Coloring Cantor sets and resolvability of pseudocompact spaces

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Dedicated to the memory of our old friend Bohus Balcar

Abstract. Let us denote by $\Phi(\lambda,\mu)$ the statement that $\mathbb{B}(\lambda)=D(\lambda)^\omega$, i.e. the Baire space of weight λ, has a coloring with μ colors such that every homeomorphic copy of the Cantor set $\mathbb{C}$ in $\mathbb{B}(\lambda)$ picks up all the μ colors.

We call a space X π-regular if it is Hausdorff and for every nonempty open set U in X there is a nonempty open set V such that $\overline{V}\subset U$. We recall that a space X is called feebly compact if every locally finite collection of open sets in X is finite. A Tychonov space is pseudocompact if and only if it is feebly compact.

The main result of this paper is the following: Let X be a crowded feebly compact π-regular space and μ be a fixed (finite or infinite) cardinal. If $\Phi(\lambda,\mu)$ holds for all $\lambda<\hat{c}(X)$ then X is μ-resolvable, i.e. X contains μ pairwise disjoint dense subsets. (Here $\hat{c}(X)$ is the smallest cardinal κ such that X does not contain κ many pairwise disjoint open sets.)

This significantly improves earlier results of [van Mill J., *Every crowded pseudocompact ccc space is resolvable*, Topology Appl. 213 (2016), 127–134], or [Ortiz-Castillo Y. F., Tomita A. H., *Crowded pseudocompact Tychonoff spaces of cellularity at most the continuum are resolvable*, Conf. talk at Toposym 2016].

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1. Introduction

It is well-known that any compact Hausdorff space X is maximally resolvable, i.e. $\Delta(X)$-resolvable, where $\Delta(X)$ is the smallest cardinality of a nonempty open set in X. The question if this is also true for countably compact regular spaces, however, is completely open.

E. G. Pytkeev proved in [9] that crowded countably compact regular spaces are ω_1-resolvable and this result was extended in [4] to crowded countably compact π-regular spaces. The space X is *π-regular* if it is Hausdorff and for every nonempty open set U in X there is a nonempty open set V such that $\overline{V}\subset U$. The terminology is explained by the fact that the latter is equivalent to the condition that the

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regular closed sets form a π-network in X. Since any π-regular crowded countably compact space X satisfies $\Delta(X) \geq \mathfrak{c}$, the natural question was raised there if ω_1 can at least be replaced here by $\mathfrak{c}$ in ZFC (the acronym for Zermelo–Fraenkel set theory with the axiom of choice).

If one weakens countably compact to pseudocompact, an even tougher problem seems to arise: It is still unknown in ZFC whether all crowded pseudocompact spaces are at least (2-)resolvable. We recall that E. Hewitt in [2] defined pseudocompact spaces as those *Tychonov* spaces on which every continuous real valued function is bounded.

S. Mardešić and P. Papić defined in [5] *feebly compact* spaces by having the property that every locally finite collection of open sets in them is finite and showed that a Tychonov space is pseudocompact if and only if it is feebly compact. In his survey paper [8] on problems on resolvability, in addition to the problem if pseudocompact spaces are resolvable, O. Pavlov also asked if crowded regular feebly compact spaces are resolvable.

The first significant advance on the problem if pseudocompact spaces are resolvable was done by J. van Mill in [6] where it was shown that any crowded pseudocompact space X is $\mathfrak{c}$-resolvable, provided that $c(X) = \omega$, i.e. X is CCC (countable chain condition). Of course, this also means that crowded CCC countably compact Tychonov spaces are $\mathfrak{c}$-resolvable.

At the 2016 TOPOSYM in Prague, A. H. Ortiz-Castillo and Y. F. Tomita announced the following (partial) improvement of J. van Mill's result: Every crowded pseudocompact space X that satisfies $c(X) \leq \mathfrak{c}$ is resolvable, see [7]. (It is not clear from [7] if their proof actually gives $\mathfrak{c}$-resolvability of X.)

Our main result improves those of J. van Mill and A. H. Ortiz-Castillo and Y. F. Tomita and, in addition, also gives a partial affirmative answer to P. Pavlov's second question mentioned above. To help to formulate it, we introduce some notation and terminology.

We denote by $\mathbb{B}(\lambda)$ the ωth power of the discrete space on λ, i.e. the Baire space of weight λ. Moreover, given some cardinals λ and μ we denote by $\Phi(\lambda, \mu)$ the statement that $\mathbb{B}(\lambda)$ has a coloring with μ colors such that every homeomorphic copy of the Cantor set $\mathbb{C}$ in $\mathbb{B}(\lambda)$ is omnichromatic, i.e. picks up all the μ colors. Using the arrow notation of partition calculus, $\Phi(\lambda, \mu)$ can be written as $\mathbb{B}(\lambda) \nrightarrow [\mathbb{C}]^1_\mu$. Finally, we recall from [3] that for any space X we denote by $\hat{c}(X)$ the smallest cardinal κ such that X does not contain κ many pairwise disjoint open sets.

Theorem 1.1. *Let X be a crowded feebly compact π-regular space and μ be a fixed cardinal. If $\Phi(\lambda, \mu)$ holds for all $\lambda < \hat{c}(X)$ then X is μ-resolvable.*

Note that the statement $\Phi(\omega, 2)$ is just Bernstein's classical decomposition theorem, stated for $\mathbb{B}(\omega)$, the space of the irrationals rather than the reals. Extensions of Bernstein's theorem to colorings of wider classes of spaces and with $\mathfrak{c}$ colors were studied by a number of authors and these works culminated in some very strong and general results in [1]. Theorem 3.5 in [1] says that if $\varrho < \mathfrak{c}$ is a regular cardinal then for every topological space X of cardinality at most $\mathfrak{c}^{+\varrho}$ there is a coloring $c\colon X \to \mathfrak{c}$ such that all T_1 subspaces of X that admit a closed continuous map onto $\mathbb{C}$ are omnichromatic. But for every $n < \omega$ we clearly have $\left(\mathfrak{c}^{+n}\right)^{\omega} = \mathfrak{c}^{+n} < \mathfrak{c}^{+\omega}$, hence $\Phi(\lambda, \mathfrak{c})$ holds for all $\lambda < \mathfrak{c}^{+\varrho}$. Consequently any crowded feebly compact π-regular space is $\mathfrak{c}$-resolvable, provided that $c(X) < \mathfrak{c}^{+\omega}$.

The same theorem combined with Corollary 1.9 of [1] implies that *every* space X has a coloring $c\colon X \to \mathfrak{c}$ as above, provided that for every singular cardinal ν of countable cofinality both $\nu^{\omega} = \nu^{+}$ and $\Box_{\nu}$ are valid. (A slightly weaker result was obtained earlier by W. Weiss in [10].) Consequently, under these conditions $\Phi(\lambda, \mathfrak{c})$ holds for *all* cardinals λ. This in turn implies the consistency of the $\mathfrak{c}$-resolvability of *all* crowded feebly compact π-regular spaces, moreover that the consistency of the negation of this requires large cardinals.

Finally we mention the following unpublished result of W. Weiss (oral communication): If ω_1 Cohen reals are added to any ground model V, note that this forcing preserves the cardinal arithmetic of the ground model, then the generic extension $V^{Fn(\omega_1,2)}$ satisfies $\Phi(\lambda, \omega_1)$ for all cardinals λ. So, in these models all crowded feebly compact π-regular spaces are ω_1-resolvable.

2. The proof of Theorem 1.1

Let us fix the crowded feebly compact π-regular space X and the cardinal $\mu \leq \mathfrak{c}$ such that $\Phi(\lambda, \mu)$ holds for all $\lambda < \kappa = \hat{c}(X)$. Note that then κ is an uncountable regular cardinal, see e.g. 4.1 of [3]. The topology of X is denoted by τ and we write $\tau^{+} = \tau \setminus \{\emptyset\}$.

We call a sequence $\vec{U} = \langle U_n \colon n < \omega \rangle$ of members of τ^{+} *strongly decreasing* (in short: SD) if $\overline{U_{n+1}} \subset U_n$ holds for every $n < \omega$. We shall denote by $\mathrm{SD}(X)$ the family of all strongly decreasing sequences.

For $\vec{U} = \langle U_n \colon n < \omega \rangle \in \mathrm{SD}(X)$ we put $\cap\vec{U} = \bigcap\{U_n \colon n < \omega\}$. Clearly, $\cap\vec{U}$ is always closed in X, moreover it is nonempty because X is feebly compact, see Lemma 2.1 below. (It can be shown that the converse of this is also true if X is π-regular.) Finally, for any $\vec{U} \in \mathrm{SD}(X)$ we shall denote by $\partial\vec{U}$ the *boundary* of $\cap\vec{U}$.

The following simple lemma will play a crucial role in our proof.

Lemma 2.1. *Assume that $\vec{U} = \langle U_n \colon n < \omega \rangle \in \mathrm{SD}(X)$ and $V \in \tau$ are such that for every $n < \omega$ we have*

$$V \cap (U_n \setminus \cap\vec{U}) \neq \emptyset.$$

Then $\overline{V} \cap \partial\vec{U} \neq \emptyset$ as well.

Proof: Let us put $W_n = V \cap (U_n \setminus \cap\vec{U})$. Then $\langle W_n \colon n < \omega \rangle$ is a decreasing sequence of nonempty open sets in X, hence there is a point x with $x \in \overline{W_n}$ for infinitely many, hence all $n \in \omega$, because X is feebly compact. But then we have $x \in \overline{W_n} \subset \overline{V}$ and also

$$x \in \cap\vec{U} \setminus \operatorname{Int}(\cap\vec{U}) = \partial\vec{U}$$

because $W_n \cap \big(\cap\vec{U}\big) = \emptyset$ implies $\overline{W_n} \cap \operatorname{Int}(\cap\vec{U}) = \emptyset$. $\square$

Next, using that X is both crowded and π-regular, we fix for every open $U \in \tau^+$ a *maximal* family $\mathcal{S}(U) \subset \tau^+$ with $|\mathcal{S}(U)| > 1$ such that

(i) $V \in \mathcal{S}(U)$ implies $\overline{V} \subset U$;
(ii) if $V, W \in \mathcal{S}(U)$ are distinct then $\overline{V} \cap \overline{W} = \emptyset$.

Clearly then the union of $\mathcal{S}(U)$ is dense in U.

Using the operation $\mathcal{S}(U)$ we now define a tree T whose nodes are members of τ^+ and the tree ordering is reverse inclusion in the following natural way. The levels T_α are defined by transfinite recursion, starting with $T_0 = \{X\}$.

If T_α has been defined then we put

$$T_{\alpha+1} = \bigcup\{\mathcal{S}(U) \colon U \in T_\alpha\}$$

where, of course, the immediate successors of any $U \in T_\alpha$ are the members of $\mathcal{S}(U)$.

Finally, if α is a limit ordinal and $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$ has been defined then T_α consists of all sets of the form $\operatorname{Int}(\cap B)$, where B is any cofinal branch of $T \restriction \alpha$ such that $\operatorname{Int}(\cap B) \neq \emptyset$.

Let θ be the height of T, i.e. the smallest ordinal for which $T_\theta = \emptyset$. Clearly, θ is a limit ordinal. Note that every branch $B = \{U_\alpha \colon \alpha < \gamma\}$ of T (where, of course, $U_\alpha \in T_\alpha$) has cardinality less than κ because

$$\{U_\alpha \setminus \overline{U_{\alpha+1}} \colon \alpha < \gamma\} \subset \tau^+$$

is a pairwise disjoint collection of open sets. This implies that $\theta \leq \kappa$.

It is also obvious that every antichain of T, in particular every level T_α, also has cardinality less than κ, hence if $\theta = \kappa$ then T is a κ-Suslin tree, however we shall not need this fact.

Let us denote by E the set of all ordinals $\alpha \leq \theta$ with $\operatorname{cf}(\alpha) = \omega$. Then for every $\alpha \in E$ we may fix a strictly increasing sequence $\langle \alpha_n \colon n < \omega \rangle$ with $\sup\{\alpha_n \colon n < \omega\} = \alpha$. We then have

$$\kappa_\alpha = \sup\{|T_{\alpha_n}| \colon n < \omega\} < \kappa$$

because κ is an uncountable regular cardinal and every level of T has size less than κ.

Consequently, for every $\alpha \in E$ there is by our assumptions a coloring $b_\alpha \colon \mathbb{B}(\kappa_\alpha) \to \mu$ such that every homeomorphic copy of the Cantor set in $\mathbb{B}(\kappa_\alpha)$ picks up all colors $\nu \in \mu$.

To simplify our notation, using that $|T_{\alpha_n}| \leq \kappa_\alpha$, we consider each level T_{α_n} of T embedded in κ_α, hence their product $\prod_{n<\omega} T_{\alpha_n}$ is embedded in ${\kappa_\alpha}^\omega = \mathbb{B}(\kappa_\alpha)$.

Then it makes sense to put

$$\mathcal{Y}_\alpha = \mathrm{SD}(X) \cap \prod_{n<\omega} T_{\alpha_n}.$$

In other words, $\mathcal{Y}_\alpha$ consists of all strongly decreasing sequences $\vec{U} = \langle U_n : n < \omega\rangle \in \mathrm{SD}(X)$ for which there is a cofinal branch B of $T \restriction \alpha$ such that for every $n < \omega$ we have $B \cap T_{\alpha_n} = \{U_n\}$. Since $\mathcal{Y}_\alpha \subset \mathbb{B}(\kappa_\alpha)$, the coloring b_α is defined, in particular, on $\mathcal{Y}_\alpha$.

Next we move back to the space X and define

$$Y_\alpha = \bigcup\{\partial\vec{U} : \vec{U} \in \mathcal{Y}_\alpha\}.$$

It is clear from our construction of the tree T that for distinct $\vec{U}, \vec{V} \in \mathcal{Y}_\alpha$ we have $\partial\vec{U} \cap \partial\vec{V} = \emptyset$. Moreover, if α and β are distinct elements of E then $Y_\alpha \cap Y_\beta = \emptyset$ as well.

Thus we may define for each $\alpha \in E$ the coloring $h_\alpha : Y_\alpha \to \mu$ by putting $h_\alpha(y) = b_\alpha(\vec{U})$ for $y \in Y_\alpha$, where $\vec{U}$ is the unique element of $\mathcal{Y}_\alpha$ such that $y \in \partial\vec{U}$. Moreover, $h = \bigcup_{\alpha\in E} h_\alpha$ is a well-defined coloring of $Y = \bigcup_{\alpha\in E} Y_\alpha$ with μ colors. Our aim now is to show that h establishes the μ-resolvability of X because $h^{-1}\{\nu\}$ is dense in X for every $\nu < \mu$. Equivalently, this means that for every open set $U \in \tau^+$ we have $h[U] = \mu$. Actually, we shall prove the following stronger statement.

Lemma 2.2. *For every $U \in \tau^+$ there is an $\alpha \in E$ such that*

$$h_\alpha[U] = \mu.$$

Proof: Let us define for every open $U \in \tau^+$ the ordinal γ_U as follows:

$$\gamma_U = \min\{\gamma \le \theta : (\cup T_\gamma) \cap U = \emptyset\}.$$

The ordinal γ_U exists because $\cup T_\theta = \emptyset$, moreover it is clear from our construction of the tree T that γ_U is a limit ordinal.

Let us say that $W \in \tau^+$ is *good* if for every $V \in \tau^+$ with $V \subset W$ we have $\gamma_V = \gamma_W$. It is clear from our assumptions that for every $U \in \tau^+$ there is a good $W \in \tau^+$ such that $\overline{W} \subset U$. Consequently, it suffices to show that for every good $W \in \tau^+$ there is an $\alpha \in E$ such that $h_\alpha\big[\,\overline{W}\,\big] = \mu$.

So, consider any good $W \in \tau^+$ and put $\gamma_W = \gamma$. We are going to define a strictly increasing sequence of ordinals $\beta_n < \gamma$ and an injective map $f : 2^{<\omega} \to T \restriction \gamma$ such that for every $n < \omega$ and $s \in 2^n$ we have $f(s) \in T_{\beta_n}$ and $f(s) \cap W \ne \emptyset$. The ordinals β_n and the map f will be defined by recursion on $n < \omega$, using the following claim.

Claim 2.2.1. *For every $V \in \tau^+$ with $V \subset W$ there is an ordinal $\beta(V) < \gamma$ such that for any $\beta \in [\beta(V), \gamma)$ we have*

$$|\{U \in T_\beta : V \cap U \neq \emptyset\}| > 1.$$

Proof: Indeed, for every $\beta < \gamma$ there is some $U \in T_\beta$ such that $U \cap V \neq \emptyset$ because $\gamma = \gamma_V$. But if no other member of T_β intersects V then $V \subset \overline{U}$ since otherwise, as W is good, $\gamma_{(V \setminus \overline{U})} = \gamma$ would result in a contradiction. Thus if we had cofinally many $\beta < \gamma$ with a unique $U_\beta \in T_\beta$ intersecting V, then there would be a cofinal branch B of $T \restriction \gamma$ with $V \subset \cap B$, contradicting the choice of $\gamma_V = \gamma$. Thus Claim 2.2.1 has been proven. $\square$

Now we define the ordinals $\beta_n < \gamma$ and the injection $f\colon 2^{<\omega} \to T \restriction \gamma$ by recursion as follows. First we choose $\beta_0 < \gamma$ arbitrarily and then put $f(\emptyset) = U_0$ where U_0 is any member of T_{β_0} such that $W \cap U_0 \neq \emptyset$. This is possible because $\beta_0 < \gamma = \gamma_W$.

If we have already defined β_n and $f(s)$ for all $s \in 2^n$ appropriately, then we may apply Claim 2.2.1 to each nonempty open subset $W \cap f(s)$ of W with $s \in 2^n$ and choose $\beta_{n+1} < \gamma$ above β_n and the finitely many values $\beta(W \cap f(s))$ for $s \in 2^n$. Clearly, this recursion goes through and results in the desired sequence of ordinals β_n and injection f.

Now, we clearly have $\sup\{\beta_n : n < \omega\} = \alpha \in E$ and we claim that $h_\alpha[\,\overline{W}\,] = \mu$. To see this, note first that for every $x \in \mathbb{C} = 2^\omega$ the sequence $\langle f(x \restriction n) : n < \omega \rangle$ determines a cofinal branch B_x of the cut off tree $T \restriction \alpha$. This branch B_x then determines the sequence $\vec{U}_x = \langle U_{x,n} : n < \omega \rangle \in \mathcal{Y}_\alpha$, where $U_{x,n}$ is the unique element of B_x on the level T_{α_n}. The injectivity of f clearly implies that if $x \neq x'$ then $B_x \neq B_{x'}$.

In this way we also obtain a map $F\colon \mathbb{C} \to \mathcal{Y}_\alpha \subset \mathbb{B}(\kappa_\alpha)$ defined by $F(x) = \vec{U}_x$. This map F is clearly injective. Moreover, F considered as a map from $\mathbb{C}$ to $\mathbb{B}(\kappa_\alpha)$ is also uniformly continuous. Indeed, for every $n < \omega$ there is $m < \omega$ such that $\alpha_n \leq \beta_m$, and then for any $x, x' \in 2^\omega$ with $x \restriction m = x' \restriction m$ we clearly have

$$F(x) \restriction n = \vec{U}_x \restriction n = \vec{U}_{x'} \restriction n = F(x') \restriction n.$$

But this means that F actually is a homeomorphism, hence we have $b_\alpha[F[\mathbb{C}]] = \mu$.

For every $x \in \mathbb{C}$ we have $F(x) = \vec{U}_x \in \mathcal{Y}_\alpha \subset \mathrm{SD}(X)$ and by our construction, for every $n < \omega$ there are $m, k < \omega$ with $\alpha_k > \beta_m > \alpha_n$ and then $\emptyset \neq W \cap (U_{x,n} \setminus U_{x,k}) \subset W \cap (U_{x,n} \setminus \cap \vec{U}_x)$. Consequently we may apply Lemma 2.1 to conclude that $\overline{W} \cap \partial(\vec{U}_x) \neq \emptyset$. So, let us pick for each $x \in \mathbb{C}$ a point $y_x \in \overline{W} \cap \partial(\vec{U}_x)$ and recall that we have $h_\alpha(y_x) = b_\alpha(\vec{U}_x) = b_\alpha(F(x))$. Consequently, we indeed have $h_\alpha[\,\overline{W}\,] = b_\alpha[F[\mathbb{C}]] = \mu$. This completes the proof of Lemma 2.2 and with that the proof of Theorem 1.1. $\square$

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