Uniqueness of means in the Cohen model

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Dedicated to the memory of Bohuslav Balcar

Abstract. We investigate the question of whether or not an amenable subgroup of the permutation group on $\mathbb{N}$ can have a unique invariant mean on its action. We extend the work of M. Foreman (1994) and show that in the Cohen model such an amenable group with a unique invariant mean must fail to have slow growth rate and a certain weakened solvability condition.

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1. Introduction

In order to place the results to be reported in this paper in context, it is worth recalling that a (discrete) group G is finite if and only if there is a unique, finitely additive, left invariant probability measure on G; in other words, if there is a unique, finitely additive measure μ on G such that $\mu(X) = \mu(gX)$ for every $X \subseteq G$ and $\mu(G) = 1$. A group is defined to be amenable if there is any finitely additive, left invariant, probability measure on G. If $\mathbb{G}$ is a group then recall that a (left) action of $\mathbb{G}$ on the set S is a mapping "$\cdot$" from $\mathbb{G} \times S$ to S satisfying the associative law $g(h \cdot s) = (gh) \cdot s$ and for which the identity element $e_{\mathbb{G}}$ satisfies $e_{\mathbb{G}} \cdot s = s$. If $X \subseteq S$ then $\{g \cdot x\colon\ x \in X\}$ will be denoted by $g \cdot X$ and a finitely additive measure λ on S is left invariant (with respect to the action) if $\lambda(Z) = \lambda(g \cdot Z)$ for all $Z \subseteq S$ and $g \in G$. Now recall Day's theorem:

Theorem 1.1 (Day)**.** *If G is a locally compact group then the following are equivalent:*

- *The group G is amenable.*
- *If "$\cdot$" is an affine action of G on a compact, convex subset K of a locally convex vector space E such that the mapping $(g, x) \mapsto g \cdot x$ from $G \times K$ to K is separately continuous, then there is $x \in K$ such that $g \cdot x = x$ for all $g \in G$.*

Note that if the discrete group G acts on X then the action on $l^*_\infty(X)$ defined by

$$g\mu(f) = \mu(gf)$$

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where $gf(x) = f(gx)$ sends the unit ball (which is weak* compact and convex) to itself, then the action is affine, and by Day's theorem there is then μ in the unit ball such that $g\mu = \mu$ for all $g \in G$. Since elements of the unit ball of $l^*_\infty(X)$ are finitely additive probability measures, it follows that μ is a G invariant such measure. In contrast to the case of a group acting on itself, though, it is not clear when there is a unique such invariant measure.

The question of the number of invariant measures with specific additional properties was the focus of J. Rosenblatt and M. Talagrand in [6]. Typical of the results they obtained there is the following:

Theorem 1.2 (J. Rosenblatt and M. Talagrand). *For an infinite amenable group G acting on itself in the natural way the following are equivalent:*

(1) *there is a left and right G-invariant mean of G that is not inversion invariant;*
(2) *there are $2^{2^{|G|}}$ such means that are mutually singular such that the failure of inversion invariance is witnessed by the same set.*

A natural extension of their results in [6] would be to the case of a group acting on an arbitrary set and J. Rosenblatt and M. Talagrand note that their results extend to general actions of a group G on a set X provided that $|G| \leq |X|$. They then asked whether there may not even be a unique G-invariant mean if $|G| > |X|$. As a partial answer, they also showed in [6] that this is not the case if G is nilpotent. S. Krasa in [4] later extended this result to apply to solvable G as well, and then M. Foreman in [3] showed that there is no analytic subgroup of $\mathrm{Sym}(\mathbb{N})$ whose natural action on $\mathbb{N}$ has a unique invariant mean.

Partially answering the question of J. Rosenblatt and M. Talagrand, Z. Yang showed in [7] that, assuming the continuum hypothesis, there is an amenable subgroup G of the full symmetric group on $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique invariant mean. Later M. Foreman showed in [3] that the same result holds under various other set theoretic hypotheses weaker than the continuum hypothesis. While Yang's mean attains all values in the interval $[0, 1]$, in Foreman's construction the unique invariant mean is an ultrafilter.

Moreover, M. Foreman also showed in [3] that in the model obtained by adding $\aleph_2$ Cohen reals to a model of the continuum hypothesis, there are no such groups that are locally finite. The significance of this results is that the groups constructed by both Z. Yang and M. Foreman are amenable by virtue of being locally finite. Nevertheless it is natural to ask whether there is any amenable group with a faithful action on $\mathbb{N}$ that has unique invariant mean in the Cohen model. Indeed, there is no model currently known in which there is no such amenable group. In this context it is of interest to know whether there are amenable groups, that are not locally finite (in some nontrivial sense) acting with unique invariant mean. The following provides some information.

Theorem 1.3 (D. Raghavan and J. Steprāns). *Assuming there is an ultrafilter on $\mathbb{N}$ generated by a tower, there is a subgroup G of the full symmetric group on $\mathbb{N}$*

whose natural action on $\mathbb{N}$ *has a unique invariant mean and that has a generating set all of whose elements have infinite order. The group is a solvable extension of a locally finite group and, hence, amenable.*

It will be shown in this paper that in the Cohen model, any group with a faithful action on $\mathbb{N}$ and a unique invariant mean must fail to have slow growth rate and weakened solvability conditions that will be defined precisely in Section 2. While it is known (pages 21–22 in [5]) that locally solvable groups and groups with subexponential growth, are amenable, there are examples of amenable groups which have neither of those properties. The Basilica group, see [1], is an example of such a group, but since it is countable its natural action on $2^{<\omega}$ cannot have a unique invariant mean by the result of M. Foreman in [3] that any analytic group of permutations of a countable set cannot have a unique invariant mean. However, this does not rule out the possibility that a group built using the Basilica group locally might not provide an absolute example of an action with a unique invariant mean.

2. Definitions

Definition 2.1. Let G be a group and $S \subseteq G$ a finite subset. Define $\gamma_G^S(n)$ to be the cardinality of the set

$$\{s_1 \cdot s_2 \cdot \ldots \cdot s_k\colon\ k \leq n \text{ and } (\forall i \leq k)\ s_i \in S\}.$$

If G is generated by S and there are d and c in $\mathbb{N}$ such that $\gamma_G^S(n) \leq cn^d$ for all n, then G is said to have polynomial growth. If $\lim_{n\to\infty}(\gamma_G^S(n))^{1/n}$ exists and is greater than 1, then G is said to have exponential growth. If $\lim_{n\to\infty}(\gamma_G^S(n))^{1/n}$ is infinite, then G is said to have ultra-exponential growth. If the limit is no greater than 1, then the group is said to have subexponential growth. Define an arbitrary group to have polynomial, exponential, or subexponential growth, if all of its finitely generated subgroups have at most the corresponding growth.

It is known that finitely generated groups with subexponential growth, or solvable groups, are amenable. Since directed limits of amenable groups are amenable, it follows that a group with subexponential growth or local solvability is amenable (pages 14, 21–22 in [5]).

The above definitions of growth can be destroyed by a direct product with a countable group with large growth, hence, for uncountable groups the following is more useful.

Definition 2.2. For $\gamma\colon \omega \longrightarrow \omega$, a finite subset H of a group G will be said to satisfy the γ-growth condition if $\gamma_G^H \leq^* \gamma$, which means that $\gamma_G^H(n) \leq \gamma(n)$ for all but finitely many $n \in \mathbb{N}$.

For functions $\gamma_j\colon \omega \longrightarrow \omega$ and $m \in \mathbb{N}$, an uncountable group G satisfies the m-$\{\gamma_j\}_{j\in\omega}$-$\kappa$-$\lambda$-growth condition if for every family $\{H_\xi\}_{\xi\in\kappa}$ of disjoint subsets of G of cardinality m, there is $S \in [\kappa]^\lambda$, such that for all infinite $B \in [S]^{<\lambda}$ there is some j where for all k and all $A \in [B]^k$, $\bigcup_{\xi\in A} H_\xi$ satisfies the γ_j-growth condition. An uncountable group G is said to satisfy the $\{\gamma_j\}_{j\in\omega}$-$\kappa$-$\lambda$-growth condition if it satisfies the m-$\{\gamma_j\}_{j\in\omega}$-$\kappa$-$\lambda$-growth condition for all $m \in \mathbb{N}$.

Note that a group G with no subset S of ultra-exponential growth, must satisfy the $\{n \mapsto j^n\}_{j\in\omega}$-$\kappa$-$\lambda$-growth condition.

Definition 2.3. Recall that for a group G the derived series $G^{(\xi)}$ for ordinals ξ is defined by setting $G^{(0)} = G$, setting $G^{(\xi+1)}$ to be the commutator group $[G^{(\xi)}, G^{(\xi)}]$ and, if ξ is a limit ordinal, letting $G^{(\xi)} = \bigcap_{\eta\in\xi} G^{(\eta)}$. A group is solvable if there is some $n \in \omega$ such that $G^{(n)}$ is trivial. For an arbitrary subset $H \subseteq G$ define $H^{[0]} = H$ and let $H^{[n+1]} = \dot{[}H^{[n]}, H^{[n]}\dot{]}$ be the set of commutators formed from $H^{[n]}$ rather than the commutator subgroup; in other words $H^{[n+1]} = \{[g,h]\colon\ g,h \in H^{[n]}\}$. Note that if $H \subseteq G$ then $H^{[n]} \subseteq G^{(n)}$. For any subset H of a group G the notation $\langle H\rangle$ will be used to denote the subgroup of G generated by H.

The following definition is a weakening of the notion of solvability.

Definition 2.4. A group G will be called (κ, λ, m)-solvable if for every family $\{H_\xi\}_{\xi\in\kappa}$ of disjoint subsets of G of cardinality m there is $S \in [\kappa]^\lambda$ such that for all $B \in [S]^{<\lambda}$ there is $A \in [B]^{\aleph_0}$ such that $\left\langle\bigcup_{\xi\in A} H_\xi\right\rangle$ is solvable. Call a group (κ, λ)-solvable if it is (κ, λ, m)-solvable for every $m \in \mathbb{N}$.

In determining if this is the correct Ramsey theoretic analogue for solvability, the following question would need to be answered:

Question 2.5. If G is a group of size $\aleph_2$ and

$$(\forall H \in [G]^{\aleph_2})(\exists\, S \in [H]^{\aleph_1})(\forall B \in [S]^{\aleph_0})(\exists\, A \in [B]^{\aleph_0})\ \langle A\rangle \ \text{ is solvable}$$

does it follow that

$$(\forall H \in [G]^{\aleph_2})(\exists\, S \in [H]^{\aleph_1})\ \langle S\rangle \ \text{ is solvable?}$$

The above question does have a positive answer if “solvable” is replaced with “abelian”; in other words, if G is such that

$$(\forall H \in [G]^{\aleph_2})(\exists\, S \in [H]^{\aleph_1})(\forall B \in [S]^{\aleph_0})(\exists\, A \in [B]^{\aleph_0})\ \langle A\rangle \ \text{ is abelian} \tag{2.1}$$

then

$$(\forall H \in [G]^{\aleph_2})(\exists\, S \in [H]^{\aleph_1})\ \langle S\rangle \ \text{ is abelian.}$$

This follows from a standard application of the Dushnik–Miller theorem, see [2]: From (2.1) it follows that for any $H \in [G]^{\aleph_2}$ there is $S \in [H]^{\aleph_1}$ such that for every infinite $B \subseteq S$ there is some infinite $A \subseteq B$ such that any two elements of A commute. Define a colouring on $[S]^2$ by sending a pair to 0 if its elements commute and to 1 otherwise. By the Dushnik–Miller theorem there is either an uncountable homogeneous set for this colouring of colour 0 or an infinite homogeneous set of colour 1. Since the second alternative is ruled out by the choice of S, it must be the case that there is an uncountable abelian subgroup of S.

Definition 2.6. If the group G acts on $\mathbb{N}$ and $0 \leq r \leq 1$, then a set $X \subseteq \mathbb{N}$ is said to be r-thick with respect to the action if for every nonempty finite $H \subseteq G$ there is $n \in \mathbb{N}$ such that

$$\frac{|\{h \in H\colon\ h(n) \in X\}|}{|H|} \geq r.$$

Lemma 2.7 (Z. Yang [7])**.** *If the group G acts on $\mathbb{N}$, then a set $X \subseteq \mathbb{N}$ is r-thick with respect to the action if and only if there is a mean $\mathfrak{m}$ on $\mathbb{N}$ invariant under the action of G such that $\mathfrak{m}(X) \geq r$.*

Observe that a consequence of Lemma 2.7 is that if $\mathfrak{m}$ is the unique invariant mean under the action of G, then a set $X \subseteq \mathbb{N}$ is r-thick precisely if $\mathfrak{m}(X) \geq r$.

3. Unique means in the Cohen model

This section will amplify Foreman's argument of Theorem 4.1 from [3] showing that there are no locally finite groups acting on $\mathbb{N}$ with a unique invariant mean in the model obtained by adding $\aleph_2$ Cohen reals. It is supposed that the ground model V satisfies the continuum hypothesis. Let $\mathbb{P}$ be the partial order for adding $\aleph_2$ Cohen reals represented as all finite functions from $\omega_2 \times \omega$ to 2 ordered by inclusion and let $\mathbb{G}$ be a $\mathbb{P}$ name for the generic subset of $\mathbb{P}$. Let Γ be name for $\bigcup \mathbb{G}$. The argument begins by assuming that there is a $\mathbb{P}$-name for a subgroup G of the symmetric group on $\mathbb{N}$ and a name $\mathfrak{m}$ such that

$$1 \Vdash_{\mathbb{P}} \text{“}\mathfrak{m} \text{ is the unique mean invariant under the natural action on } \mathbb{N}\text{”}.$$

Notation 3.1. For any set of permutations H of $\mathbb{N}$ and $n \in \mathbb{N}$ let $H\langle n\rangle = \{h(n)\colon\ h \in H\}$.

For each $\xi \in \omega_2$ let $c_\xi = \{i \in \omega\colon\ \Gamma(\xi, i) = 0\}$ be the ξth Cohen real. Either $\aleph_2$ many Cohen reals have measure less than 1 or $\aleph_2$ many of their complements do, so by symmetry it can be assumed the first case holds. Using Lemma 2.7 and the uniqueness of the mean, there are $\aleph_2$ many $\xi \in \omega_2$ for which there is a finite $H \subseteq G$ with $H\langle n\rangle \not\subseteq c_\xi$ for each $n \in \mathbb{N}$. Using the continuum hypothesis, a Δ-system argument can then be used to find $\{(D_\eta, f_\eta, H_\eta, \xi(\eta))\}_{\eta<\omega_2}$ such that for $\eta < \omega_2$:

(1) the set D_η is a countable subset of ω_2 with $\xi(\eta) \in D_\eta$;
(2) if $\mathbb{D}_\eta$ is defined to be the partially ordered subset of $\mathbb{P}$ whose conditions have support in $D_\eta \times \omega$, then $f_\eta \in \mathbb{D}_\eta$ and H_η is a $\mathbb{D}_\eta$-name;
(3) there is a countable $D \subseteq \omega_2$ such that $\{D_\zeta\}_{\zeta<\omega_2}$ is a Δ-system with root D;
(4) if $\mathbb{D}$ is defined to be the partially ordered subset of $\mathbb{P}$ whose conditions have support in $D \times \omega$ then there is $f \in \mathbb{D}$ such that $f_\eta \restriction D \times \omega = f$ for each η;
(5) there is $T \in \mathbb{N}$ not depending on η such that $f_\eta \Vdash_{\mathbb{D}_\eta}$ “$|H_\eta| = \check{T}$”;

(6) for all $n \in \mathbb{N}$, $f_\eta \Vdash_{\mathbb{D}_\eta}$ "$H_\eta\langle \check{n}\rangle \not\subseteq c_{\xi(\eta)}$"; and
(7) there is a $\mathbb{D}$ name H such that $f_\eta \Vdash_{\mathbb{D}_\eta}$ "$H_\eta \cap V[\mathbb{D} \cap \mathbb{G}] = H$" for each η.

Note that $\mathbb{D}_\eta$ is the forcing for adding a single Cohen real. Without loss of generality, by arguing in the model $V[\mathbb{G} \cap \mathbb{D}]$, it can be assumed that $D = \emptyset$. Also, by adding functions to H_η we may assume T is arbitrarily large.

Definition 3.1. Given any $\mathbb{P}$ name for a subgroup $\overline{H}$ of G and $f \in \mathbb{P}$, let $A(\overline{H}, f, k, m)$ be the following statement:

$$(\forall\, a \subseteq \mathbb{N})(\forall\, l \in \mathbb{N}) \text{ if } |a| \leq m \text{ and } \min(a) > k \text{ then } f \not\Vdash_{\mathbb{P}} \text{“}(\exists\, u \in \check{a})\ \max(\overline{H}\langle u\rangle) \leq \check{l}\text{”}.$$

Lemma 3.2. *Given $\eta \in \omega_2$, $f_\eta \subseteq f \in \mathbb{D}_\eta$, and $m \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $A(H_\eta, f, k, m)$ holds.*

Proof: If the lemma fails for some i, f and m, then it is possible to construct a sequence $\{(k_j, l_j, a_j)\}_{j<\omega}$ such that:
(i) $N < k_j, l_j \in \mathbb{N}$;
(ii) $a_j \subseteq \mathbb{N}$ and $|a_j| \leq m$;
(iii) $k_j < \min(a_j) \leq \max(a_j) < k_{j+1}$; and
(iv) $f \Vdash$ "$(\exists\, u \in a_j)\ \max(H_\eta\langle u\rangle) \leq l_j$".

Let $L > T|f|$ and let d be so large that $d > \max_{j\leq L}(l_j)$. Define $g \in \mathbb{D}_\eta$ by setting

$$\mathbf{domain}(g) = (\{\xi(\eta)\} \times d) \cup \mathbf{domain}(f)$$

and letting

$$g(u,v) = \begin{cases} f(u,v) & \text{if } (u,v) \in \mathbf{domain}(f), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma \subseteq \mathbb{D}_\eta$ be generic such that $g \in \Gamma$. In $V[\Gamma]$ note that

$$\{h^{-1}(u)\colon\ (\xi(\eta), u) \in \mathbf{domain}(f) \text{ and } h \in H_\eta\}$$

has cardinality no greater than $T|f|$ and so there must be some $j \leq L$ such that $(\xi(\eta) \times H_\eta\langle u\rangle) \cap\ \mathbf{domain}(f) = \emptyset$ for each $u \in a_j$. Using (iv) and the fact that $g \supseteq f$ it follows that $V[\Gamma]$ satisfies $\max(H_\eta\langle u\rangle) \leq l_j \leq d$ for some $u \in a_j$. But

$$g \Vdash \text{“}\{u \in d\colon\ (\xi(\eta), u) \notin \mathbf{domain}(f)\} \subseteq c_{\xi(\eta)}\text{”}$$

contradicting the hypothesis (6) since $g \supseteq f_\eta$. □

Claim 3.3. *Without loss of generality*

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“}\{H_\eta\}_{\eta\in\omega_2} \textit{ is a pairwise disjoint family”}.$$

Proof: Keeping in mind that we are now arguing in $V[\mathbb{G}\cap\mathbb{D}]$, let $H_{\mathbb{G}}$ be the interpretation of H in $V[\mathbb{G}\cap\mathbb{D}]$. It suffices to show that Lemma 3.2 is still satisfied if one replaces H_η with $H_\eta \setminus H_{\mathbb{G}}$ for each $\eta \in \omega_2$. To see that this is the case, note that by the genericity of $\mathbb{G} \cap \mathbb{D}_\eta \setminus \mathbb{D}$ over the model $V[\mathbb{G} \cap \mathbb{D}]$ it follows that there are infinitely many $n \in \mathbb{N}$ such that $H_{\mathbb{G}}\langle n \rangle \subseteq c_\eta$. In other words, the elements of $H_{\mathbb{G}}$ are never used to satisfy the conclusion of Lemma 3.2. So henceforth it will be assumed that $H_\eta = H_\eta \setminus H_{\mathbb{G}}$. □

3.1 κ-solvability in the Cohen model.

Lemma 3.4. *Suppose that*

- $i_0, \ldots, i_{N-1} \in \omega$;
- $\bigcup_{j<N} f_{i_j} \subseteq f \in \prod_{j<N} \mathbb{D}_{i_j}$;
- $2^m \leq N$.

There is $k \in \mathbb{N}$ such that $A\big(\big(\bigcup_{j<N} H_{i_j}\big)^{[m]}, f, k, 1\big)$ holds.

Proof: Proceed by induction on m; the case $m = 0$ is true by Lemma 3.2. Assume the lemma is true for m and let $2^{m+1} \leq N$, $i_0, \ldots, i_{N-1} \in \omega$,

$$\bigcup_{j<N} f_{i_j} \subseteq f \in \prod_{j<N} \mathbb{D}_{i_j}.$$

To see that there is some k such that $A\big(\big(\bigcup_{j<N} H_{i_j}\big)^{[m+1]}, f, k, 1\big)$ is true let l be given. By the inductive hypothesis, there is k_1 such that

$$A\Bigg(\Bigg(\bigcup_{2^m \leq j<N} H_{i_j}\Bigg)^{[m]}, f \restriction \prod_{2^m \leq j<N} \mathbb{D}_{i_j}, k_1, 1\Bigg) \tag{3.1}$$

holds. Let $n > k_1$ be arbitrary and extend $f \restriction \prod_{j<2^m} \mathbb{D}_{i_j}$ to $f' \in \prod_{j<2^m} \mathbb{D}_{i_j}$ so there is some $L \in \omega$ such that

$$f' \Vdash_{\prod_{j<2^m} \mathbb{D}_{i_j}} \text{“}(\forall\, i \leq \check{l}) \Bigg(\bigcup_{j<2^m} H_{i_j}\Bigg)^{[m]} \langle i \rangle < L\text{”}. \tag{3.2}$$

By the inductive hypothesis, there is k_2 such that $A\big(\big(\bigcup_{j<2^m} H_{i_j}\big)^{[m]}, f', k_2, 1\big)$ holds, and since (3.1) holds, it is possible to find $h \in \big(\bigcup_{2^m \leq j<N} H_{i_j}\big)^{[m]}$, $n' \in \omega$, and $f'' \in \prod_{2^m \leq j<N} \mathbb{D}_{i_j}$ extending $f \restriction \prod_{2^m \leq j<N} \mathbb{D}_{i_j}$ such that

$$f'' \Vdash_{\prod_{2^m \leq j<N} \mathbb{D}_{i_j}} \text{“}n' = h(n) > \check{k}_2\text{”}. \tag{3.3}$$

Since $A((\bigcup_{j<2^m} H_{i_j})^{[m]}, f', k_2, 1)$ holds,

$$(\forall K \in \omega)\ f' \nVdash_{\mathbb{P}} \text{“}\max\Bigg(\Bigg(\bigcup_{j<2^m} H_{i_j}\Bigg)^{[m]} \langle n' \rangle\Bigg) \leq K\text{”}$$

so there are infinitely many possible $K \in \omega$ for which there are elements $g_K \in \big(\bigcup_{j<2^m} H_{i_j}\big)^{[m]}$ and $f'_K \in \prod_{j<2^m} \mathbb{D}_{i_j}$ extending f' such that

$$f'_K \Vdash \text{“}g_K(n') = K. \tag{3.4}$$

This implies there is K as above with $f'' \nVdash h^{-1}(K) \leq L$, so we can extend f'' to $f''' \in \prod_{2^m \leq j<N} \mathbb{D}_{i_j}$ deciding $h^{-1}(K) > L$. Therefore combining (3.2), (3.3), and (3.4),

$$f'_K \cup f''' \Vdash l < g_K^{-1} h^{-1} g_K h(n) = [g_K, h](n),$$

and this proves $A\big(\big(\bigcup_{j<N} H_{i_j}\big)^{[m+1]}, f, k, 1\big)$ holds. □

Now it will be proven that G cannot be $(\aleph_2, \aleph_1)$-solvable.

Theorem 3.5. *In the $\aleph_2$ Cohen real model, every group acting faithfully on $\mathbb{N}$ with a unique invariant mean is not $(\aleph_2, \aleph_1)$-solvable.*

Proof: If G is a counterexample, let $\{(D_\eta, f_\eta, H_\eta, \xi(\eta))\}_{\eta<\omega_2}$ and T be as in (1) to (7) of Section 3. The set $\Lambda = \{\eta\colon\ f_\eta \in \mathbb{G}\}$ must have size $\aleph_2$. Suppose that

$$\mathbb{1} \Vdash \text{“}S \in [\Lambda]^{\omega_1} \ \text{ and } \ (\forall B \in [S]^{\aleph_0})(\exists\, A \in [B]^{\aleph_0}) \left\langle \bigcup_{\eta\in A} H_\eta \right\rangle \ \text{is solvable”}.$$

Extend each f_η such that $f_\eta \nVdash$ “$\eta \notin S$” to $\bar{f}_\eta$ so that $\bar{f}_\eta \Vdash$ “$\eta \in S$”, and extend D_η to $\overline{D}_\eta$ so that if $\overline{\mathbb{D}}_\eta$ is defined accordingly then $\bar{f}_\eta \in \overline{\mathbb{D}}_\eta$. Let $E = \{\eta \in \omega_2\colon\ \bar{f}_\eta \Vdash \eta \in S\}$. The set E must be uncountable, and so refine E so that $\{\operatorname{supp}(\bar{f}_\eta)\}_{\eta\in E}$ forms a Δ-system. As in Lemma 3.2 it may be assumed that $\{\operatorname{supp}(\bar{f}_\eta)\}_{\eta\in E}$ and $\{\overline{D}_\eta\}_{\eta\in E}$ are pairwise disjoint. Without loss of generality, by re-labelling the first ω indices in E, assume $\omega \subseteq E$ so that $(\forall\, i \in \mathbb{N})\ f_i \Vdash i \in S$. It will be shown that $B = \omega \cap S$ satisfies that

$$\mathbb{1} \Vdash_{\mathbb{P}} \text{“}|B| = \omega \ \text{ and } \ (\forall A \in [B]^{\aleph_0}) \left\langle \bigcup_{i\in A} H_i \right\rangle \ \text{cannot be solvable.”}$$

To see $|B| = \omega$, note that for any $n \in \omega$, $f \in \mathbb{P}$ there is i such that $n < i < \omega$ and $\operatorname{supp}(f_i) \cap \operatorname{supp}(f) = \emptyset$; hence, $f \cup f_i \Vdash_{\mathbb{P}}$ “$i \in B$”. Suppose $\mathbb{1} \Vdash A \in [B]^{\aleph_0}$, and let $\widetilde{G} = \big\langle \bigcup_{i\in A} H_i \big\rangle$. The proof follows as a corollary from Lemma 3.4. Suppose $m \in \omega$ and f is some condition forcing $\widetilde{G}^{[m]}$ is trivial. Let $N \in \omega$ be larger than 2^m and extend f to force that $i_0, \ldots, i_{N-1}$ are distinct elements of A. Since $A \subseteq \Lambda$, f must extend $\bigcup_{j<N} f_{i_j}$. Lemma 3.4 yields some $k \in \omega$ with the property $A\big(\big(\bigcup_{j<N} H_{i_j}\big)^{[m]}, f, k, 1\big)$, and so there are $g \in \big(\bigcup_{j<N} H_{i_j}\big)^{[m]} \subseteq \widetilde{G}^{[m]}$ and $f' \in \prod_{j<N} \mathbb{D}_{i_j}$ extending f such that

$$f' \Vdash g(k+1) > k+1.$$

In other words, the condition f' forces a contradiction since g is the identity but $g(k+1) > k+1$. □

3.2 Subexponential growth in the Cohen model.

Notation 3.2. In the next two lemmas the notation n^m for n and m elements of $\mathbb{N}$ will be used to denote both the set of all functions from $m = \{0, 1, \ldots, m-1\}$ to $n = \{0, 1, \ldots, n-1\}$, as well as the cardinality of this set of functions. However, this potential ambiguity should cause no distress to the careful reader.

Lemma 3.6. *Let $K, J \in \mathbb{N}$ with $K \geq JT^2$, let $\mathbb{Q} = \prod_{i \leq K} \mathbb{D}_i$, and let $q \in \mathbb{Q}$ be a condition with $q(i) \leq f_i$. There are $\{(q_n, \{(a_t, b_t)\}_{t \in K^n}, k_n^0, k_n^1)\}_{n \in \omega}$ such that for $n \in \omega$:*

(1) *$q_0 = q$;*
(2) *$q_{n+1}(i) \supseteq q_n(i)$ for each $i \leq K$;*
(3) *the property $A(H_0, q_n(0), k_n^0, K^n)$ of Lemma 2.1 holds;*
(4) *the property $A(H_{i+1}, q_n(i+1), k_n^1, K^n)$ of Lemma 3.2 holds for $i \in K - 1$;*
(5) *$a_t, b_t \subseteq \mathbb{N}$, and $h_t^i \in H_i$ for $i \leq K$;*
(6) *$k_n^0 < a_t < k_{n+1}^0$ and $k_n^1 < b_t < k_{n+1}^1$ for each $t \in K^n$;*
(7) *$q_{n+1}(0) \Vdash$ "$b_t \in H_0\langle a_t \rangle$" for each $t \in K^n$;*
(8) *$q_{n+1}(i+1) \Vdash$ "$a_{t^\frown i} \in H_{i+1}\langle b_t \rangle$" for each $t \in K^n$ and $i \in K - 1$;*
(9) *if t and s are in K^n and $j \in K - 1$ then $a_{t^\frown j} < a_{s^\frown j+1}$;*
(10) *$|\{a_t\}_{t \in K^n}| \geq J^n$.*

Proof: Proceed by induction on n. To begin, let $q_0 = q$ and use Lemma 3.2 to find k_0 sufficiently large that the property $A(H_i, q_0(i), k_0, 1)$ holds for each $i \leq K$ and let $k_0^0 = k_0^1 = k_0$. Let $a_\emptyset \in \mathbb{N}$ be arbitrary such that $a_\emptyset > k_0$. Then using $A(H_0, q_0(0), k_0^0, 1)$ for $l = k_0$ let $q_1(0) \supseteq q(0)$ be such that there is $h_\emptyset^0 \in H_0$ with

$$q_1(0) \Vdash \text{“}h^0(a_\emptyset) \ \text{ is decided and above } k_0\text{”}.$$

Set $b_\emptyset = h^0(a_\emptyset)$.

Then let $k_1^0 > a_\emptyset$ be so large that property $A(H_0, q_1(0), k_1^0, K)$ holds. Using property $A(H_1, q_0(1), k_0^1, 1)$ with $l = k_1^0$ let $q_1(1) \supseteq q_0(1)$ and $a_{\emptyset^\frown 0}$ be such that

$$q_1(1) \Vdash \text{“}a_{\emptyset^\frown 0} \in H_1\langle b_\emptyset \rangle \ \text{ and } \ a_{\emptyset^\frown 0} > k_1^0\text{”}.$$

Using property $A(H_2, q_0(2), k_0^1, 1)$ with $l = a_{\emptyset^\frown 0}$ let $q_1(2) \supseteq q_0(2)$ and $a_{\emptyset^\frown 1}$ be such that

$$q_1(2) \Vdash \text{“}a_{\emptyset^\frown 1} \in H_2\langle b_\emptyset \rangle \ \text{ and } \ a_{\emptyset^\frown 1} > a_{\emptyset^\frown 0} > k_1^0\text{”}.$$

Proceed inductively to use property $A(H_i, q_0(i), k_0^1, 1)$ with $l = a_{\emptyset^\frown i-1}$ to let $q_1(i) \supseteq q_0(i)$ and $a_{\emptyset^\frown i-1}$ be such that

$$q_1(i) \Vdash \text{“}a_{\emptyset^\frown i-1} \in H_i\langle b_\emptyset \rangle \ \text{ and } \ a_{\emptyset^\frown i-1} > a_{\emptyset^\frown i-2}\text{”}$$

for each $i \leq K-1$. Then let $k_1^1 > b_\emptyset$ sufficiently large that $A(H_{i+1}, q_1(i+1), k_1^1, K)$ holds for $i \leq K-1$. The values of the condition $q_1(i)$ have been defined for each $i \in K$ and, noting $|\{a_i\}_{i \in K}| = K \geq J$, it is easy to check that the induction hypotheses are satisfied.

Now assume that q_m, $\{(a_t, b_t)\}_{t\in K^m}$, k^0_m and k^1_m are all given satisfying the induction hypotheses. Using (3) it follows that the property $A(H_0, q_m(0), k^0_m, K^m)$ holds. Note that, in the notation of Lemma 3.2 setting $a = \{a_t\}_{t\in K^m}$, it is the case that $a > k^0_m$. Hence it is possible to apply this property to $l = k^1_m$ and a to find $q_{m+1}(0) \supseteq q_m(0)$, such that for all $t \in K^m$ there is $g^0_t \in H_0$ with

$$q_{m+1}(0) \Vdash \text{“}g^0_t(a_t) \text{ is decided and above } k^1_m\text{”}.$$

Set $b_t = g^0_t(a_t)$. By pigeonholing, there must be some $h^0_t \in \{g^0_t\}_{t\in K^m}$ that is forced by $q_{m+1}(0)$ to map at least $|a|/T$ elements of a above k^1_m, and since h^0_t is injective, $|\{b_t\}_{t\in K^m}| \geq |\{a_t\}_{t\in K^m}|/T$.

Let $k^0_{m+1} > \max_{t\in K^m} a_t$ be so large that property $A(H_0, q_{m+1}(0), k^0_{m+1}, K^{m+1})$ holds. Using property $A(H_1, q_m(1), k^1_m, K^m)$ with $l = k^0_{m+1}$, let $q_{m+1}(1) \supseteq q_m(1)$ be such that for every $t \in K^m$ there is $g^1_t \in H_1$ with

$$q_{m+1}(1) \Vdash \text{“}g^1_t(b_t) \text{ is decided and above } k^0_{m+1}\text{”}.$$

Set $a_{t\frown 0} = g^1_t(b_t)$. By pigeonholing, there must be some $h^1_t \in \{g^1_t\}_{t\in K^m}$ that is forced by $q_{m+1}(0)$ to map at least $|\{b_t\}_{t\in K^m}|/T \geq |\{a_t\}_{t\in K^m}|/T^2$ elements of $\{b_t\}_{t\in K^m}$ above k^1_m, and since h^i_t is injective, $|\{a_{t\frown 0}\}_{t\in K^m}| \geq |\{a_t\}_{t\in K^m}|/T^2$.

Proceeding by the induction using the property $A(H_i, q_m(i), k^1_m, K^m)$ with $l = \max_{t\in K^m} a_{t\frown i-2}$ let $q_{m+1}(i) \supseteq q_m(i)$ and $a_{t\frown i-1}$ be such that for every $t \in K^m$ there is $g^i_t \in H_i$ with

$$q_{m+1}(i) \Vdash \text{“}a_{t\frown i-1} = g^i_t(b_t) > \max_{t\in K^m} a_{t\frown i-2}.$$

Again there must be some $h^i_t \in \{g^i_t\}_{t\in K^m}$ that is forced by $q_{m+1}(i)$ to map at least $|\{b_t\}_{t\in K^m}|/T \geq |\{a_t\}_{t\in K^m}|/T^2$ elements of $\{b_t\}_{t\in K^m}$ above $\max_{t\in K^m} a_{t\frown i-2}$, and since h^i_t is injective, $|\{a_{t\frown i-1}\}_{t\in K^m}| \geq |\{a_t\}_{t\in K^m}|/T^2$. Let $k^1_{m+1} > \max_{t\in K^m} b_t$ be sufficiently large that $A(H_{i+1}, q_{m+1}(i+1), k^1_{m+1}, K^{m+1})$ holds for each $i \in K-1$. Noting that

$$|\{a_t\}_{t\in K^{m+1}}| \geq \sum_{i\in K} |\{a_{t\frown i}\}_{t\in K^m}| \geq K\frac{|\{a_t\}_{t\in K^m}|}{T^2} \geq K\frac{J^m}{T^2} \geq J^{m+1},$$

it is again routine to check that the induction hypotheses are all satisfied. $\square$

Corollary 3.7. *Given $K, J, N \in \mathbb{N}$ with $K \geq JT^2$, and $q \in \mathbb{Q} = \prod_{i\leq K} \mathbb{D}_i$ with $q(i) \leq f_i$, there is $B \subseteq K^N$ with $|B| = J^N$ and $\{a_t\}_{t\in B} \subseteq \mathbb{N}$, $a_\emptyset \in \mathbb{N}$, $\{h^i_t\}_{i\leq K, t\in K^{\leq N}}$, and $q' \in \mathbb{Q}$ extending q such that*

(1) *$(\forall\, t, s \in B)$ $t \neq s$ then $a_t \neq a_s$;*

(2) *$(\forall\, t \in K^{\leq N})(\forall\, i \leq K)$ $f \Vdash$ “$h^i_t \in H_i$”;*

(3) *$(\forall\, t \in B)$ $q' \Vdash$ “$a_t = h^{t(N-1)+1}_t h^0_{t\restriction(N-1)} \circ \cdots \circ h^{t(1)+1}_{t\restriction 2} h^0_{t\restriction 1} \circ h^{t(0)+1}_{t\restriction 1} h^0_\emptyset(a_\emptyset)$”.*

In particular, for $\gamma_j = n \mapsto j^n$, $\bigcup_{i\leq K} H_i$ does not satisfy the $\gamma_{(J-1)}$-growth condition.

Proof: Using Lemma 3.6 set $q' = q_N$, and pick J^N distinct elements

$$\{a_t\}_{t\in B} \subseteq \{a_t\}_{t\in K^N}.$$

$\square$

Theorem 3.8. *In the $\aleph_2$ Cohen real model, every group acting faithfully on $\mathbb{N}$ with a unique invariant mean does not have the $\{n \mapsto j^n\}_{j\in\omega}$-$\aleph_2$-$\aleph_1$-growth condition.*

Proof: If G is a counterexample, let $\{(D_\eta, f_\eta, H_\eta, \xi(\eta))\}_{\eta<\omega_2}$ and T be as in (1) to (7) of Section 3. Define the function γ_j on $\mathbb{N}$ by $\gamma_j(n) = j^n$. Let Γ be a generic filter for $\mathbb{P}$ and set $\Lambda = \{\eta\colon f_\eta \in \Gamma\}$, noting that this set must have size $\aleph_2$. Suppose that

$$\mathbb{1} \Vdash \text{“}S \in [\Lambda]^{\omega_1}\text{”} \text{ and } \mathbb{1} \Vdash \text{“}(\forall B \in [S]^{\aleph_0})(\exists\, j \in \omega)(\forall A \in [B]^{(j+1)T^2}) \bigcup_{\eta\in A} H_\eta \text{ satisfies the } \gamma_j\text{-growth condition”}.$$

As in Theorem 3.5, extend each f_η such that $f_\eta \not\Vdash$ “$\eta \notin S$” to $\bar{f}_\eta$ so that $\bar{f}_\eta \Vdash$ “$\eta \in S$”, and extend D_η to $\overline{D}_\eta$ so that if $\overline{\mathbb{D}}_\eta$ is defined accordingly then $\bar{f}_\eta \in \overline{\mathbb{D}}_\eta$. Let $E = \{\eta \in \omega_2\colon\ \bar{f}_\eta \Vdash \eta \in S\}$. The set E must be uncountable, and so refine E so that $\{\mathrm{supp}(\bar{f}_\eta)\}_{\eta\in E}$ forms a Δ-system. As in Lemma 3.2 it may be assumed that $\{\mathrm{supp}(\bar{f}_\eta)\}_{\eta\in E}$ and $\{\overline{D}_\eta\}_{\eta\in E}$ are pairwise disjoint. Without loss of generality, by re-labelling the first ω indices in E, assume $\omega \subseteq E$ so that $(\forall\, i \in \mathbb{N})\ f_i \Vdash i \in S$. As in Theorem 3.5, $\mathbb{1} \Vdash_{\mathbb{P}}$ “$|B| = \omega$”.

Suppose for some $p \in \mathbb{P}$, $J \in \omega$ that

$$p \Vdash \text{“}(\forall A \in [B]^{(J+1)T^2}) \bigcup_{\eta\in A} H_\eta \text{ satisfies the } \gamma_J\text{-growth condition”}.$$

Let $K = (J+1)T^2$. There are $f_l, \dots, f_{l+K}$ such that for $i \le K$, $\mathrm{supp}(f_{l+i}) \cap \mathrm{supp}(p) = \emptyset$. For $q = p \cup \bigcup_{i\le K} f_{l+i}$ apply Corollary 3.7 to get $q' \le q$ such that

$$q' \Vdash \text{“} \bigcup_{i\le K} H_{l+i} \text{ does not satisfy the } \gamma_J\text{-growth condition”}.$$

Since $q' \Vdash l, \dots, l+K \in B$ it is the case that

$$q' \Vdash \text{“} \bigcup_{i\le K} H_{l+i} \text{ satisfies the } \gamma_J\text{-growth condition”},$$

yielding a contradiction. $\square$

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