On maps preserving connectedness and/or compactness

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This paper is dedicated to the memory of Bohuslav Balcar

Abstract. We call a function $f\colon X \to Y$ P-preserving if, for every subspace $A \subset X$ with property P, its image $f(A)$ also has property P. Of course, all continuous maps are both compactness- and connectedness-preserving and the natural question about when the converse of this holds, i.e. under what conditions such a map is continuous, has a long history.

Our main result is that any nontrivial product function, i.e. one having at least two nonconstant factors, that has connected domain, T_1 range, and is connectedness-preserving must actually be continuous. The analogous statement badly fails if we replace in it the occurrences of "connected" by "compact". We also present, however, several interesting results and examples concerning maps that are compactness-preserving and/or continuum-preserving.

Keywords: compactness; connectedness; preserving compactness; preserving connectedness

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1. Introduction

Let P be any topological property and X, Y be topological spaces. We call a function $f\colon X \to Y$ P-*preserving* if the image $f(A)$ of every subspace $A \subset X$ with property P has property P as well. A *preserving* function is one that is both compactness- and connectedness-preserving. Of course, all continuous maps are preserving and the natural question about when the converse of this holds, i.e. under what conditions is a preserving map continuous, has been treated by numerous authors. More details about the history of this question can be found by J. Gerlits, I. Juhász, L. Soukup and Z. Szentmiklóssy in [1].

D. J. White in [6] proved that if the Tychonov space X is not locally connected at a point $p \in X$, then there exists a preserving function from X into the interval $\mathbb{I} = [0, 1]$ which is not continuous at p.

E. R. McMillan in [3] proved that if X is a locally connected Fréchet T_2-space, then every preserving map of X into a T_2-space is continuous. (The Hausdorffness of X is actually superfluous, as was shown in [1].)

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The main result of [1] is that if X is a product of connected linearly ordered spaces and Y is regular, then each preserving function from X to Y is continuous. The authors asked in Problem 4.1 whether every preserving function from a locally connected space that is either compact or sequential into a T_2-space is continuous. These basic problems are still unsolved.

In this paper we are interested in the question what happens when a nontrivial product of functions is preserving. We were surprised to see that just the preservation of connectedness for such a product function with a connected domain already yields its continuity. For product functions that preserve compactness this is not so, as we will demonstrate by several counterexamples.

We then proceed by considering functions that preserve compactness or compactness and continua, as well as products of such functions.

2. Connectedness-preserving product functions

Assume that $f_i\colon X_i \to Y_i$ is a function for every element i of an index set I. Then $f = \prod_{i\in I} f_i$ denotes the product of the functions $\{f_i\colon i \in I\}$. In other words, f maps the cartesian product $X = \prod_{i\in I} X_i$ to the product $Y = \prod_{i\in I} Y_i$ so that for every $x = \langle x_i\colon i \in I\rangle \in X$ we have $f(x) = \langle f_i(x_i)\colon i \in I\rangle$.

Before formulating and proving the main result of this section, we present a simple technical lemma about products of connected spaces.

Lemma 2.1. *Assume that X_i for $i \in I$ are connected topological spaces and $X = \prod_{i\in I} X_i$. As usual, $\pi_i\colon X \to X_i$ denotes the ith projection map for every $i \in I$. Assume, moreover, that we have $J \subset I$ with $|J| \geq 2$ and for every index $i \in J$ a fixed nonempty subset $A_i \subset X_i$. Then the set*

$$A = \bigcup_{i\in J} \pi_i^{-1}(A_i) \subset X$$

is connected.

PROOF: Clearly, for any index $i \in I$ and for any point $a \in Y_i$ the set $\pi_i^{-1}(a) \subset X$ is connected. Thus if we pick for each $i \in J$ a point $a_i \in A_i$ then the set

$$B = \{x \in X\colon \exists\, i \in J(\pi_i(x) = a_i)\} = \bigcup_{i\in J} \pi_i^{-1}(a_i)$$

is also connected because $\bigcap_{i\in J} \pi_i^{-1}(a_i) \neq \emptyset$. Obviously, this connected set B is included in A.

Now, for each $i \in J$ and $a \in A_i$ the connected set $\pi_i^{-1}(a) \subset A$ intersects $B \subset A$, as is witnessed by any point $x \in X$ that satisfies $\pi_i(x) = a$ and $\pi_j(x) = a_j$ for some $j \in J$ with $i \neq j$. (This is the point where we use $|J| \geq 2$.) But A is the union of all the connected sets $\pi_i^{-1}(a)$ with $i \in J$ and $a \in A_i$, consequently A is indeed connected. □

Now we are ready to present our main result.

Theorem 2.2. *Let X_i and Y_i be topological spaces for every $i \in I$, where every X_i is connected and every Y_i is T_1. Moreover, $f_i \colon X_i \to Y_i$ be a function for every $i \in I$ such that f_i is nonconstant for at least two different indices $i \in I$. Then the product function $f = \prod_{i\in I} f_i \colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ is connectedness-preserving if and only if f, or equivalently, every f_i is continuous.*

PROOF: Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For every $i \in I$, let $\pi_i \colon X \to X_i$ denote the projection as above, and ϱ_i be the ith projection from Y to Y_i. Of course, we may assume without any loss of generality that $|I| \geq 2$ and f_i is nonconstant for every $i \in I$. Obviously, we only have to prove that if f is connectedness-preserving then it is continuous.

So, assume that the function f is connectedness-preserving and take a point $x = \langle x_i \colon i \in I\rangle \in X$ and a set $S \subset X$ such that $x \in \overline{S}$. We have to show that then $f(x) \in \overline{f(S)}$. Assume, on the contrary, that $f(x) \notin \overline{f(S)}$, i.e. $f(x)$ has an open neighborhood U in Y that misses $f(S)$.

We may assume that $U = \bigcap_{i\in J} \varrho_i^{-1}(U_i)$ where J is a finite subset of I with $|J| \geq 2$ and U_i is an open proper subset of Y_i for each $i \in J$. The latter assumption is justified because for every $i \in J$ there is a point $z_i \in X_i$ with $f_i(z_i) \neq f_i(x_i)$ as f_i is nonconstant, and hence $Y_i \setminus \{f_i(z_i)\}$ is a neighborhood of $f_i(x_i)$ as Y_i is T_1.

We may then apply our Lemma 2.1 to the finite index set J and the nonempty sets $A_i = f_i^{-1}(Y_i \setminus U_i)$ to conclude that

$$A = \bigcup_{i\in J} \pi_i^{-1}(A_i) \subset X$$

is connected.

But then $f(S) \cap U = f(S) \cap \bigcap_{i\in J} \varrho_i^{-1}(U_i) = \emptyset$ clearly implies that $S \subset A$, consequently we have $x \in \overline{A}$, hence the set $A \cup \{x\}$ is connected as well. On the other hand, we have $U \cap f(A \cup \{x\}) = \{f(x)\}$, hence $f(x)$ is an isolated point of the non-singleton set $f(A \cup \{x\})$. So, $f(A \cup \{x\})$ is disconnected, contradicting our assumption that f is connectedness-preserving. $\square$

It is an immediate consequence of this theorem that if a function f with connected domain and T_1 range is connectedness-preserving but not continuous then its square $f^2 = f \times f$ cannot be connectedness-preserving. The following simple example illustrates this.

Example 2.3. Let $f \colon \mathbb{I} \to [-1, 1]$ be the function defined by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Then f is connectedness-preserving but f^2 is not.

It is very easy to check that f is connectedness-preserving and it is trivially not continuous. But a direct verification of the fact that f^2 is not connectedness-preserving requires some effort.

Of course, the condition in Theorem 2.2 that there are at least two factors on which the respective functions are nonconstant is essential. If all but one of them are constant then we are actually dealing with just one connectedness-preserving map and from that nothing new can be concluded.

Also, any map whose domain is hereditarily disconnected is trivially connectedness-preserving, hence the assumption in Theorem 2.2 that all the factor spaces X_i are connected is very natural.

3. Compactness-preserving functions

In this section we present several interesting examples of functions and product functions which, instead of connectedness preserving, are compactness-preserving. Let us note first that Theorem 2.2 badly fails if we replace in it both occurrences of "connected" by "compact". This is because clearly any function f with finite range is compactness-preserving, moreover so is any finite power f^n of it, hence any discontinuous map with finite range yields a counterexample.

We do have, however, a result that gives a simple condition under which compactness-preserving maps are automatically continuous.

Theorem 3.1. *Assume that $f\colon X \to Y$ is a compactness-preserving function with compact fibers such that both X and Y are T_2 and X is compact. Then f is continuous.*

Proof: Clearly, we may assume that f is a surjection, hence Y is also compact. Let U be any open subset of Y and assume that $f^{-1}(U)$ is not open in X. This means that there is a point $x \in f^{-1}(U)$ such that $x \in \overline{A}$ where $A = X \setminus f^{-1}(U)$.

Using that X is compact T_2, and hence T_4, the family $\mathscr{C}$ of all closed, hence compact, neighborhoods of the compact set $f^{-1}(f(x))$ in X satisfies $\bigcap \mathscr{C} = f^{-1}(f(x))$. But every neighborhood of $f^{-1}(f(x))$ is also a neighborhood of x, hence we have $C \cap A \neq \emptyset$ for all $C \in \mathscr{C}$. It follows that $\{f(C)\colon C \in \mathscr{C}\} \cup \{Y \setminus U\}$ is a centered collection of closed sets in Y, hence there is a point y in its intersection. But $f^{-1}(y) \subset X$ is also compact and clearly $f^{-1}(f(x)) \cap f^{-1}(y) = \emptyset$, hence there is some $C \in \mathscr{C}$ such that $C \cap f^{-1}(y) = \emptyset$ as well. But this implies $y \notin f(C)$, contradicting our choice of y and thus completing the proof. $\square$

We now present an example which, in analogy with Example 2.3, shows that the square of a compactness-preserving function may fail to be compactness-preserving.

Example 3.2. There is a compactness-preserving surjection $f\colon \mathbb{I} \to \omega{+}1$ such that its square $f^2\colon \mathbb{I}^2 \to (\omega{+}1)^2$ is not compactness-preserving.

Proof: Let $t_0 = 0$ and for every $n \geq 1$, let $t_n = 1 - 1/(n+1)$. In addition, for every $n < \omega$ let $D_n \subseteq [t_n, 1)$ be a countable dense set such that if $n \neq m$ then $D_n \cap D_m = \emptyset$. Define $f\colon \mathbb{I} \to \omega{+}1$, as follows:

$$f(s) = \begin{cases} n, & s \in D_n, \\ 0, & s \in [0,1) \setminus \bigcup_{n<\omega} D_n, \\ \omega, & s = 1. \end{cases}$$

We first show that f is compactness-preserving. To see this, let us note that any $C \subset \omega+1$ is compact if $\omega \in C$. Consequently, if K is a compact subset of $\mathbb{I}$ for which $1 \in K$ then $f(K)$ is compact. But if $1 \notin K$ then the compactness of K implies that K is bounded below 1, hence it may intersect only finitely many of the sets D_n. Thus, in this case $f(K)$ is finite.

We next will show that f^2 is not compactness-preserving. Pick a sequence $(d_n)_n$ in D_0 such that $d_n \to 1$. Moreover, for every n let $e_n \in D_n$ be arbitrary. Observe that the sequence $(e_n)_n$ converges to 1 as well. Hence the sequence $(d_n, e_n)_n$ converges to (1,1) in $\mathbb{I}^2$. Put $K = \{(d_n, e_n)\colon n < \omega\} \cup \{(1,1)\}$. Then K is compact, but the infinite set

$$f^2(K) \subset \{(0,n)\colon n \in \omega\} \cup \{(\omega,\omega)\}$$

is clearly not. □

We have noted above that any function $f\colon \mathbb{I} \to \mathbb{I}$ which is both connectedness- and compactness-preserving is continuous. So if we want to detect 'bad behavior' in such functions which are connectedness-preserving, we have to relax the compactness-preserving condition. It seems that the best we can hope for in this respect is to consider functions that are *continuum-preserving.* (That is, functions f having the property that the image under f of any continuum is a continuum.)

Clearly, any function $f\colon \mathbb{I} \to \mathbb{I}$ is connectedness-preserving if and only if the image of any interval is an interval, moreover it is continuum-preserving if and only if the image of any closed interval is a closed interval. As any subinterval of $\mathbb{I}$ is an increasing union of closed intervals, it is easy to show that every continuum-preserving function $f\colon \mathbb{I} \to \mathbb{I}$ is automatically connectedness-preserving. Our next example yields a function $f\colon \mathbb{I} \to \mathbb{I}$ that has the stronger of these two properties but its square misses even the weaker one.

Example 3.3. There is a function $f\colon \mathbb{I} \to \mathbb{I}$ that is continuum-preserving but for some arc $J \subseteq \mathbb{I}^2$ we have that $f^2(J)$ is not compact. Consequently, f^2 is not connectedness-preserving.

Proof: Let $\{(r_n, s_n)\colon n < \omega\}$ enumerate all nontrivial open intervals in $(0,1)$ with rational endpoints. For every n let $K_n \subseteq (r_n, s_n)$ be a Cantor set such that $K_n \cap K_m = \emptyset$ if $n \neq m$, moreover $f_n\colon K_n \to \mathbb{I}$ be a surjection. We then define $f\colon \mathbb{I} \to \mathbb{I}$, as follows:

$$f(s) = \begin{cases} f_n(s), & s \in K_n,\ n < \omega, \\ 0, & s \in \mathbb{I} \setminus \bigcup_{n<\omega} K_n. \end{cases}$$

Since every non-degenerate subinterval of $\mathbb{I}$ contains one of the K_n's it is mapped by f onto $\mathbb{I}$, hence f is trivially continuum-preserving.

Let us fix a strictly increasing sequence $(t_n)_{n<\omega}$ in $(0,1/2)$ converging to $1/2$. We shall construct a polygonal arc J in $[0,1/2]^2$ as the union of countably many vertical segments V_n and countably many horizontal segments H_n for $n < \omega$ and of a compactifying point as follows. The segment V_0 has the form $\{0\} \times [0, \alpha_0]$, where $\alpha_0 < 1/4$ and $f(\alpha_0) = t_0$. Since $f((0, 1/4)) = \mathbb{I}$, it is clear that such a value α_0 exists. Observe that $f^2(V_0) = \{0\} \times \mathbb{I}$. The segment H_0 has the form $[0, \beta_0] \times \{\alpha_0\}$, where $\beta_0 < 1/4$ and $f(\beta_0) = t_0$. Observe that $f^2(H_0) = \mathbb{I} \times \{t_0\}$. The segment V_1 has the form $\{\beta_0\} \times [\alpha_0, \alpha_1]$, where $\alpha_1 - \alpha_0 < 1/8$ and such that $f(\alpha_1) = t_1$. Observe that $f^2(V_1) = \{t_0\} \times \mathbb{I}$. The segment H_1 has the form $[\beta_0, \beta_1] \times \{\alpha_1\}$, where $\beta_1 - \beta_0 < 1/8$ and $f(\beta_1) = t_1$. Observe that $f^2(H_1) = \mathbb{I} \times \{t_1\}$.

It should be clear how to continue this for all $n < \omega$ to obtain the strictly increasing sequences of values α_n, β_n with $f(\alpha_n) = f(\beta_n) = t_n$ satisfying both $\alpha_{n+1} - \alpha_n < 1/2^{n+3}$ and $\beta_{n+1} - \beta_n < 1/2^{n+3}$, and then put $V_n = \{\beta_{n-1}\} \times [\alpha_{n-1}, \alpha_n]$ and $H_n = [\beta_{n-1}, \beta_n] \times \{\alpha_n\}$. It is also clear that we shall then have $f^2(V_n) = \{t_{n-1}\} \times \mathbb{I}$ and $f^2(H_n) = \mathbb{I} \times \{t_n\}$ for all n.

Let $\beta = \sup\{\beta_n : n < \omega\}$ and $\alpha = \sup\{\alpha_n : n < \omega\}$, respectively. Then clearly $\beta, \alpha < 1/2$, and the point (β, α) compactifies $\bigcup_{n<\omega} H_n \cup \bigcup_{n<\omega} V_n$, i.e.

$$J = \bigcup_{n<\omega} H_n \cup \bigcup_{n<\omega} V_n \cup \{(\beta, \alpha)\}$$

is an arc.

Its image $f(J)$ contains the points $\{(1, t_n) : n < \omega\}$ which converge to $(1, 1/2)$ and it also contains the points $\{(t_n, 1) : n < \omega\}$ which converge to $(1/2, 1)$. It is also clear from our construction that $\{(1, 1/2), (1/2, 1)\} \cap f(J \setminus \{(\beta, \alpha)\}) = \emptyset$. This, however, implies that $f^2(J)$ does not contain both $(1, 1/2)$ and $(1/2, 1)$ and therefore is not compact.

Consequently, f^2 is not continuum-preserving, and therefore it is not continuous. But then by Theorem 2.2 it is not connectedness-preserving either. □

Next we show, using a slight variation of Example 3.3, that a connectedness-preserving function $f : \mathbb{I} \to \mathbb{I}$ is not necessarily continuum-preserving.

Example 3.4. There is a function $f : \mathbb{I} \to \mathbb{I}$ which is connectedness-preserving but not continuum-preserving.

Proof: As in the proof of Example 3.3, let $\{K_n : n < \omega\}$ be a pairwise disjoint sequence of Cantor sets in $(0, 1)$ such that every non-degenerate interval in $\mathbb{I}$ contains one of them. Now, for every n let f_n be a surjection of K_n not onto $\mathbb{I}$ but onto the open interval $(0, 1)$, and then define $f : \mathbb{I} \to \mathbb{I}$ as follows:

$$f(s) = \begin{cases} f_n(s), & s \in K_n,\ n < \omega, \\ 0, & s = 0, \\ 1, & s = 1, \\ \frac{1}{2}, & s \in (0, 1) \setminus \bigcup_{n<\omega} K_n. \end{cases}$$

This function is connectedness-preserving for the same reason as the function from Example 3.3 is, but not continuum preserving since $f([0, 1/2]) = [0, 1)$. □

Our above observation that every continuum-preserving function $f\colon \mathbb{I} \to \mathbb{I}$ is automatically connectedness-preserving obviously generalizes to any map from one connected LOTS to another. As we may expect, this implication fails in general, even within the class of metric compacta. We present next two examples that demonstrate this.

Example 3.5. For every non-degenerate indecomposable metrizable continuum K there is a map $f\colon K \to \mathbb{I}$ which is continuum-preserving but not connectedness-preserving.

Proof: Let K be any non-degenerate indecomposable metrizable continuum (for example, the pseudo-arc). For every point $x \in K$, let K_x denote the composant of x in K. That is, K_x is the union of all proper subcontinua of K that contain x. It is known that every K_x is dense in K and that every K_x as well as the collection $\{K_x\colon x \in K\}$ has cardinality $\mathfrak{c}$. For details, see [5].

Pick two distinct points x and y in K. It is clear that there is a function $f\colon K \to \mathbb{I}$ which has the following properties:

(1) f is constant on each K_z and, in particular, $f(K_x) = \{0\}$ and $f(K_y) = \{1\}$,

(2) f is surjective.

Then f maps every proper subcontinuum of K to a single point, while $f(K) = \mathbb{I}$. On the other hand, it is clear from the density and connectivity of K_x that $K_x \cup K_y$ is connected as well, while it is mapped onto $\{0, 1\}$. Hence f is as required. □

Example 3.6. Let K be any non-degenerate metric continuum such that no countable subset of K separates K (for example, the square $\mathbb{I}^2$). Then there is a map $f\colon K \to \mathbb{I}$ which is continuum-preserving but not connectedness-preserving.

Proof: First split K into two sets A and B which meet every Cantor subset of K (a Bernstein decomposition). Then A is dense and connected. Dense is clear. If A were disconnected, we could split it into two disjoint nonempty relatively open sets U and V. Then there are open subsets E and F of K such that $E \cap A = U$ and $F \cap A = V$. Of course, E and F are disjoint because A is dense. But the closed set $S = K \setminus (E \cup F)$ separates K, and therefore has to be uncountable. Hence S contains a Cantor set, and so S meets A, a contradiction.

Now split A into two nonempty disjoint sets A_0 and A_1. Let $\mathscr{C}$ be the family of all Cantor sets in K. Then $C \cap B$ has size $\mathfrak{c}$ for every $C \in \mathscr{C}$. By Kuratowski's disjoint refinement theorem, then there is a family $\mathscr{S}$ consisting of $\mathfrak{c}$ many pairwise disjoint subsets of B, each of size $\mathfrak{c}$, such that every member of $\mathscr{C}$ includes an element of $\mathscr{S}$.

Now define the function $f\colon K \to \mathbb{I}$ by the following stipulations: $f(A_0) = \{0\}$, $f(A_1) = \{1\}$, and each member of $\mathscr{S}$ is mapped onto $\mathbb{I}$ by f. Then f is continuum-preserving since each non-degenerate subcontinuum of K contains a Cantor set and hence a member of $\mathscr{S}$. But f is not connectedness-preserving because $f(A) = \{0, 1\}$. □

Theorem 1.8 in [1] says that if X is a locally connected and Fréchet space then any preserving, i.e. both compactness- and connectedness-preserving map on X

into a T_2 space is continuous. Its proof is based on Lemma 1.7 in that paper which is easily seen to be true for continuum-preserving functions in the special case in which the domain of the map is also locally compact. Hence the following is true:

Theorem 3.7. *Assume that X is locally compact, locally connected and Fréchet, moreover Y is T_2. Then every map $f\colon X \to Y$ that is both continuum-preserving and compactness-preserving is continuous.*

Let us note that by the result of D. J. White in [6] mentioned in the introduction, the conclusion of Theorem 3.7 fails for any Tychonov space X that is not locally connected. Indeed, this is because any preserving map is trivially continuum-preserving.

Our last example shows that, at least under the Continuum hypothesis, we can even get a locally connected and compact T_2 space for which the conclusion of Theorem 3.7 fails. Of course, such a space cannot have the Fréchet property, and our example, not surprisingly, will be kind of "anti-Fréchet", namely it has no nontrivial convergent sequences.

Example 3.8 (Continuum hypothesis)**.** There are a locally connected Hausdorff continuum X and a function $f\colon X \to \mathbb{I}$ such that f is both compactness- and continuum-preserving but not connectedness-preserving.

Proof: Let X be the locally connected T_2 continuum of weight $\mathfrak{c}$ containing no convergent sequences that was constructed, using Continuum hypothesis, by J. van Mill in [4]. (Under the stronger assumption $\Diamond$, even a 1-dimensional, locally connected, and hereditarily separable continuum without convergent sequences was constructed by J. E. Hart and K. Kunen in [2].)

It is well-known that then every infinite compact subspace of X has cardinality $2^{\omega_1} = 2^{\mathfrak{c}}$ and there are at most $2^{\mathfrak{c}}$ such compacta since the weight of X is $\mathfrak{c}$. We can consequently partition X into a collection $\{A_t : t \in \mathbb{I}\}$ of sets A_t intersecting every infinite closed subset of X. (In fact, such a family can even be chosen to be of cardinality $2^{\mathfrak{c}}$ by Kuratowski's disjoint refinement theorem.) Let then $f\colon X \to \mathbb{I}$ be the function that sends every A_t onto the point t. Clearly, f sends every infinite compact subset of X onto $\mathbb{I}$, and hence is both compactness- and continuum-preserving. Since every nonempty open set in X includes an infinite compact subset, it is clear that every A_t is dense in X.

We claim that by the construction in [4] it follows that our space X cannot be separated by any finite subset (the same is true for the construction in [2]). But if this is true, then every dense set A_t intersects every closed set that separates X, and this means that A_t is connected. This implies then that $A_0 \cup A_1$ is connected, while its image $f(A_0 \cup A_1) = \{0, 1\}$ is not.

To verify our claim, let us assume that the space X in [4] has a cut point, say x. (The same proof will show that no finite set separates X.) We adopt the notation in [4]. Hence $X = M = M_{\omega_1}$ is the limit of the inverse system of length ω_1 presented in [4], in which M_α is the Hilbert cube for each $\alpha < \omega_1$.

Since all bonding maps of this inverse system are monotone, so is the projection $\pi = \pi_0^{\omega_1} \colon M \to M_0$.

Since x is a cut point of M, we may write $M \setminus \{x\}$ as $U \cup V$, where U and V are nonempty pairwise disjoint open sets. Then for every $q \in M_0 \setminus \{\pi(x)\}$ we have that $\pi^{-1}(q)$ is a continuum contained either in U or in V. So, if we put $E = \{q \in M_0 \setminus \{p\} \colon \pi^{-1}(q) \subseteq U\}$ and $F = \{q \in M_0 \setminus \{p\} \colon \pi^{-1}(q) \subseteq V\}$, respectively, then E and F would demonstrate that $\pi(x)$ is a cut point of the Hilbert cube M_0, which is absurd. □

We do not know if such an example exists in Zermelo–Fraenkel set theory with the axiom of choice (ZFC). In fact, even the existence in ZFC of a locally connected continuum with no nontrivial convergent sequences is an interesting open problem.

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