathbb{N} \times \mathbb{N}$ such that $\lim\{z_n^k\colon (k,n) \in E\} = \mathbf{0}$. For each k there are only finitely many n with $(k,n) \in E$; this is because $\lim_n z_n^k = a_k$. Hence there is an infinite subset M of $\mathbb{N}$ and a function G on M such that $\lim\{z_{G(m)}^m\colon m \in M\} = \mathbf{0}$. Let $M = \{m(0) < m(1) < \cdots < m(k) < \cdots : k \in \mathbb{N}\}$ and let $F(k) = G(m(k))$. Then we have for every k

$$x_{F(k)}^k \le y_{F(k)}^k \le y_{F(k)}^{m(k)} \le z_{G(m(k))}^{m(k)}$$

and so $\lim_k x_F^k(k) = \mathbf{0}$. $\square$

Definition 3.4. The sequential topology given by a convergence ideal I is *uniformly Fréchet* if there exist choice functions F_k acting on infinite sequences such that, whenever $\{x_n^k\}_n$, $k \in \mathbb{N}$, are sequences converging to $\mathbf{0}$, then

$$\lim_k F_k(\{x_n^k\}_n) = \mathbf{0}$$

(the kth term is the term of the kth sequence chosen by the kth function.)

This property is the uniform version of the diagonal property, that is, the element chosen from the kth sequence depends only on k and the sequence, not on the other sequences.

Proposition 3.5. *Let m be a strictly positive exhaustive submeasure on B, and let I be the ideal of all sequences $\{a_n\}_n$ in B such that $\lim_n m(a_n) = 0$. Then I is a convergence ideal and the topology is uniformly Fréchet.*

Proof: For $\{a_n\}_n \in I$, F_k chooses some a_n such that $m(a_n) < 1/k$. $\square$

Theorem 3.6. *A Boolean algebra B carries a strictly positive exhaustive submeasure if and only if it has a uniformly Fréchet convergence ideal.*

We prove the theorem in the next section.

4. Fragmentations

If m is a measure (or an exhaustive submeasure) on a Boolean algebra B, let for each n

$$C_n = \left\{a \colon m(a) \geq \frac{1}{n}\right\}.$$

Every antichain in every C_n is finite (or of bounded size if m is uniformly exhaustive) and so (in modern terminology), B is σ-finite cc, or σ-bounded cc. These properties were considered in [8] and [7]. More recently, approximations of B by a chain $\{C_n\}_n$ of sets with various properties have proved useful, see e.g. [4, page 260]. In Balcar's terminology, such approximations are called fragmentations.

Definition 4.1. A *fragmentation* of a Boolean algebra B is a sequence of subsets $C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots$ such that $\bigcup_n C_n = B^+$ and for every n, if $a \in C_n$ and $a \leq b$ then $b \in C_n$.

A fragmentation is *σ-finite* cc if for every n, every antichain in C_n is finite.

A fragmentation is *σ-bounded* cc if for every n there is a constant K_n such that every antichain $A \subset C_n$ has size less than or equal to K_n.

A fragmentation is *graded* if for every n, whenever $a \cup b \in C_n$ then either $a \in C_{n+1}$ or $b \in C_{n+1}$.

In this section we prove that if B has a uniformly Fréchet ideal I then B has a graded σ-finite cc fragmentation.

Let I be a uniformly Fréchet ideal and let F_k, $k \in \mathbb{N}$, be choice functions from Definition 3.4. For each n, let

$$V_n = \{a\colon a \le F_n(\vec{x}) \text{ for some } \vec{x} \in I\}, \qquad U_n = V_1 \cap \cdots \cap V_n, \qquad C_n = B - U_n.$$

We claim that $\{C_n\}_n$ is a fragmentation. It is enough to show that each $a \neq \mathbf{0}$ is in some C_n. If $a \notin C_n$ for all n, then for every n there exists some $\vec{x}^{\,n} \in I$ such that $a \le F_n(\vec{x}^{\,n})$. By the uniform diagonality, $\{F_n(\vec{x}^{\,n})\}_n \in I$ and it follows that $a = \mathbf{0}$.

A similar argument shows

Lemma 4.2. *If $\{a_n\}_n$ is such that $a_n \notin C_n$ for each n then $\{a_n\}_n \in I$.*

Proof: For each n there exists some $\vec{x}^{\,n} \in I$ such that $a_n \le F_n(\vec{x}^{\,n})$. Because $\{F_n(\vec{x}^{\,n})\}_n \in I$, we have $\{a_n\}_n \in I$. □

Lemma 4.3. *For every n, no $\vec{a} \in I$ is a subset of C_n.*

Proof: If some $\vec{a} \in I$ is a subset of C_n then for some $k \le n$, some infinite subsequence $\vec{b}$ of $\vec{a}$ is disjoint from V_k. But $F_k(\vec{b}) \in V_k$. □

Note that the fragmentation is σ-finite cc because every infinite antichain is in I.

Lemma 4.4. *For every n there exists a k such that $U_k \vee U_k \subset U_n$. Hence $\{C_n\}_n$ has a subfragmentation that is σ-finite* cc *and graded.*

Proof: Let $n \in \mathbb{N}$ and assume that for each k there exist a_k and b_k in U_k such that $c_k = a_k \vee b_k \in C_n$. By Lemma 4.2, $\{a_k\}_k$ and $\{b_k\}_k$ are in I and so $\{c_k\}_k \in I$. But $\{c_k\}_k \subset C_n$, a contradiction. □

5. Construction of an exhaustive submeasure

Let B be a Boolean algebra and assume that B has a graded σ-finite cc fragmentation $\{C_n\}_n$. We shall define a submeasure on B.

For each n let $U_n = B - C_n$ and $U_0 = B$; we have $U_0 \supset U_1 \supset \cdots \supset U_n \supset \cdots$ and $\bigcap_n U_n = \{\mathbf{0}\}$. For $X, Y \subset B$, let $X \vee Y$ denote the set $\{x \cup y\colon x \in X,\ y \in Y\}$. As the fragmentation is graded, we have

$$U_{n+1} \vee U_{n+1} \subset U_n$$

for all $n \ge 0$. Consequently, if $n_1 < \cdots < n_k$ then

$$U_{n_1+1} \vee \cdots \vee U_{n_k+1} \subset U_{n_1}.$$

This is proved by induction on k.

Let D be the set of all $r = \sum_{i=1}^{k} 1/2^{n_i}$ where $0 < n_1 < \cdots < n_k$. For each $r \in D$ we let $V_r = U_{n_1} \vee \cdots \vee U_{n_k}$ and $V_1 = U_0 = B$. For each $a \in B$ define

$$m(a) = \inf\{r \in D \cup \{1\}\colon a \in V_r\}.$$

Using the above property of the U_n, it follows that $V_r \subset V_s$ if $r < s$, and $V_r \vee V_s \subset V_{r+s}$ when $r + s \leq 1$. From this we have $m(a) \leq m(b)$ if $a \subset b$, and $m(a \cup b) \leq m(a) + m(b)$.

The submeasure m is strictly positive because if $a \neq \mathbf{0}$ then for some $n, a \notin U_n$, and hence $m(a) \geq 1/2^n$.

Finally, m is exhaustive: If A is an infinite antichain, then for each n only finitely many elements of A belong to C_n and hence only finitely many have $m(a) \geq 1/2^n$.

6. Measures and submeasures

Following D. Maharam's work [16], the early work on the von Neumann problem and the Control measure problem isolated the following classes of Boolean algebras: complete Boolean algebras that carry a σ-additive measure (measure algebras), complete Boolean algebras that carry a continuous submeasure (Maharam algebras), Boolean algebras that carry a finitely additive measure and Boolean algebras that carry exhaustive submeasure. (For a detailed analysis, see D. H. Fremlin's article [6] in the Handbook of Boolean algebras [18], and B. Balcar et al. [1].)

The Control measure problem was reduced to the question whether every exhaustive submeasure is equivalent to a finitely additive measure. The result of N. J. Kalton and J. W. Roberts in [13] shows that every uniformly exhaustive submeasure is equivalent to a finitely additive measure, and in 2007, M. Talagrand answered the question in [22] by constructing an exhaustive measure (on a countable Boolean algebra) that is not uniformly exhaustive.

Returning to the characterization of algebras with submeasure by convergence, we have the following:

Definition 6.1. Let I be a convergence ideal on a Boolean algebra B. The algebra B is *I-concentrated* if for every sequence $\{A_n\}_n$ of finite antichains with $|A_n| \geq n$ there exists a sequence $\{a_n\}_n \in I$ such that $a_n \in A_n$ for all n.

Theorem 6.2. *A Boolean algebra carries a strictly positive finitely additive measure if and only if it has a uniformly Fréchet convergence ideal I and is I-concentrated.*

Proof: Let $\{C_n\}_n$ be the fragmentation constructed in Section 4. We claim that each C_k has a bound on the size of antichains. Otherwise, let A_n be antichains in C_k of size at least n, and let $\{a_n\}_n \in I$ be such that $a_n \in A_n$. This contradicts Lemma 4.3.

It follows that B has a σ-bounded cc graded fragmentation. Using the Kalton–Roberts method, one can verify Kelley's condition, see [15], for the existence of a strictly positive measure; for details, see [11]. $\square$

We wish to mention a subtle point here. In the proof of Theorem 6.2 we use a graded σ-bounded cc fragmentation. It is not enough to have a σ-bounded cc algebra with a graded fragmentation because the metric completion of Talagrand's example is a Maharam algebra that does not carry a measure but is σ-bounded cc.

7. Continuity and weak distributivity

Let B be a complete Boolean algebra. The algebra B has a σ-additive measure if and only if it has finitely additive measure and is weakly distributive. This is explicitly stated in J. L. Kelley's work [15] but was already known to Pinsker, see [14]. Moreover, the proof of continuity is exactly the same when applied to submeasures, see D. H. Fremlin [6, page 946]:

Theorem 7.1. *Let B be a complete Boolean algebra.*

(a) *The algebra B carries a strictly positive σ-additive measure if and only if it is weakly distributive and carries a strictly positive finitely additive measure.*
(b) *The algebra B carries a strictly positive continuous submeasure if and only if it is weakly distributive and carries a strictly positive exhaustive submeasure.*

We note that (for ccc algebras) weak distributivity is equivalent to the Fréchet property for the algebraic convergence (and uniform weak distributivity is equivalent to uniformly Fréchet). Thus:

Theorem 7.2. *Let B be a complete Boolean algebra.*

(a) *The algebra B is a Maharam algebra if and only if it is uniformly weakly distributive, see* [4], [5].
(b) *The algebra B is a measure algebra if and only if it is uniformly weakly distributive and concentrated, see* [10].

8. Fréchet and uniformly Fréchet

The Fréchet property does not (provably) imply uniform Fréchet. This is because a Suslin algebra (the algebra obtained from a Suslin tree) is weakly distributive but not σ-finite cc. This counterexample is only consistent with ZFC (Zermelo–Fraenkel set theory with the axiom of choice), see [23], [9], and does not exist in some models of set theory, see [19]. Moreover, by [3] it is consistent that every weakly distributive ccc complete Boolean algebra is a Maharam algebra (and hence uniformly weakly distributive). This shows that Fréchet could imply uniformly Fréchet for the algebraic convergence but the proof does not seem to work in general. Thus we end with the following open problem:

Problem 8.1. Is there (in ZFC) a Boolean algebra with a convergence ideal I that is Fréchet but not uniformly Fréchet?

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