Ultrafilter extensions of asymptotic density

Jan Grebík

In memory of Bohuslav Balcar

Abstract. We characterize for which ultrafilters on ω is the ultrafilter extension of the asymptotic density on natural numbers σ-additive on the quotient boolean algebra $\mathcal{P}(\omega)/d_{\mathcal{U}}$ or satisfies similar additive condition on $\mathcal{P}(\omega)/\mathrm{fin}$. These notions were defined in [Blass A., Frankiewicz R., Plebanek G., Ryll-Nardzewski C., *A Note on extensions of asymptotic density*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3313–3320] under the name $\boldsymbol{AP}$(null) and $\boldsymbol{AP}$(*). We also present a characterization of a P- and semiselective ultrafilters using the ultraproduct of σ-additive measures.

Keywords: asymptotic density; measure; ultrafilter; P-ultrafilter

Classification: 28A12, 03E05, 03E35, 11B05

This paper is based on the author's Bachelor thesis that was supervised by Bohuslav Balcar and defended in 2014. We investigate additive properties of measures on $\mathcal{P}(\omega)$ that are extensions of asymptotic density as defined in [2]. More concretely in Section 2 we give a necessary and sufficient combinatorial condition for an ultrafilter $\mathcal{U}$ on ω for the extension of asymptotic density given by $\mathcal{U}$ to satisfy $\boldsymbol{AP}$(null) or $\boldsymbol{AP}$(*). In Section 3 we characterize P- and semiselective ultrafilters by relations between some ideals in an ultraproduct of measures.

We note that since 2014 there has been made some progress in similar direction of density measures and additivity properties (see [4]).

1. Introduction

Let B be a boolean algebra and $m\colon B \to [0,1]$. We say that m is

- *monotone* if $m(a) \le m(b)$ whenever $a \le b \in B$;
- *strictly positive* if $m(a) = 0$ implies that $a = 0$;
- a *measure* if m is monotone, $m(1) = 1$ and $m\big(\bigvee_{i<n} a_i\big) = \sum_{i<n} m(a_i)$ for every finite antichain $\{a_i\}_{i<n} \subseteq B$;
- *σ-additive* if m is a measure and $m\big(\bigvee_{i<\omega} a_i\big) = \sum_{i<\omega} m(a_i)$ for every antichain $\{a_i\}_{i<\omega} \subseteq B$.

DOI 10.14712/1213-7243.2015.279
The author was supported by the GAČR project 17-33849L and RVO: 67985840.

If m is a measure on B, then define $\mathcal{N}(m) = \{a \in B \colon m(a) = 0\}$. The quotient boolean algebra $B/\mathcal{N}(m)$ carries a unique strictly positive measure that is naturally derived from m. We will abuse the notation and write B/m for the quotient algebra, m for the unique induced measure on B/m and $[a]$ for the equivalence class of $a \in B$. The following theorem is in fact a corollary of a stronger statement from [5] but this version is sufficient for our purposes. Recall that a boolean algebra B is σ-complete if every countable subset of B has a supremum in B.

Theorem 1.1 ([5])**.** *Let m be a measure on a σ-complete boolean algebra B. Then B/m is a countable chain condition (c.c.c.) complete boolean algebra.*

We use ω for the set of natural numbers. We write n for the set $\{0, 1, \dots, n-1\}$ and $[r, s]$ for the set $\{n \in \omega \colon r \leq n \leq s\}$ where $r, s \in \mathbb{R}$. Recall that a set $A \subset \omega$ has an asymptotic density if

$$\lim_{n \to \infty} \frac{|A \cap n|}{n}$$

exists, and in that case we denote the value of the limit as $d(A)$. We say that a measure m on $\mathcal{P}(\omega)$ is a *density* if it extends the asymptotic density, i.e. $m(A) = d(A)$ for every $A \subseteq \omega$ for which the asymptotic density exists. Note that a density m cannot be σ-additive on $\mathcal{P}(\omega)$ because it has the value 0 on each atom. Since the algebra $\mathcal{P}(\omega)/m$ is σ-complete by Theorem 1.1, it is natural to ask whether the density m is σ-additive on $\mathcal{P}(\omega)/m$. This question was considered in [2] where the authors define two additive properties for measures on $\mathcal{P}(\omega)$.

Definition 1.2 ([2])**.** A measure m on $\mathcal{P}(\omega)$ satisfies $\boldsymbol{AP}(\text{null})$ if for every inclusion increasing sequence $\{A_n\}_{n<\omega}$ of subsets of ω there is $B \subseteq \omega$ such that

- ∘ $\lim_{n\to\infty} m(A_n) = m(B)$;
- ∘ $m(A_n \setminus B) = 0$ for every $n < \omega$.

If we can moreover find such B that also satisfies

- ∘ $|A_n \setminus B| < \omega$ for every $n < \omega$,

then we say that m satisfies $\boldsymbol{AP}(*)$.

One can easily check that $\boldsymbol{AP}(\text{null})$ is equivalent with the σ-additivity of m on $\mathcal{P}(\omega)/m$. It is known (see [2]) that there are densities that satisfy $\boldsymbol{AP}(\text{null})$ but there are also densities that fail to have $\boldsymbol{AP}(\text{null})$. The question about $\boldsymbol{AP}(*)$ is more complicated since there is a model of Zermelo–Fraenkel set theory with the axiom of choice (ZFC) in which no density satisfies $\boldsymbol{AP}(*)$. On the other hand it is also consistent that densities satisfying $\boldsymbol{AP}(*)$ do exist, for example the existence of a P-ultrafilter is sufficient.

Definition 1.3. Let $\mathcal{U}$ be an ultrafilter on ω. Define

$$d_{\mathcal{U}}(A) = \mathcal{U}\text{-}\lim \frac{|A \cap n|}{n}$$

for every $A \subseteq \omega$.

We call densities of the form $d_{\mathcal{U}}$ ultrafilter densities. All examples presented in [2] are in fact ultrafilter densities. The aim of this paper is to give a complete combinatorial characterization of ultrafilters for which the ultrafilter densities satisfy $\boldsymbol{AP}(\text{null})$ or $\boldsymbol{AP}(*)$. Let us state here the case of $\boldsymbol{AP}(\text{null})$ and postpone the more technical case of $\boldsymbol{AP}(*)$ until the end of Section 2.

Definition 1.4. We say that an ultrafilter $\mathcal{U}$ on ω is *$\times$-invariant* if for all $U \in \mathcal{U}$ there is $1 < k \in \omega$ such that

$$kU = \bigcup_{n\in U} [kn, (k+1)n] \in \mathcal{U}.$$

The following is the main result of this paper and Section 2 is devoted to the proof of this statement.

Theorem 1.5. *Let $\mathcal{U}$ be an ultrafilter on ω. The following are equivalent*

- *$d_{\mathcal{U}}$ is σ-additive on $\mathcal{P}(\omega)/d_{\mathcal{U}}$ (i.e. satisfies $\boldsymbol{AP}(\text{null})$);*
- *$\mathcal{U}$ is not $\times$-invariant.*

2. Ultrafilter densities

In this section we present the proof of Theorem 1.5. We start with some general facts about ultrafilters on ω. All ultrafilters considered in this section are non-principal.

Claim 2.1. *Let $\mathcal{U}$ be a $\times$-invariant ultrafilter (see Definition 1.4). Then for every $U \in \mathcal{U}$ there are infinitely many $k < \omega$ such that*

$$kU = \bigcup_{n\in U} [kn, (k+1)n] \in \mathcal{U}.$$

Proof: Assume that for a given $U \in \mathcal{U}$ there is some maximal k such that $kU \in \mathcal{U}$. Then there must be some $2 \leq l < \omega$ such that

$$l(kU) = \bigcup_{m\in kU} [lm, (l+1)m] \subseteq \bigcup_{n\in U} [lkn, (l+1)(k+1)n] \in \mathcal{U}.$$

Because $\mathcal{U}$ is an ultrafilter, there must be some $p < \omega$ such that $lk \leq p \leq (l+1)(k+1) - 1$ and $pU \in \mathcal{U}$. Now $2k \leq lk \leq p$ contradicts the maximality of k. □

In order to prove our main result we need to investigate which ultrafilters give rise to the same ultrafilter densities.

Definition 2.2. Let $\mathcal{U}, \mathcal{V}$ be ultrafilters. We say that $\mathcal{U}$ *is close to* $\mathcal{V}$ if for every $U \in \mathcal{U}$ and for every $\varepsilon > 0$ there is $V \in \mathcal{V}$ such that

- for all $x \in U$ there is $y \in V$ such that $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \varepsilon$;
- for all $x \in V$ there is $y \in U$ such that $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \varepsilon$.

Claim 2.3. *Let $\mathcal{U},\mathcal{V}$ be ultrafilters. Then $\mathcal{U}$ is close to $\mathcal{V}$ if and only if*

$$U_\varepsilon = \left\{x<\omega\colon\ \exists n\in U \quad \max\left\{\left|1-\frac{n}{x}\right|,\left|1-\frac{x}{n}\right|\right\}<\varepsilon\right\}\in\mathcal{V}$$

for every $\varepsilon>0$.

Proposition 2.4. *The relation of being close is an equivalence relation on the set of ultrafilters.*

Proof: Suppose that $\mathcal{U}$ is close to $\mathcal{V}$ but $\mathcal{V}$ is not close to $\mathcal{U}$. Then there is $\delta>0$ and $V\in\mathcal{V}$ such that $V_\delta\notin\mathcal{U}$. Therefore $B=\omega\setminus V_\delta\in\mathcal{U}$. Then $B_\delta\cap V=\emptyset$ because if $x\in B_\delta\cap V$, then there exists $y\in B$ such that $\max\left\{\left|1-\frac{x}{y}\right|,\left|1-\frac{y}{x}\right|\right\}<\delta$ and also $x\in V$ implies $y\in\omega\setminus B$. Claim 2.3 gives us that $B_\delta\cap V=\emptyset\in\mathcal{V}$, a contradiction.

In order to prove that the relation is transitive first notice that

$$U_\varepsilon=\bigcup_{n\in U}\left[n(1-\varepsilon),\frac{n}{(1-\varepsilon)}\right].$$

Assume now that $\mathcal{U}$ is close to $\mathcal{V}$, $\mathcal{V}$ is close to $\mathcal{W}$ and take $U\in\mathcal{U}$. We know that $U_\varepsilon\in\mathcal{V}$ and $(U_\varepsilon)_\varepsilon\in\mathcal{W}$ but

$$U_{2\varepsilon-\varepsilon^2}=\bigcup_{n\in U}\left[n(1-\varepsilon)^2,\frac{n}{(1-\varepsilon)^2}\right]\supseteq(U_\varepsilon)_\varepsilon\in\mathcal{W}.$$

Since $\varepsilon>0$ was arbitrary we see that $\mathcal{U}$ is close to $\mathcal{W}$. □

Once we have established Proposition 2.4 we can write that a pair of ultrafilters $\mathcal{U},\mathcal{V}$ *is close* since the relation $\mathcal{U}$ is close to $\mathcal{V}$ is symmetric. Note also that $\mathcal{U},\mathcal{V}$ are close if and only if

$$\langle\{U_\varepsilon\colon U\in\mathcal{U},\ \varepsilon>0\}\rangle=\langle\{V_\varepsilon\colon V\in\mathcal{V},\ \varepsilon>0\}\rangle,$$

where $\langle\mathcal{A}\rangle$ denotes the filter generated by $\mathcal{A}\subseteq\mathcal{P}(\omega)$.

Theorem 2.5. *Let $\mathcal{U},\mathcal{V}$ be close ultrafilters. Then $d_\mathcal{U}=d_\mathcal{V}$ and $\mathcal{U}$ is $\times$-invariant if and only if $\mathcal{V}$ is $\times$-invariant.*

Proof: Let $A\subseteq\omega$ and $\varepsilon>0$ be given. Find a set $U\in\mathcal{U}$ such that

$$\left|d_\mathcal{U}(A)-\frac{|A\cap n|}{n}\right|<\varepsilon$$

holds for every $n\in U$. Since $\mathcal{U},\mathcal{V}$ are close, we have that $U_\varepsilon\in\mathcal{V}$. Let $x\in U_\varepsilon$ and $n\in U$ such that $\max\left\{\left|1-\frac{n}{x}\right|,\left|1-\frac{x}{n}\right|\right\}<\varepsilon$. We have

$$\left|d_\mathcal{U}(A)-\frac{|A\cap x|}{x}\right|\le\left|d_\mathcal{U}(A)-\frac{|A\cap n|}{n}\right|+\left|\frac{|A\cap n|}{n}-\frac{|A\cap x|}{x}\right|<3\varepsilon$$

because if for example $n \leq x$, then

$$\left|\frac{|A \cap n|}{n} - \frac{|A \cap x|}{x}\right| \leq \frac{|A \cap n|}{n}\left|1 - \frac{n}{x}\right| + \frac{x-n}{x} < \varepsilon + \varepsilon < 2\varepsilon.$$

We may conclude that $d_{\mathcal{V}}(A) = d_{\mathcal{U}}(A)$.

Next suppose that $\mathcal{U}$ is $\times$-invariant and let $V \in \mathcal{V}$ be given. We know from Claim 2.3 that $V_{1/4} = \big\{y\colon \exists\, n \in V \ \max\big\{\big|1 - \frac{n}{y}\big|, \big|1 - \frac{y}{n}\big|\big\} < \frac{1}{4}\big\} \in \mathcal{U}$. Therefore using Claim 2.1 there exists $4 \leq k < \omega$ such that $kV_\varepsilon \in \mathcal{U}$. We show that there are $\delta > 0$ and $3 \leq b < \omega$ such that

$$(kV_\varepsilon)_\delta \subseteq \bigcup_{n \in V} [2n, bn].$$

Once we have this the proof is finished because $(kV_\varepsilon)_\delta \in \mathcal{V}$. We describe how to find δ and $3 \leq b < \omega$. By a simple computation it follows that

$$V_\varepsilon = \bigcup_{n \in V} \Big[n\Big(1 - \frac{1}{4}\Big), \frac{n}{\big(1 - \frac{1}{4}\big)}\Big],$$

therefore

$$(kV_\varepsilon)_\delta = \bigcup_{n \in V} \Big[kn\Big(1 - \frac{1}{4}\Big)(1-\delta), \frac{(k+1)n}{\big(1 - \frac{1}{4}\big)(1-\delta)}\Big].$$

We see that if we choose $\delta < \frac{1}{3}$ and $b \geq \frac{(k+1)}{(1-1/4)(1-\delta)}$, we have the desired conclusion. $\square$

Next we show that close to any given ultrafilter there is a thin ultrafilter. Recall that an ultrafilter $\mathcal{V}$ is thin if

$$\inf_{V \in \mathcal{V}} \Big\{ \limsup_{n \to \infty} \frac{F_V(n)}{F_V(n+1)} \Big\} = 0,$$

where $F_A(n)$ is the nth element of A, i.e. F_A is the enumerating function of A. Note that an ultrafilter $\mathcal{V}$ is thin if and only if there is a set $V \in \mathcal{V}$ such that

$$\limsup_{n \to \infty} \frac{F_V(n)}{F_V(n+1)} < 1.$$

Denote $I_n = [2^n, 2^{n+1})$ for every $n < \omega$.

Proposition 2.6. *Let $\mathcal{U}$ be an ultrafilter. For every $\varepsilon, \delta > 0$ there is a set $U \in \mathcal{U}$ such that for every $x < y \in U$*

$$\frac{x}{y} < \varepsilon \qquad \text{or} \qquad \frac{x}{y} > 1 - \delta.$$

Proof: Let $\alpha\colon \omega \to \{0,1\}$. Inductively define intervals $I_n^{\alpha\restriction k}$ for $k \in \omega$ as

- $I_n^{\alpha\restriction 0} := I_n$;

- for $0 < k \leq n$ if $\alpha(k-1) = 0$ put $I_n^{\alpha\restriction k}$ to be the left half of the interval $I_n^{\alpha\restriction k-1}$;
- for $0 < k \leq n$ if $\alpha(k-1) = 1$ put $I_n^{\alpha\restriction k}$ to be the right half of the interval $I_n^{\alpha\restriction k-1}$;
- for $k > n$ put $I_n^{\alpha\restriction k} := I_n^{\alpha\restriction n}$.

There exists $\alpha_{\mathcal{U}} \colon \omega \to \{0,1\}$ such that for every $k \in \omega$

$$\bigcup_{n\in\omega} I_n^{\alpha_{\mathcal{U}}\restriction k} \in \mathcal{U}.$$

Let $x < y \in I_n^{\alpha_{\mathcal{U}}\restriction k}$. Since $|I_n^{\alpha_{\mathcal{U}}\restriction k}| = 2^{\max\{n-k,0\}}$ we have that

$$\frac{x}{y} > \frac{2^n}{2^n + |I_n^{\alpha_{\mathcal{U}}\restriction k}|} = \frac{2^n}{2^n + 2^{n-k}} = 1 - \frac{2^{n-k}}{2^n + 2^{n-k}} > 1 - \frac{1}{2^k}.$$

Finally it is enough to observe that for every $k < \omega$ and $\mathcal{U}$ there is $A \subseteq \omega$ such that $\bigcup_{n\in A} I_n \in \mathcal{U}$ and $(A + j) \cap A = \emptyset$ for every $j < k$. If $n < m \in A$, $x \in I_n$ and $y \in I_m$, then

$$\frac{x}{y} < \frac{2^{n+1}}{2^m} \leq \frac{2^{n+1}}{2^{n+k}} \leq \frac{1}{2^{k+1}}.$$

To finish the proof it is enough to combine the two estimates. □

We use the function $\alpha_{\mathcal{U}}$ that was defined in the proof of Proposition 2.6 for the next definition.

Definition 2.7. Let $\mathcal{U}$ be an ultrafilter on ω. Define the function $\alpha_{\mathcal{U}}$ as in the proof of Proposition 2.6. Let

$$A_{\mathcal{U}} = \bigcap_{k<\omega} \bigcup_{n<\omega} I_n^{\alpha_{\mathcal{U}}\restriction k}.$$

The ultrafilter $G(\mathcal{U})$ is defined by $U \in G(\mathcal{U})$ if

$$\bigcup \{I_n \colon I_n \cap U \cap A_{\mathcal{U}} \neq \emptyset\} \in \mathcal{U}.$$

Proposition 2.8. *Let $\mathcal{U}$ be an ultrafilter. Then $G(\mathcal{U})$ is a thin ultrafilter and $\mathcal{U}, G(\mathcal{U})$ are close.*

Proof: From the definition it follows that $G(\mathcal{U})$ is a non-principal ultrafilter and we have $\limsup_{n\to\infty} \frac{F_{A_{\mathcal{U}}}(n)}{F_{A_{\mathcal{U}}}(n+1)} < 1$. Since $A_{\mathcal{U}} \in G(\mathcal{U})$, it follows that $G(\mathcal{U})$ is thin.

Let $\varepsilon > 0$ and $V \in G(\mathcal{U})$ be given. We may assume that $V \subseteq A_{\mathcal{U}}$. Find $k < \omega$ such that $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \varepsilon$ for every $n < \omega$ and every $x, y \in I_n^{\alpha\restriction k}$. Then

$$V_\varepsilon \supseteq U = \bigcup \{I_n^{\alpha\restriction k} \colon V \cap I_n^{\alpha\restriction k} \neq \emptyset\} \in \mathcal{U}.$$

□

Corollary 2.9. *Let $\mathcal{U}$ be an ultrafilter. Then $d_{\mathcal{U}} = d_{G(\mathcal{U})}$ and $\mathcal{U}$ is $\times$-invariant if and only if $G(\mathcal{U})$ is $\times$-invariant.*

The last ingredient needed for the proof of Theorem 1.5 is the ultraproduct of measures. Let us define for a non-principal ultrafilter $\mathcal{U}$ a measure $m_{\mathcal{U}}$ on the set $\prod_{n\in\omega}\mathcal{P}(n)$ by putting

$$m_{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim_{n\to\infty}\frac{|f(n)|}{n},$$

i.e. we are taking the measure ultraproduct of the sequence $(\mathcal{P}(n))_{n<\omega}$ where each $\mathcal{P}(n)$ is endowed with the normalized counting measure. Next we consider the embedding $e\colon \mathcal{P}(\omega) \to \prod_{n\in\omega}\mathcal{P}(n)$ defined for $A \subseteq \omega$ as $e(A)(n) = A \cap n$. Immediately from the definitions we have $m_{\mathcal{U}}(e(A)) = d_{\mathcal{U}}(A)$. Therefore the embedding e lifts to the quotients, i.e.

$$e\colon \mathcal{P}(\omega)/d_{\mathcal{U}} \to \prod_{n\in\omega}\mathcal{P}(n)/m_{\mathcal{U}}.$$

It is well-known that the measure $m_{\mathcal{U}}$ on $\prod_{n\in\omega}\mathcal{P}(n)/m_{\mathcal{U}}$ is σ-additive (see [3]).

Proposition 2.10. *Let $\mathcal{U}$ be a thin ultrafilter. Then the density $d_{\mathcal{U}}$ is σ-additive if and only if the embedding e is isomorphism.*

Proof: Let $f \in \prod_{n\in\omega}\mathcal{P}(n)$ and $\varepsilon > 0$ be given. We show that there is $A \subseteq \omega$ such that $|m_{\mathcal{U}}(e(A)\triangle f)| < \varepsilon$. Because $\mathcal{U}$ is thin, there is $U \in \mathcal{U}$ such that

$$\frac{F_U(n)}{F_U(n+1)} < \varepsilon.$$

We define

$$A := \bigcup_{n<\omega}([F_{U(n)}, F_{U(n+1)}] \cap f(F_U(n+1))).$$

We have for every $n < \omega$ that

$$\Big|\frac{|(e(A)(F_U(n+1)))\triangle f(F_U(n+1))|}{F_U(n+1)}\Big| \le \frac{F_U(n)}{F_U(n+1)} < \varepsilon.$$

This implies that $e(\mathcal{P}(\omega)/d_{\mathcal{U}})$ is dense in $\prod_{n\in\omega}\mathcal{P}(n)/m_{\mathcal{U}}$, therefore $d_{\mathcal{U}}$ is σ-additive if and only if e is surjective. $\square$

We are now ready to prove our main result.

Proof of Theorem 1.5: Assume first that $\mathcal{U}$ is thin and not $\times$-invariant. We show that e is onto. Let $f \in \prod_{n\in\omega}\mathcal{P}(n)$. We find a set A, $A \subseteq \omega$, such that $|m_{\mathcal{U}}(e(A)\triangle f)| = 0$. Let $U \in \mathcal{U}$ such that for every $3 \le k < \omega$ is

$$U_k = \Big(\omega \setminus \bigcup_{n\in U}[2n, kn]\Big) \cap U \in \mathcal{U}$$

and $\frac{F_U(n)}{F_U(n+1)} < \frac{1}{2}$. Define

$$A = \bigcup_{n<\omega} ([F_U(n), F_U(n+1)] \cap f(F_U(n+1))).$$

Let $m \in U_k$. Choose the largest $n \in U$ such that $n < m$. Then by definition of U_k we have that $\frac{n}{m} < \frac{1}{k}$. Note that $m \in U$. Therefore by the definition of a set A we have the estimate

$$\frac{|e(A)(m) \triangle f(m)|}{m} \leq \frac{n}{m} < \frac{1}{k},$$

and the claim follows.

Assume on the other hand that $\mathcal{U}$ is thin and $\times$-invariant. There is a decreasing sequence $\{U_k\}_{k<\omega} \subseteq \mathcal{U}$ such that $\frac{F_{U_k}(n)}{F_{U_k}(n+1)} < \frac{1}{2^{k+1}}$. Define

$$A_k = \bigcup_{n \in U_k} \left[\frac{n}{2^{k+1}}, \frac{n}{2^k}\right].$$

We have $d_{\mathcal{U}}(A_k) < \frac{1}{2^k}$. Assume that there is $A \subseteq \omega$ such that $d_{\mathcal{U}}(A_k \setminus A) = 0$ and $d_{\mathcal{U}}(A) < \frac{1}{8}$ for every $3 < k < \omega$, i.e. A is a candidate for the upper bound of the sequence $\{A_k\}_{3<k<\omega}$. Let $U = \left\{n \colon \frac{|A \cap n|}{n} \leq \frac{1}{8}\right\}$. There must be $16 \leq l < \omega$ such that

$$W = \bigcup_{n \in U} [ln, (l+1)n] \in \mathcal{U}.$$

Consider now the smallest $k < \omega$ such that $l + 1 < 2^k$. Define $V = U_k \cap W \in \mathcal{U}$. Since for $n \in V$ there is $m \in U$ such that $lm \leq n \leq (l+1)m < 2^k m$ and $\left[\frac{n}{2^{k+1}}, \frac{n}{2^{k-1}}\right] \subseteq A_{k-1} \cup A_k$, we have

$$\frac{n}{2^{k+1}} \leq \frac{m}{2},\ m \leq \frac{n}{2^{k-1}}.$$

Therefore $\left[\frac{m}{2}, m\right] \subseteq A_{k-1} \cup A_k$. Since $m \in U$, we must have

$$\frac{|A \cap m|}{m} \leq \frac{1}{8},$$

and therefore

$$\left|\left[\frac{m}{2}, m\right] \setminus A\right| \leq \frac{3m}{8}.$$

Finally we can conclude that

$$\frac{|((A_{k-1} \cup A_k) \setminus A) \cap n|}{n} \geq \frac{3m}{8n} \geq \frac{3}{8(l+1)}$$

for $n \in V$. This is a contradiction with the properties of A. We conclude that there is no upper bound for $\{A_k\}_{3<k<\omega}$ such that its measure is less than $\frac{1}{8}$, consequently $d_{\mathcal{U}}$ is not σ-additive. $\square$

Corollary 2.11 ([2]). *Let $\mathcal{U}$ be an ultrafilter that contains a thin set, i.e. a set A such that $\lim_{n\to\infty} \frac{F_A(n)}{F_A(n+1)} = 0$. Then $d_{\mathcal{U}}$ satisfies $\boldsymbol{AP}$(null).*

An example of an ultrafilter $\mathcal{U}$ such that $d_{\mathcal{U}}$ does not satisfy $\boldsymbol{AP}$(null) was presented in [2] (the construction is due to D. H. Fremlin).

Our aim is now to characterize those ultrafilters $\mathcal{U}$ such that $d_{\mathcal{U}}$ satisfies $\boldsymbol{AP}(*)$. For that we need the following observation. Recall that an ultrafilter $\mathcal{U}$ is a *P-ultrafilter* if every decresaing sequence $\{U_i\}_{i<\omega} \subseteq \mathcal{U}$ has a pseudointersection $U \in \mathcal{U}$,

Proposition 2.12 ([2]). *Let $\mathcal{U}$ be an ultrafilter that contains a thin set. Then $d_{\mathcal{U}}$ has $\boldsymbol{AP}(*)$ if and only if $\mathcal{U}$ is a P-ultrafilter.*

Claim 2.13. *Let $\mathcal{U}$ be a thin P-ultrafilter. Then $\mathcal{U}$ contains a thin set.*

PROOF: Let $\{U_k\}_{k<\omega} \subseteq \mathcal{U}$ be a decreasing sequence such that $\frac{F_{U_k}(n)}{F_{U_k}(n+1)} < \frac{1}{k}$ for every $k < \omega$. Take the pseudointersection U of $\{U_k\}_{k<\omega}$. Then for every $k < \omega$ there is $n_0 < \omega$ such that for every $n > n_0$

$$\frac{F_U(n)}{F_U(n+1)} < \frac{1}{k}.$$

□

Proposition 2.14. *Let $\mathcal{U}$ be an ultrafilter. Then the following are equivalent*

- *$G(\mathcal{U})$ is a P-ultrafilter;*
- *$d_{\mathcal{U}}$ has $\boldsymbol{AP}(*)$.*

PROOF: Assume that $G(\mathcal{U})$ is a P-ultrafilter. By the Claim 2.13 it must contain a thin set and by Proposition 2.13 $d_{\mathcal{U}}$ has $\boldsymbol{AP}(*)$.

Assume that $d_{\mathcal{U}}$ has $\boldsymbol{AP}(*)$. Again by Proposition 2.13 it is enough to show that $G(\mathcal{U})$ contains a thin set. Fix a decreasing sequence $\{U_k\}_{k<\omega} \subseteq G(\mathcal{U})$ such that

$$\frac{F_{U_k}(n)}{F_{U_k}(n+1)} < \frac{1}{k+1}$$

and define

$$A_k = \bigcup_{n\in U_k} \left[\frac{n}{2}, n\right].$$

One can easily verify that a sequence $\{A_k\}_{k<\omega}$ is a decreasing sequence such that $\lim_{k\to\infty} d_{\mathcal{U}}(A_k) = \frac{1}{2}$. By the property $\boldsymbol{AP}(*)$ there is a set $A \subseteq \omega$ such that $|A \setminus A_k| < \omega$ and $d_{\mathcal{U}}(A) = \frac{1}{2}$ (here we use the property $\boldsymbol{AP}(*)$ for decreasing rather than increasing sequences). Define

$$U = \left\{n \in U_3 \colon \left[\frac{n}{2}, n\right] \cap A \neq \emptyset\right\}.$$

We must show that $U \in G(\mathcal{U})$ and U is thin. Assume that $U \notin G(\mathcal{U})$. Then $U_3 \setminus U \in G(\mathcal{U})$. For $n \in U_3 \setminus U$ we have

$$\frac{|A \cap n|}{n} \leq \frac{1}{4},$$

which is a contradiction with $d_{\mathcal{U}}(A) = \frac{1}{2}$. To prove that U is thin it is enough to observe that $|A \setminus A_k| < \omega$ implies $|U \setminus U_k| < \omega$. □

Definition 2.15. We say that ultrafilter $\mathcal{U}$ is *close to a P-ultrafilter* if for every decreasing sequence $\{U_k\}_{k \in \mathbb{N}} \subseteq \mathcal{U}$ and every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $|U \setminus (U_k)_\varepsilon| < \omega$ for all $k \in \mathbb{N}$.

Note that the ambiguity in the Definition 2.15 with respect to the Definition 2.2 is justified by the following claims. It follows that if $\mathcal{U}$ is close to a P-ultrafilter, then we can find a P-ultrafilter $\mathcal{V}$ such that $\mathcal{U}$ is close to $\mathcal{V}$, in particular we can take $\mathcal{V} = G(\mathcal{U})$.

Claim 2.16. *Let $\mathcal{U}$ be thin and close to a P-ultrafilter. Then $\mathcal{U}$ is a P-ultrafilter.*

Proof: Let $\{U_k\}_{k<\omega} \subseteq \mathcal{U}$ be a decreasing sequence. Assume that $\frac{F_{U_0}(n)}{F_{U_0}(n+1)} < \frac{1}{2}$. Find a pseudointersection U of $\{(U_k)_{1/4}\}_{k<\omega}$. We claim that $V = U \cap U_0$ is a pseudointersection of $\{U_k\}_{k<\omega}$. To see this fix $k < \omega$. We know that there is some m such that $U \setminus m \subseteq (U_k)_\varepsilon$. Let $x \in U_0 \cap (U \setminus m)$. There is $y \in (U_k)_{1/4}$ such that $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \frac{1}{4}$. Note that $y \in U_0$ because the sequence is decreasing. From the properties of U_0 we have that $x = y$. This implies that $V \setminus m \subseteq U_k$ which finishes the proof. □

Claim 2.17. *Let $\mathcal{U}, \mathcal{V}$ be close ultrafilters. Then $\mathcal{U}$ is close to a P-ultrafilter if and only if $\mathcal{V}$ is close to a P-ultrafilter.*

Proof: Assume that $\mathcal{U}, \mathcal{V}$ are close and $\mathcal{U}$ is close to a P-ultrafilter. Let $\varepsilon > 0$ and $\{V_k\}_{k<\omega} \subseteq \mathcal{V}$ are given. Choose $\delta_0, \delta_1, \delta_2 > 0$ such that $1 - \varepsilon < (1 - \delta_0) \times (1 - \delta_1)(1 - \delta_2)$. Then by simple computation we have for every $A \subseteq \omega$

$$\begin{aligned}((A_{\delta_0})_{\delta_1})_{\delta_2} &= \bigcup_{n \in A} \left[(1-\delta_0)(1-\delta_1)(1-\delta_2)n, \frac{n}{(1-\delta_0)(1-\delta_1)(1-\delta_2)}\right] \\ &\subseteq \bigcup_{n \in A} \left[(1-\varepsilon)n, \frac{n}{(1-\varepsilon)}\right] = A_\varepsilon.\end{aligned}$$

Because $\mathcal{U}, \mathcal{V}$ are close, we have $\{(V_k)_{\delta_0}\}_{k<\omega} \subseteq \mathcal{U}$. By the assumption on $\mathcal{U}$ there is a pseudointersection V of $\{((V_k)_{\delta_0})_{\delta_1}\}_{k<\omega}$. One can easily check that V_{δ_2} is a pseudointersection of $\{(((V_k)_{\delta_0})_{\delta_1})_{\delta_2}\}_{k<\omega}$. Since $\mathcal{U}, \mathcal{V}$ are close, $V_{\delta_2} \in \mathcal{V}$ and $\{(((V_k)_{\delta_0})_{\delta_1})_{\delta_2}\}_{k<\omega} \subseteq \mathcal{V}$. So V_{δ_2} is also a pseudointersection of $\{(V_k)_\varepsilon\}_{k<\omega} \subseteq \mathcal{V}$. □

Theorem 2.18. *An ultrafilter $\mathcal{U}$ is close to a P-ultrafilter if and only if $d_{\mathcal{U}}$ has* $\boldsymbol{AP}(*)$.

Proof: Combine Proposition 2.14, Claim 2.16 and Claim 2.17. □

Corollary 2.19. *There is a P-ultrafilter if and only if there exists a ultrafilter density that satisfies* $\boldsymbol{AP}(*)$.

Question 2.20. Does the existence of a density that satisfies $\boldsymbol{AP}(*)$ imply the existence of a P-ultrafilter?

3. Ultraproducts

In the last section we show how certain special properties of ultrafilters may affect properties of some ideals in the measure ultraproduct. Recall that for a sequence $(B_i, m_i)_{i<\omega}$ of σ-complete boolean algebras with measures (not necessarily strictly positive or σ-additive) and for $\mathcal{U}$ an ultrafilter on ω we define the ultraproduct measure $m_{\mathcal{U}}$ on $\prod_{i<\omega} B_i$ as

$$m_{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim m_i(f(i))$$

for $f \in \prod_{i<\omega} B_i$.

There are several natural ideals that one may assign to the product. In order to keep the presentation as straightforward as possible we make the assumption that $(B_i, m_i) = (B, m)$ for every $i < \omega$ where B is a σ-complete boolean algebra with a measure m. Given an ultrafilter $\mathcal{U}$ on ω we define

- $\mathcal{N}_{\mathcal{U}} = \{f \in B^\omega \colon m_{\mathcal{U}}(f) = 0\}$;
- $\mathcal{Z} = \{f \in B^\omega \colon \lim_{i<\omega} m(f(i)) = 0\}$;
- $\mathcal{M}_{\mathcal{U}} = \{f \in B^\omega \colon \{i \colon m(f(i)) = 0\} \in \mathcal{U}\}$;
- $\mathcal{I}_{\mathcal{U}} = \big\{f \in B^\omega \colon \bigwedge_{U\in\mathcal{U}} \bigvee_{i\in U} f(i)\big\}$.

We summarize basic relations between these ideals.

Proposition 3.1. *Let (B, m) be a σ-complete boolean algebra with a σ-additive and strictly positive measure. Then $\mathcal{Z} \subseteq \mathcal{N}_{\mathcal{U}}$, $\mathcal{M}_{\mathcal{U}} \subseteq \mathcal{N}_{\mathcal{U}}$ and $\mathcal{M}_{\mathcal{U}} \subseteq \mathcal{I}_{\mathcal{U}} \subseteq \mathcal{N}_{\mathcal{U}}$.*

Proof: The only case that does not follow immediately from the definitions is $\mathcal{I}_{\mathcal{U}} \subseteq \mathcal{N}_{\mathcal{U}}$. Let $f \notin \mathcal{N}_{\mathcal{U}}$. Then

$$\inf_{U\in\mathcal{U}} m\Bigg(\bigvee_{i\in U} f(i)\Bigg) = c > 0.$$

Take a decreasing sequence $\{U_k\}_{k<\omega} \subseteq \mathcal{U}$ such that

$$\lim_{k\to\infty} m\Bigg(\bigvee_{i\in U_k} f(i)\Bigg) = c.$$

Since the sequence $\big\{\bigvee_{i\in U_k} f(i)\big\}_{k<\omega}$ is also decreasing there must be some $b \in B$ such that $b \leq \bigvee_{i\in U_k} f(i)$ for every $k < \omega$ and $m(b) = c$. We show that $d \leq$

$\bigvee_{i\in U} f(i)$ for every $U \in \mathcal{U}$, this finishes the proof. Assume that there is some $U \in \mathcal{U}$ such that $b \not\leq \bigvee_{i\in U} f(i) = a$. Then $m(b \setminus a) = \varepsilon > 0$ and therefore

$$\lim_{k\to\infty} m\Big(\bigvee_{i\in U_k\cap U} f(i)\Big) = c - \varepsilon$$

which is a contradiction. □

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. We say that $\mathcal{U}$ is

- *semi-selective* if for every $\{a_n\}_{n<\omega}$ of positive real numbers such that $\mathcal{U}\text{-}\lim_{n\to\infty} a_n = 0$ there is $U \in \mathcal{U}$ such that $\sum_{n\in U} a_n < \infty$.

Theorem 3.2. *Let (B, m) be a σ-complete infinite boolean algebra with a σ-additive strictly positive measure and $\mathcal{U}$ an ultrafilter on ω. Then the following hold*

- *$\mathcal{U}$ is a P-ultrafilter if and only if $\mathcal{N}_\mathcal{U} = \mathcal{Z} + \mathcal{M}_\mathcal{U} = \{f \vee g \colon f \in \mathcal{Z},\ g \in \mathcal{M}_\mathcal{U}\}$;*
- *$\mathcal{U}$ is semi-selective if and only if $\mathcal{I}_\mathcal{U} = \mathcal{N}_\mathcal{U}$.*

Proof: To prove the first claim notice that it is enough for each $f \in \mathcal{N}_\mathcal{U}$ find a set $U \in \mathcal{U}$ such that $\lim_{i\in U} m(f(i)) = 0$. Under the assumption that B is infinite, this is possible if and only if $\mathcal{U}$ is P-ultrafilter.

Let $\mathcal{U}$ be a semi-selective ultrafilter and $f \in \mathcal{N}_\mathcal{U}$. Then there is $U \in \mathcal{U}$ such that $\sum_{i\in U} m(f(i)) < \infty$ and therefore

$$\bigwedge_{n<\omega} \bigvee_{i\in(U\setminus n)} f(i) = 0.$$

Let $\mathcal{U}$ be not semi-selective. There must be a sequence $\{a_i\}_{i<\omega}$ of positive real numbers such that $\mathcal{U}\text{-}\lim a_i = 0$ and for every $U \in \mathcal{U}$ is $\sum_{i\in U} a_i = \infty$. Take a sequence $\{b_i\}_{i<\omega} \subseteq B$ such that $m(b_i) = a_i$ and $\{b_i\}_{i<\omega}$ is independent (see for example [1]). We have for every $U \in \mathcal{U}$ that

$$m\Big(1 - \bigvee_{i\in U} f(i)\Big) = m\Big(\bigwedge_{i\in U}(1 - f(i))\Big) = \prod_{i\in U} m(1 - f(i)) = 0.$$

Therefore $\bigvee_{i\in U} f(i) = 1$ and $f \in \mathcal{N}_\mathcal{U} \setminus \mathcal{I}_\mathcal{U}$. □

References

[1] Bartoszyński T., Judah H., *Set Theory: On the Structure of the Real Line*, A. K. Peters, Wellesley, 1995.

[2] Blass A., Frankiewicz R., Plebanek G., Ryll-Nardzewski C., *A Note on extensions of asymptotic density*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3313–3320.

[3] Fremlin D. H., *Measure Theory, Vol. 3: Measure Algebras*, Torres Fremlin, Colchester, 2004.

[4] Kunisada R., *Density measures and additive property*, J. Number Theory **176** (2017), 184–203.

[5] Smith E. C. Jr., Tarski A., *Higher degrees of distributivity and completeness in Boolean algebras*, Trans. Amer. Math. Soc. **84** (1957), 230–257.

J. Grebík:
Institute of Mathematics, Academy of Sciences of the Czech Republic,
Žitná 609/25, 110 00 Praha 1 - Nové Město, Czech Republic

E-mail: grebik@math.cas.cz

(*Received* December 29, 2017*, revised* August 18, 2018)