Another ordering of the ten cardinal characteristics in Cichoń's diagram

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Dedicated to the memory of Bohuslav Balcar (1943–2017)

Abstract. It is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

Assuming four strongly compact cardinals, it is consistent that

$$\aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{M}) = \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.$$

Keywords: set theory of the reals; Cichoń's diagram; forcing; compact cardinal

Classification: 03E17

Introduction

We assume that the reader is familiar with basic properties of Amoeba, Hechler, random and Cohen forcing, and with the cardinal characteristics in Cichoń's diagram, given in Figure 1: An arrow between $\mathfrak{x}$ and $\mathfrak{y}$ indicates that Zermelo–Fraenkel set theory (ZFC) proves $\mathfrak{x} \leq \mathfrak{y}$. Moreover, $\max(\mathfrak{d}, \mathrm{non}(\mathcal{M})) = \mathrm{cof}(\mathcal{M})$ and $\min(\mathfrak{b}, \mathrm{cov}(\mathcal{M})) = \mathrm{add}(\mathcal{M})$. These (in)equalities are the only one provable. More precisely, all assignments of the values $\aleph_1$ and $\aleph_2$ to the characteristics in Cichoń's diagram are consistent, provided they do not contradict the above (in)equalities. (A complete proof can be found in [2, Chapter 7].)

In the following, we will only deal with the ten "independent" characteristics listed in Figure 2 (they determine $\mathrm{cof}(\mathcal{M})$ and $\mathrm{add}(\mathcal{M})$).

Regarding the left hand side, it was shown in [8] that consistently

$$(\mathrm{left}_{\mathrm{old}}) \quad \aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

(This corresponds to λ_1 to λ_5 in Figure 3.) The proof is repeated in [7], in a slightly different form which is more convenient for our purpose. Let us call this construction the "old construction".

DOI 10.14712/1213-7243.2015.273

Supported by Austrian Science Fund (FWF): P26737 & P30666 (first author), European Research Council grant ERC-2013-ADG 338821 (second author). The third author is recipient of a DOC Fellowship of the Austrian Academy of Sciences at the Institute of Discrete Mathematics and Geometry, TU Wien. This is publication number 1131 of the second author.

$$
\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

FIGURE 1. Cichoń's diagram.

$$
\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \boxed{} \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \boxed{} \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

FIGURE 2. The ten "independent" characteristics.

In this paper, building on [16], we give a construction to get a different order for these characteristics, where we swap $\mathrm{cov}(\mathcal{N})$ and $\mathfrak{b}$:

$$(\mathrm{left_{new}}) \quad \aleph_1 < \mathrm{add}(\mathcal{N}) < \mathrm{add}(\mathcal{M}) = \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}.$$

(This corresponds to λ_1 to λ_5 in Figure 4.)

This construction is more complicated than the old one. Let us briefly describe the reason: In both constructions, we assign to each of the cardinal characteristics of the left hand side a relation R. E.g., we use the "eventually different" relation $\mathrm{R}_4 \subseteq \omega^\omega \times \omega^\omega$ for $\mathrm{non}(\mathcal{M})$. We can then show that the characteristic remains "small" (i.e., is at most the intended value λ in the final model), because all single forcings we use in the iterations are either small (i.e., smaller than λ) or are "R-good". However, $\mathfrak{b}$ (with the "eventually dominating" relation $\mathrm{R}_2 \subseteq \omega^\omega \times \omega^\omega$) is an exception: We do not know any variant of an eventually different forcing (which we need to increase $\mathrm{non}(\mathcal{M})$) which satisfies that all of its subalgebras are R_2-good. Accordingly, the main effort (in both constructions) is to show that $\mathfrak{b}$ remains small.

In the old construction, each non-small forcing is a (σ-centered) subalgebra of the eventually different forcing $\mathbb{E}$. To deal with such forcings, ultrafilter limits of sequences of $\mathbb{E}$-conditions are introduced and used (and we require that all

$$\begin{tikzcd}
 & \lambda_2 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
 & & \lambda_3 \arrow[u] \arrow[r] & \lambda_6 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_7 \arrow[uu] &
\end{tikzcd}$$

FIGURE 3. The old order.

$$\begin{tikzcd}
 & \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_9 \\
 & & \lambda_2 \arrow[u] \arrow[r] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

FIGURE 4. The new order.

$\mathbb{E}$-subforcings are basically $\mathbb{E}$ intersected with some model, and thus closed under limits of sequences in the model). In the new construction, we have to deal with an additional kind of "large" forcing: (subforcings of) random forcing. Ultrafilter limits do not work any more, but, similarly to [16], we can use finite additive measures (FAMs) and interval-FAM-limits of random conditions. But now $\mathbb{E}$ doesn't seem to work with interval-FAM-limits any more, so we replace it with a creature forcing notion $\widetilde{\mathbb{E}}$.

We also have to show that $\operatorname{cov}(\mathcal{N})$ remains small. In the old construction, we could use a rather simple (and well understood) relation $\mathrm{R}^{\mathrm{old}}$ and use the fact that all σ-centered forcings are $\mathrm{R}^{\mathrm{old}}$-good: As all large forcings are subalgebras of either eventually different forcing or of Hechler forcing, they are all σ-centered. In the new construction, the large forcings we have to deal with are subforcings of $\widetilde{\mathbb{E}}$. But $\widetilde{\mathbb{E}}$ is not σ-centered, just (ϱ, π)-linked for a suitable pair (ϱ, π) (a property between σ-centered and σ-linked, first defined in [15], see Definition 1.18). So we use a different (and more cumbersome) relation R_3, introduced in [15], where it is also shown that (ϱ, π)-linked forcings are R_3-good.

Regarding the whole diagram, in [7], starting with the iteration for ($\mathrm{left}_{\mathrm{old}}$), a new iteration is constructed to get simultaneously different values for all characteristics: Assuming four strongly compact cardinals, the following is consistent (cf. Figure 3):

$$\begin{aligned}\aleph_1 &< \mathrm{add}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})\\ &< \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.\end{aligned}$$

The essential ingredient is the concept of the Boolean ultrapower of a forcing notion.

In exactly the same way we can expand our new version ($\mathrm{left}_{\mathrm{new}}$) to the right hand side, where also the characteristics dual to $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ are swapped. So we get: If four strongly compact cardinals are consistent, then so is the following (cf. Figure 4):

$$\begin{aligned}\aleph_1 &< \mathrm{add}(\mathcal{N}) < \mathfrak{b} < \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})\\ &< \mathrm{non}(\mathcal{N}) < \mathfrak{d} < \mathrm{cof}(\mathcal{N}) < 2^{\aleph_0}.\end{aligned}$$

We closely follow the presentation of [7]. Several times, we refer to [7] and to [16] for details in definitions or proofs. We thank M. Goldstern and D. A. Mejía for valuable discussions, and an anonymous referee for a very detailed and helpful report pointing out (and even fixing) several mistakes in the first version of the paper.

1. Finitely additive measure limits and the $\widetilde{\mathbb{E}}$-forcing

1.1 FAM-limits and random forcing. We briefly list some basic notation and facts around finite additive measures. (A bit more details can be found in Section 1 of [16].)

Definition 1.1.
- A "partial FAM" (finitely additive measure) Ξ' is a finitely additive probability measure on a sub-Boolean algebra $\mathcal{B}$ of $\mathcal{P}(\omega)$, the power set of ω, such that $\{n\} \in \mathcal{B}$ and $\Xi'(\{n\}) = 0$ for all $n \in \omega$. We set $\mathrm{dom}(\Xi') = \mathcal{B}$.
- Ξ is a FAM if it is a partial FAM with $\mathrm{dom}(\Xi) = \mathcal{P}(\omega)$.
- For every FAM Ξ and bounded sequence of non-negative reals $\bar{a} = (a_n)_{n\in\omega}$ we can define in the natural way the average (or: integral) $\mathrm{Av}_\Xi(\bar{a})$, a non-negative real number.

[16, 1.2] lists several results that informally say:

(*) There is a FAM Ξ that assigns the values a_i to the sets A_i (for all i in some index set I) if and only if for each $I' \subseteq I$ finite and $\varepsilon > 0$ there is an arbitrary large[1] finite $u \subseteq \omega$ such that the counting measure on u for A_i approximates a_i with an error of at most ε for all $i \in I'$.

[1] Equivalently: "a finite u with arbitrary large minimum", which is the formulation actually used in most of the results.

For the size of such an "ε-good approximation" u to some FAM Ξ we can give an upper bound for $|u|$ which only depends on $|I'|$ and ε (and not on Ξ):

Lemma 1.2. *Given $N, k^* \in \omega$ and $\varepsilon > 0$, there is an $M \in \omega$ such that: For all* FAMs Ξ *and $(A_n)_{n<N}$ there is a nonempty $u \subseteq \omega$ of size less than or equal to M such that* $\min(u) > k^*$ *and $\Xi(A_n) - \varepsilon < |A_n \cap u|/|u| < \Xi(A_n) + \varepsilon$ for all $n < N$.*

Proof: We can assume that $\varepsilon = 1/L$ for an integer L. The set $\{A_n : n \in N\}$ generates the set algebra $\mathfrak{B} \subseteq \mathcal{P}(\omega)$. Let $\mathcal{X}$ be the set of atoms of $\mathfrak{B}$. So $\mathcal{X}$ is a partition of ω of size less than or equal to 2^N. Set $\mathcal{X}' = \{x \in \mathcal{X} : \Xi(x) > 0\}$. Every $x \in \mathcal{X}'$ is infinite, and $\sum_{x\in\mathcal{X}'} \Xi(x) = 1$.

Round $\Xi(x)$ to some number $\Xi^\varepsilon(x) = l_x/(L \cdot 2^N)$ for some integer $0 \le l_x \le L \cdot 2^N$, such that $|\Xi(x) - \Xi^\varepsilon(x)| < (L \cdot 2^N)^{-1}$ and $\sum_{x\in\mathcal{X}'} \Xi^\varepsilon(x)$ is still 1. So $\sum_{x\in\mathcal{X}'} l_x = L \cdot 2^N$, and we construct u consisting of l_x many points that are bigger than k^* and in x (for each $x \in \mathcal{X}'$). □

We will use the following variants of $(*)$, regarding the possibility to extend a partial FAM Ξ' to a FAM Ξ. The straightforward, if somewhat tedious, proofs are given in [16, 1.3 (G) and 1.7].

Fact 1.3. Let Ξ' be a partial FAM, and I some index set.

(a) Fix for each $i \in I$ some $A_i \subseteq \omega$.
If $A \cap \bigcap_{i\in I'} A_i \neq \emptyset$ for all $I' \subseteq I$ finite and $A \in \mathrm{dom}(\Xi')$ with $\Xi'(A) > 0$, *then* Ξ' can be extended to a FAM Ξ such that $\Xi(A_i) = 1$ for all $i \in I$.

(b) Fix for each $i \in I$ some real b^i and some bounded sequence of non-negative reals $\bar{a}^i = (a^i_k)_{k\in\omega}$.
If for each finite partition $(B_m)_{m<m^*}$ of ω into elements of $\mathrm{dom}(\Xi')$ for each $\varepsilon > 0$, $k^* \in \omega$, and $I' \subseteq I$ finite there is a finite $u \subseteq \omega \setminus k^*$ such that
   - for all $m < m^*$, $\Xi'(B_m) - \varepsilon \le |B_m \cap u|/|u| \le \Xi'(B_m) + \varepsilon$, and
   - for all $i \in I'$, $|u|^{-1} \sum_{k\in u} a^i_k \ge b^i - \varepsilon$,

then Ξ' can be extended to a FAM Ξ such that $\mathrm{Av}_\Xi(\bar{a}^i) \ge b^i$ for all $i \in I$.

We first define what it means for a forcing Q to have FAM limits.

Remark 1.4. Intuitively, this means (in the simplest version): Fix a FAM Ξ. We can define for each sequence q_k of conditions that are all "similar" (e.g., have the same stem and measure) a limit $\lim_\Xi \bar{q}$. And we find in the Q-extension a FAM Ξ' extending Ξ, such that $\lim_\Xi(\bar{q})$ forces that the set of k satisfying $P(k) \equiv$ "$q_k \in G$" has "large" Ξ'-measure. Up to here, we get the notion used in [8] and [7] (but there we use ultrafilters instead of FAMs, and "large" means being in the ultrafilter). However, we need a modification: Instead of single conditions q_k we use a finite sequence $(p_l)_{l\in I_k}$ (where I_k is a fixed, finite interval); and the condition $P(k)$, which we want to satisfy on a large set, now is "$|\{l \in I_k : p_l \in G\}|/|I_k| > b$" for some suitable b. This is the notion used implicitly in [16].

Notation. Let T^* be a compact subtree of $\omega^{<\omega}$, for example $T^* = 2^{<\omega}$. Let $s, t \in T^*$. Let S be a subtree of T^*.

- $t \rhd s$ means "t is immediate successor of s".
- $|s|$ is the length of s (i.e.: the height, or level, of s).
- $[t]$ is the set of nodes in T^* comparable with t.
- We set $\lim(S) = \{x \in \omega^\omega \colon (\forall n \in \omega)\ x \restriction n \in S\}$.
- $\operatorname{trunk}(S)$ is the smallest splitting node of S. With "$t \in S$ above the stem" we mean that $t \in S$ and $t \geq \operatorname{trunk}(S)$; or equivalently: $t \in S$ and $|t| \geq |\operatorname{trunk}(S)|$.
- Leb is the canonical measure on the Borel subsets of $\lim(T^*)$. We also write $\operatorname{Leb}(S)$ instead of $\operatorname{Leb}(\lim(S))$.[2]

We fix for the rest of the paper an interval partition $\bar{I} = (I_k)_{k\in\omega}$ of ω such that $|I_k|$ converges to infinity. We will use forcing notions Q satisfying the following setup:

Assumption 1.5.
- $Q' \subseteq Q$ is dense and the domain of functions trunk and loss, where $\operatorname{trunk}(q) \in H(\aleph_0)$ and $\operatorname{loss}(q)$ is a non-negative rational.
- For each $\varepsilon > 0$ the set $\{q \in Q' \colon \operatorname{loss}(q) < \varepsilon\}$ is dense (in Q' and thus in Q).
- $\{p \in Q' \colon (\operatorname{trunk}(p), \operatorname{loss}(p)) = (\operatorname{trunk}^*, \operatorname{loss}^*)\}$ is $\lfloor 1/\operatorname{loss}^* \rfloor$-linked. I.e., each $\lfloor 1/\operatorname{loss}^* \rfloor$ many such conditions are compatible.[3]

In this paper, Q will be one of the following two forcing notions: random forcing, or $\widetilde{\mathbb{E}}$ (as defined in Definition 1.12). We will now specify the instance of random forcing that we will use:

Definition 1.6.
- A random condition is a tree $T \subseteq 2^{<\omega}$ such that the measure $\operatorname{Leb}(T \cap [t]) > 0$ for all $t \in T$.
- $\operatorname{trunk}(T)$ is the stem of T (i.e., the shortest splitting node).
- If $\operatorname{Leb}(T) = \operatorname{Leb}([\operatorname{trunk}(T)])$, we set $\operatorname{loss}(T) = 0$. Otherwise, let m be the maximal natural number such that
$$\operatorname{Leb}(T) > \operatorname{Leb}([\operatorname{trunk}(T)])\Big(1 - \frac{1}{m}\Big)$$
and set[4] $\operatorname{loss}(T) = 1/m$.

Note that $\operatorname{Leb}(T) \geq 2^{-|\operatorname{trunk}(T)|}(1 - \operatorname{loss}(T))$ (and the inequality is strict if $\operatorname{loss}(T) > 0$).

Note that this definition of random forcing satisfies Assumption 1.5 (with $Q' = Q$).

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[2]I.e., we define $\operatorname{Leb}([s])$ by induction on the height of $s \in T^*$ as follows: $\operatorname{Leb}(T^*) = 1$, and if s has n many immediate successors in T^*, then $\operatorname{Leb}([t]) = \operatorname{Leb}([s])/n$ for any such successor. This defines a measure on each basic clopen set, which in turn defines a (probability) measure on the Borel subsets of $\lim(T^*)$ (a closed subset of ω^ω).

[3]In [16, 2.9], trunk and loss are called h_2 and h_1; and instead of I_k the interval is called $[n^*_k, n^*_{k+1} - 1]$. Moreover, in [16] the sequence $(n^*_k)_{k\in\omega}$ is one of the parameters of a "blueprint", whereas we assume that the I_k are fixed.

[4]In [16], this is implicit in 2.11 (f).

Definition 1.7. Fix Q and functions (trunk, loss) as in Assumption 1.5, a FAM Ξ and a function $\lim_\Xi\colon Q^\omega \to Q$. Let us call the objects mentioned so far a "limit setup". Let a $(\mathrm{trunk}^*, \mathrm{loss}^*)$-sequence be a sequence $(q_l)_{l\in\omega}$ of Q-conditions such that $\mathrm{trunk}(q_l) = \mathrm{trunk}^*$ and $\mathrm{loss}(q_l) = \mathrm{loss}^*$ for all $l \in \omega$.

We say "$\lim_\Xi$ is a strong FAM limit for intervals", if the following is satisfied: Given

- a pair $(\mathrm{trunk}^*, \mathrm{loss}^*)$, $j^* \in \omega$, and $(\mathrm{trunk}^*, \mathrm{loss}^*)$-sequences $\bar q^j$ for $j < j^*$;
- $\varepsilon > 0$, $k^* \in \omega$;
- $m^* \in \omega$ and a partition of ω into sets B_m, $m \in m^*$; and
- a condition q stronger than all $\lim_\Xi(\bar q^j)$ for all $j < j^*$;

there is a finite $u \subseteq \omega \setminus k^*$ and a q' stronger than q such that

- $\Xi(B_m) - \varepsilon < |u \cap B_m|/|u| < \Xi(B_m) + \varepsilon$ for $m < m^*$;
- $|u|^{-1} \sum_{k\in u} |\{l \in I_k : q' \le q^j_l\}|/|I_k| \ \ge\ 1 - \mathrm{loss}^* - \varepsilon$ for $j < j^*$.

(We are only interested in $\lim_\Xi(\bar q)$ for $\bar q$ as above, so we can set $\lim_\Xi(\bar q)$ to be undefined or some arbitrary value for other $\bar q \in Q^\omega$.)

The motivation for this definition is the following:

Lemma 1.8. *Assume that* $\lim_\Xi$ *is such a limit. Then there is a Q-name Ξ^+ such that for every* $(\mathrm{trunk}^*, \mathrm{loss}^*)$*-sequence $\bar q$ the limit* $\lim_\Xi(\bar q)$ *forces* $\Xi^+(A_{\bar q}) \ge 1 - \sqrt{\mathrm{loss}^*}$, *where*

$$A_{\bar q} = \{k \in \omega\colon |\{l \in I_k : q_l \in G\}| \ge |I_k|(1 - \sqrt{\mathrm{loss}^*})\}. \tag{1.9}$$

Proof: Work in the Q-extension. Now Ξ is a partial FAM. Let J enumerate all suitable sequences $\bar q \in V$ with $\lim_\Xi(\bar q) \in G$, and for such a sequence $\bar q^j$ set $a^j_k = |\{l \in I_k : q^j_l \in G\}|/|I_k|$, and $b^j = 1-\mathrm{loss}^*$. Using that Ξ satisfies Definition 1.7, we can apply Fact 1.3 (b), we can extend Ξ to some FAM Ξ^+ such that $\mathrm{Av}_{\Xi^+}(\bar a^j) \ge 1 - \mathrm{loss}^*$ for $j < j^*$. So $\Xi^+(A_{\bar q^j}) + (1 - \Xi^+(A_{\bar q^j}))(1 - \sqrt{\mathrm{loss}^*}) \ge \mathrm{Av}_{\Xi^+}(a^j_k) \ge 1 - \mathrm{loss}^*$, and thus $\Xi^+(A_{\bar q^j}) \ge 1 - \sqrt{\mathrm{loss}^*}$. □

Definition 1.10. $(Q, \mathrm{trunk}, \mathrm{loss})$ as in Assumption 1.5 "has strong FAM limits for intervals", if for every FAM Ξ there is a function $\lim_\Xi$ that is a strong FAM limit for intervals.

Lemma 1.11 ([16])**.** *Random forcing has strong FAM-limits for intervals.*

Proof: $\lim_\Xi$ is implicitly defined in [16, 2.18], in the following way: Given a sequence r_l with $(\mathrm{trunk}(p_l), \mathrm{loss}(p_l)) = (\mathrm{trunk}^*, \mathrm{loss}^*)$, we can set $r^* = [\mathrm{trunk}^*]$ and $b = 1 - \mathrm{loss}^*$; and we set n^*_k such that $I_k = [n^*_k, n^*_{k+1} - 1]$. We now use these objects to apply [16, 2.18] (note that (c)($*$) is satisfied). This gives $r^\otimes$, and we define $\lim_\Xi(\bar r)$ to be $r^\otimes$.

In [16, 2.17], it is shown that this $r^\otimes$ satisfies Definition 1.7, i.e., is a limit: If r is stronger than all limits $r^{\otimes i}$, then r satisfies [16, 2.17 ($*$)]. □

1.2 The forcing $\widetilde{\mathbb{E}}$. We now define $\widetilde{\mathbb{E}}$, a variant of the forcing notion Q^2 defined in [9]:

Definition 1.12. By induction on the height $h \geq 0$, we define a compact homogeneous tree $T^* \subset \omega^{<\omega}$, and set

(1.13) $\varrho(h) := \max(|T^* \cap \omega^h|, h+2)$ and $\pi(h) := ((h+1)^2 \varrho(h)^{h+1})^{\varrho(h)^h}$,

we set Ω_s to be the set $\{t \rhd s : t \in T^*\}$, i.e., the set of immediate successors of s, and define for each s a norm μ_s on the subsets of Ω_s. In more detail:

- The unique element of T^* of height 0 is $\langle\rangle$, i.e., $T^* \cap \omega^0 = \{\langle\rangle\}$.
- We set

$$a(h) = \pi(h)^{h+2}, \qquad M(h) = a(h)^2, \qquad \text{and} \qquad \mu_h(n) = \log_{a(h)}\Big(\frac{M(h)}{M(h)-n}\Big)$$

  for natural numbers $0 \leq n < M(h)$, and we set $\mu_h(M(h)) = \infty$.
- For any $s \in T^* \cap \omega^h$, we set $\Omega_s = \{s^\frown l : l \in M(h)\}$ (which defines $T^* \cap \omega^{h+1}$). For $A \subset \Omega_s$, we set $\mu_s(A) := \mu_h(|A|)$. So $|\Omega_s| = M(h)$, $\mu_s(\emptyset) = 0$ and $\mu_s(\Omega_s) = \infty$. Note that $|A| = |\Omega_s|(1 - a(h)^{-\mu_s(A)})$.

We can now define $\widetilde{\mathbb{E}}$:

Definition 1.14.

- For a subtree $p \subseteq T^*$, the stem of p is the smallest splitting node. For $s \in p$, we set $\mu_s(p) = \mu_s(\{t \in p : t \rhd s\})$.
  The set $\widetilde{\mathbb{E}}$ consists of subtrees p with some stem s^* of height h^* such that $\mu_t(p) \geq 1 + 1/h^*$ for all $t \in p$ above the stem. (So the only condition with $h^* = 0$ is the full condition, where all norms are ∞.)
  The set $\widetilde{\mathbb{E}}$ is ordered by inclusion.
- trunk(p) is the stem of p.
  loss(p) is defined if there is an $m \geq 2$ satisfying the following, and in that case loss$(p) = 1/m$ for the maximal such m:
  - p has stem s^* of height $h^* > 3m$,
  - $\mu_s(p) \geq 1 + 1/m$ for all $s \in p$ of height greater than or equal to h^*.

  We set $Q' = \mathrm{dom}(\mathrm{loss})$.

By simply extending the stem, we can find for any $p \in \widetilde{\mathbb{E}}$ and $\varepsilon > 0$ some $q \leq p$ in Q' with loss$(q) < \varepsilon$; i.e., one of Assumptions 1.5 is satisfied. (The other one is dealt with in Lemma 1.19 (a).) In particular $Q' \subseteq \widetilde{\mathbb{E}}$ is dense.

We list a few trivial properties of the loss function:

Facts 1.15. Assume $p \in Q'$ with $s = \mathrm{trunk}(p)$ of height h.

(a) loss$(p) < 1$, $\mu_s(p) \geq 1 + \mathrm{loss}(p)$ for any s above the stem, and loss$(p) > 3/h$.

(b) If q is a subtree of p such that all norms above the stem are greater than or equal to $1 + \mathrm{loss}(p) - 2/h$, then q is a valid $\widetilde{\mathbb{E}}$-condition.

(c) $\prod_{l=h}^{\infty}(1 - 1/l^2) = 1 - 1/h > 1 - \mathrm{loss}(p)/3$.

Lemma 1.16. *Let $s \in T^*$ be of height h and $A \subset \Omega_s$.*

(a) *If $\mu_s(A) \geq 1$, then $|A| \geq |\Omega_s|(1 - 1/h^2)$.*

(b) *If $A \subsetneq \Omega_s$, i.e., A is a proper subset, then $\mu_s(A \setminus \{t\}) > \mu_s(A) - 1/h$ for $t \in A$.*

(c) *For $i < \pi(h)$, assume that $A_i \subseteq \Omega_s$ satisfies $\mu_s(A_i) \geq x$. Consequently $\mu_s\big(\bigcap_{i\in\pi(h)} A_i\big) > x - 1/h$.*

(d) *For $i < I$ (an arbitrary finite index set) pick proper subsets $A_i \subsetneq \Omega_s$ such that $\mu_s(A_i) \geq x$, and assign weighs a_i to A_i such that $\sum_{i\in I} a_i = 1$. Then*

$$\mu_s(B) > x - \frac{1}{h} \qquad \text{for } B := \Big\{t \in \Omega_s \colon \sum_{t\in A_i} a_i > 1 - \frac{1}{h^2}\Big\}. \tag{1.17}$$

Proof: (a) Trivial, as $a(h)^{-\mu_s(A)} \leq 1/a(h) < 1/h^2$.

(b)
$$\begin{aligned}\mu_s(A \setminus \{t\}) &= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(|\Omega_s| - |A| + 1)\\ &\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)}(2(|\Omega_s| - |A|))\\ &\geq \mu_s(A) - \log_{a(h)}(2) > \mu_s(A) - 1/h.\end{aligned}$$

(c)
$$\begin{aligned}\mu_s\big(\textstyle\bigcap_{i\in\pi(h)} A_i\big) &= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}\big(|\Omega_s| - \big|\textstyle\bigcap_{i\in\pi(h)} A_i\big|\big)\\ &= \log_{a(h)}(|\Omega_s|) - \log_{a(h)}\big(\big|\textstyle\bigcup_{i\in\pi(h)}(\Omega_s - A_i)\big|\big)\\ &\geq \log_{a(h)}(|\Omega_s|) - \log_{a(h)}\big(\pi(h)\cdot \max_{i\in\pi(h)}|\Omega_s - A_i|\big)\\ &\geq x - \log_{a(h)}(\pi(h)) > x - 1/h.\end{aligned}$$

(d) Set $y = \sum_{i\in I} a_i|A_i|$. On the one hand, $y \geq |\Omega_s|(1 - a(h)^{-x})$. On the other hand, $y = \sum_{t\in\Omega_s}\sum_{t\in A_i} a_i \leq |B| + (|\Omega_s \setminus B|)(1 - 1/h^2)$.
So $|B| \geq |\Omega_s|(1 - h^2 a(h)^{-x}) > |\Omega_s|(1 - a(h)^{-(x-1/h)})$, as $a(h)^{1/h} > \pi(h) > h^2$.

□

The set $\widetilde{\mathbb{E}}$ is not σ-centered, but it satisfies a property, first defined in [15], which is between σ-centered and σ-linked:

Definition 1.18. Fix f, g functions from ω to ω converging to infinity. Set Q is (f, g)-linked if there are $g(i)$-linked $Q^i_j \subseteq Q$ for $i < \omega$, $j < f(i)$ such that each $q \in Q$ is in every $\bigcup_{j<f(i)} Q^i_j$ for sufficiently large i.

Recall that we have defined ϱ and π in (1.13).

Lemma 1.19. (a) *If $\pi(h)$ many conditions $(p_i)_{i\in\pi(h)}$ have a common node s above their stems, $|s| = h$, then there is a q stronger than each p_i.*

(b) *The set $\widetilde{\mathbb{E}}$ is (ϱ, π)-linked (in particular it is countable chain condition (ccc)).*

(c) *The $\widetilde{\mathbb{E}}$-generic real η is eventually different (from every real in $\lim(T^*)$, and therefore from every real in ω^ω as well).*

(d) $\mathrm{Leb}(p) \geq \mathrm{Leb}([\mathrm{trunk}(p)])(1 - \mathrm{loss}(p)/2)$; *more explicitly: for any $h > |\mathrm{trunk}(p)|$,*

$$\frac{|p \cap \omega^h|}{|T^* \cap \omega^h \cap [\mathrm{trunk}(p)]|} \geq 1 - \frac{1}{2}\,\mathrm{loss}(p).$$

(e) *The set Q' (which is a dense subset of $\widetilde{\mathbb{E}}$) is an incompatibility-preserving subforcing of random forcing, where we use the variant*[5] *of random forcing on* $\lim(T^*)$ *instead of* 2^ω*. Let B' be the sub-Boolean-algebra of* Borel/Null *generated by* $\{\lim(q)\colon q \in Q'\}$*. Then Q' is dense in B'.*

(Here, Borel refers to the set of Borel subsets of $\lim(T^*)$. In the following proof, we will denote the equivalence class of a Borel set A by $[A]_{\mathcal{N}}$.)

Proof: (a) Set $S = [s] \cap \bigcap_{i<\pi(h)} p_i$. According to 1.16 (c), for each $t \in S$ of height $h' \geq h$, the successor set has norm bigger than $1 + 1/h - 1/h' > 1$, so in particular there is a branch $x \in S$, and $S \cap [x \restriction 2h]$ is a valid condition stronger than all p_i.

(b) For each $h \in \omega$, enumerate $T^* \cap \omega^h$ as $\{s^h_1, \dots, s^h_{\varrho(h)}\}$, and set $Q^h_i = \{p \in \widetilde{\mathbb{E}}\colon s^h_i \in p \text{ and } |\operatorname{trunk}(p)| \leq h\}$. So for all h, Q^h_i is $\pi(h)$-linked, and $p \in \bigcup_{i<\varrho(h)} Q^h_i$ for all $p \in Q$ with $|\operatorname{trunk}(p)| \leq h$.

(c) Use 1.16 (b).

(d) Use 1.16 (a) and the definition of loss.

(e) As in the previous item, we get that $\operatorname{Leb}(p \cap [t]) > 0$ whenever $p \in Q'$ and $t \in p$. So Q' is a subset of random forcing. As both sets are ordered by inclusion, Q' is a subforcing. If $q_1, q_2 \in Q'$ and q_1, q_2 are compatible as random conditions, then $q_1 \cap q_2$ has arbitrary high nodes, in particular a node above both stems, which implies that q_1 is compatible with q_2 in $\widetilde{\mathbb{E}}$ and therefore in Q'. It remains to show that Q' is dense in B'. It is enough to show: If $x \neq 0$ in B' has the form $x = \bigwedge_{i<i^*} [\lim(q_i)]_{\mathcal{N}} \wedge \bigwedge_{j<j^*} [\lim(T^*) \setminus \lim(q_j)]_{\mathcal{N}}$ then there is some $q \in Q'$ with $[\lim(q)]_{\mathcal{N}} < x$. Note that $0 \neq x = [A]_{\mathcal{N}}$ for $A = \lim\left(\bigcap_{i<i^*} q_i\right) \setminus \bigcup_{j<j^*} \lim(q_j)$, so pick some $r \in A$ and pick $h > i^*$ large enough such that $s = r \restriction h$ is not in any q_j. Then any $q \in Q'$ stronger than all $q_i \cap [s]$ for $i < i^*$ is as required. □

Lemma 1.20. *The set $\widetilde{\mathbb{E}}$ has strong FAM-limits for intervals.*

Proof: Let $(p_l)_{l\in\omega}$ be a (s^*, loss^*)-sequence, s^* of height h^*. Set $\widetilde{\zeta}^{h^*} = 0$ and

$$\widetilde{\zeta}^{h} := 1 - \prod_{m=h^*}^{h-1} \left(1 - \frac{1}{m^2}\right) \qquad \text{for } h > h^*.$$

This is a strictly increasing sequence below $\text{loss}^*/3$, cf. Fact 1.15 (c). Also, all norms in all conditions of the sequence are at least $1 + \text{loss}^*$, cf. Fact 1.15 (a).

We will first construct $(q_k)_{k\in\omega}$ with stem s^* and all norms greater than $1 + \text{loss}^* - 1/h^*$ such that q_k forces $|\{l \in I_k : p_l \in G\}|/|I_k| > 1 - \text{loss}^*/3$. We will then use $\bar{q}$ to define $\lim_\Xi(\bar{p})$, and in the third step show that it is as required.

Step 1: So let us define q_k. Fix $k \in \omega$.

[5] We can use Definition 1.6, replacing 2^ω with $\lim(T^*)$.

- Set $X_t = \{l \in I_k : t \in p_l\}$ and $Y_h = \{t \in [s^*] \cap \omega^h : |X_t| \geq |I_k|(1 - \widetilde{\zeta}^{\,h})\}$.
- We define q_k by induction on the level, such that $q_k \cap \omega^h \subseteq Y_h$. The stem is s^*. (Note that $X_{s^*} = I_k$ and so $s^* \in Y_{h^*}$.) For $s \in q_k \cap \omega^h$ (and thus, by induction hypothesis, in Y_h), we set $q_k \cap [s] \cap \omega^{h+1} = [s] \cap Y_{h+1}$, i.e., a successor t of s is in q_k if and only if it is Y_{h+1}. Then $\mu_s(q_k) > 1 + \text{loss}^* - 1/h$.

  Proof: Set $I = X_s$. By induction, $|X_s| \geq |I_k|(1 - \widetilde{\zeta}^{\,h})$. For $l \in I$, set $A_l = p_l \cap [s] \cap \omega^{h+1}$, i.e., the immediate successors of s in p_l. Obviously $\mu_s(A_l) \geq 1 + \text{loss}^*$. We give each A_l equal weight $a_l = 1/|I|$. According to (1.17), the set $B = \{t \rhd s : |\{l \in X_s : t \in A_l\}| \geq |I|(1 - 1/h^2)\}$ has norm greater than $1 + \text{loss}^* - 1/h$. □

- The condition q_k forces that $p_l \in G$ for $\geq |I_k|(1 - \text{loss}^*/2)$ many $l \in I_k$.

  Proof: Let $r < q_k$ have stem s' of length h', without loss of generality $h' > |I_k| + 1$. As $s' \in Y_{h'}$, there are greater than $|I_k|(1 - \text{loss}^*/3)$ many $l \in I_k$ such that $s' \in p_l$. So we can find a condition r' stronger than r and all these p_l (as these are at most $|I_k| + 1 \leq h'$ many conditions all containing s' above the stem). □

Step 2: Now we use $(q_k)_{k\in\omega}$ to construct by induction on the height $q^* = \lim_\Xi(\bar{p})$, a condition with stem s^* and all norms greater than or equal to $1 + \text{loss}^* - 2/h$ such that for all $s \in q^*$ of height $h \geq h^*$,

$$(*)\quad \Xi(Z_s) \geq 1 - \tilde{\zeta}^h \qquad \text{for } Z_s := \{k \in \omega : s \in q_k\}. \qquad \text{So } \ \Xi(Z_s) > 1 - \frac{1}{3}\,\text{loss}^*\,.$$

Note that $Z_{s^*} = \omega$, so $(*)$ is satisfied for s^*. Fix an $s \geq s^*$ satisfying $(*)$. Set $A(k)$ to be the s-successors in q_k for each $k \in Z_s$. Enumerate the (finitely many) $A(k)$ as $(A_i)_{i\in I}$. Clearly $\mu_s(A_i) > 1 + \text{loss}^* - 1/h$. Assign to A_i the weight $a_i = (1/\Xi(Z_s))\Xi(\{k \in Z_s : A(k) = A_i\})$. Again using (1.17), $\mu_s(B) \geq 1 + \text{loss}^* - 2/h$, where B consists of those successors t of s such that

$$1 - \frac{1}{h^2} < \sum_{t\in A_i} a_i = \frac{1}{\Xi(Z_s)}\Xi(\{k \in Z_s : t \in q_k\}) \leq \frac{1}{\Xi(Z_s)}\Xi(Z_t).$$

So every $t \in B$ satisfies $\Xi(Z_t) > \Xi(Z_s)(1 - 1/h^2) \geq \widetilde{\zeta}^{\,h+1}$, i.e., satisfies $(*)$. So we can use B as the set of s-successors in q^*.

This defines q^*, which is a valid condition by Fact 1.15 (b).

Step 3: We now show that this limit works: As in Definition 1.7, fix m^*, $(B_m)_{m<m^*}$, ε, k^*, i^* and sequences $(p^i_l)_{l<\omega}$ for $i < i^*$, such that $(\text{trunk}(p^i_l), \text{loss}(p^i_l)) = (\text{trunk}^*, \text{loss}^*)$.

For each $i < i^*$, $\bar{q}^{\,i} = (q^i_k)_{k\in\omega}$ is defined from $\bar{p}^{\,i} = (p^i_l)_{l\in\omega}$, and in turn defines the limit $\lim_\Xi(\bar{p}^{\,i})$. Let q be stronger than all $\lim_\Xi(\bar{p}^{\,i})$.

Let M be as in Lemma 1.2 for $N = m^* + i^*$. So for any N many sets there is a u of size at most M (above k^*) which approximates the measure well. We use the following N many sets:

- ∘ B_m for $m < m^*$.
- ∘ Fix an $s \in q$ of height $h > M \cdot i^*$, and use the i^* many sets $Z^i_s \subseteq \omega$ defined in $(*)$.

Accordingly, there is a u (starting above k^*) of size less than or equal to M with

- ∘ $\Xi(B_m) - \varepsilon \leq |B_m \cap u|/|u| \leq \Xi(B_m) + \varepsilon$ for each $m < m^*$, and
- ∘ $|Z^i_s \cap u|/|u| \geq 1 - \text{loss}^*/3 - \varepsilon$ for each $i < i^*$.

So for each $i \in i^*$ there are at least $|u|(1 - \text{loss}^*/2 - \varepsilon)$ many $k \in u$ with $s \in q^i_k$. There is a condition r stronger than q and all those q^i_k (as less than or equal to $Mi^* + 1$ many conditions of height $h > M \cdot i^*$ with common node s above their stems are compatible). So r forces for all $i < i^*$ and $k \in u \cap Z^i_s$ that $q^i_k \in G$ and therefore that $|\{l \in I_k : p^i_l \in G\}| \geq |I_k|(1 - \text{loss}^*/3)$. By increasing r to some q', we can assume that r decides which p^i_l are in G and that r is actually stronger than each p^i_l decided to be in G. So all in all we get $q' \leq q$ such that

$$\frac{1}{|u|}\sum_{k\in u}\frac{|\{l \in I_k : q' \leq p^j_l\}|}{|I_k|} \;\geq\; \frac{1}{|u|}|\{k \in u : k \in Z^j_s\}|\Big(1 - \frac{1}{3}\,\text{loss}^*\Big) \;>\; 1 - \text{loss}^* - \varepsilon,$$

as required. □

2. The left hand side of Cichoń's diagram

We write $\mathfrak{x}_1$ for $\text{add}(\mathcal{N})$, $\mathfrak{x}_2$ for $\mathfrak{b}$ (which will also be $\text{add}(\mathcal{M})$), $\mathfrak{x}_3$ for $\text{cov}(\mathcal{N})$ and $\mathfrak{x}_4$ for $\text{non}(\mathcal{M})$.

2.1 Good iterations and the LCU property. We want to show that some forcing $\mathbb{P}^5$ results in $\mathfrak{x}_i = \lambda_i$ for $i = 1, \dots, 4$. So we have to show two "directions", $\mathfrak{x}_i \leq \lambda_i$ and $\mathfrak{x}_i \geq \lambda_i$.

For $i = 1, 3, 4$ (i.e., for all the characteristics on the left hand side apart from $\mathfrak{b} = \text{add}(\mathcal{M})$), the direction $\mathfrak{x}_i \leq \lambda_i$ will be given by the fact that $\mathbb{P}^5$ is (R_i, λ_i)-good for a suitable relation R_i. (For $i = 2$, i.e., the unbounding number, we will have to work more.)

We will use the following relations:

Definition 2.1.
1. Let $\mathcal{C}$ be the set of strictly positive rational sequences $(q_n)_{n\in\omega}$ such that $\sum_{n\in\omega} q_n \leq 1$.[6] Let $\mathrm{R}_1 \subseteq \mathcal{C}^2$ be defined by: $f \,\mathrm{R}_1\, g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
2. $\mathrm{R}_2 \subseteq (\omega^\omega)^2$ is defined by: $f \,\mathrm{R}_2\, g$ if $(\forall^* n \in \omega)\, f(n) \leq g(n)$.
4. $\mathrm{R}_4 \subseteq (\omega^\omega)^2$ is defined by: $f \,\mathrm{R}_4\, g$ if $(\forall^* n \in \omega)\, f(n) \neq g(n)$.

[6] It is easy to see that $\mathcal{C}$ is homeomorphic to ω^ω, when we equip the rationals with the discrete topology and use the product topology.

So far, these relations fit the usual framework of goodness, as introduced in [10] and [3] and summarized, e.g., in [2, 6.4] or [8, Section 3] or [13, Section 2]. For $\mathfrak{x}_3$, i.e., $\mathrm{cov}(\mathcal{N})$, we will use a relation R_3 that does not fit this framework (as the range of the relation is not a Polish space). Nevertheless, the property "(R_3, λ)-good" behaves just as in the usual framework (e.g., finite support limits of good forcings are good, etc.). The relation R_3 was implicitly used by S. Kamo and N. Osuga in [15], who investigated (R_3, λ)-goodness.[7] It was also used in [4]; a unifying notation for goodness (which works for the usual cases as well as relations such as R_3) is given in [5, Section 4].

Definition 2.2. We call a set $\mathcal{E} \subset \omega^\omega$ an R_3-parameter, if for all $e \in \mathcal{E}$

- $\lim e(n) = \infty$, $e(n) \leq n$, $\lim(n - e(n)) = \infty$,
- there is some $e' \in \mathcal{E}$ such that $(\forall^* n)\, e(n) + 1 \leq e'(n)$, and
- for all countable $\mathcal{E}' \subseteq \mathcal{E}$ there is some $e \in \mathcal{E}$ such that for all $e' \in \mathcal{E}'$ $(\forall^* n)\, e(n) \geq e'(n)$.

Note that such an R_3-parameter of size $\aleph_1$ exists. This is trivial if we assume continuum hypothesis (CH), which we could in this paper, but also true without this assumption, see [5, 4.20]. Recall that ϱ and π were defined in equation (1.13).

Definition 2.3. We fix for the rest of the paper, an R_3-parameter $\mathcal{E}$ of size $\aleph_1$, and set

$$b(h) = (h+1)^2 \varrho(h)^{h+1}, \quad \mathcal{S} = \left\{ \psi \in \prod\nolimits_{h \in \omega} P(b(h)) \colon (\forall\, h \in \omega)\, |\psi(h)| \leq \varrho(h)^h \right\},$$

$$\mathcal{S}_e = \left\{ \varphi \in \prod\nolimits_{h \in \omega} P(b(h)) \colon (\forall\, h \in \omega)\, |\varphi(h)| \leq \varrho(h)^{e(h)} \right\} \qquad \text{and} \qquad \widehat{\mathcal{S}} = \bigcup_{e \in \mathcal{E}} \mathcal{S}_e.$$

We can now define the relation for $\mathrm{cov}(\mathcal{N})$:

3. $\mathrm{R}_3 \subseteq \mathcal{S} \times \widehat{\mathcal{S}}$ is defined by: $\psi \,\mathrm{R}_3\, \varphi$ if and only if $(\forall^* n \in \omega)\, \varphi(n) \not\subseteq \psi(n)$.

Note that $\mathcal{S}_e \subset \widehat{\mathcal{S}} \subset \mathcal{S}$ and that $\mathcal{S}_e$ and $\mathcal{S}$ are Polish spaces. Assume that M is a forcing extension of V by either a ccc forcing (or by a σ-closed forcing). Then $\mathcal{E}$ is an "R_3-parameter" in M as well, and we can evaluate in M for each $e \in \mathcal{E}$ the sets $\mathcal{S}_e^M$ and $\mathcal{S}^M$, as well as $\widehat{\mathcal{S}}^M = \bigcup_{e \in \mathcal{E}} \mathcal{S}_e^M$. Absoluteness gives $\mathcal{S}_e^V = \mathcal{S}_e^M \cap V$ and $\widehat{\mathcal{S}}^V = \widehat{\mathcal{S}}^M \cap V$.

Definition 2.4. Fix one of these relations $R \subseteq X \times Y$.

- We say "f is bounded by g" if $f \,\mathrm{R}\, g$, and for $\mathcal{Y} \subseteq \omega^\omega$ "f is bounded by $\mathcal{Y}$" if $(\exists\, y \in \mathcal{Y})\, f \,\mathrm{R}\, y$. We say "unbounded" for "not bounded". (I.e., f is unbounded by $\mathcal{Y}$ if $(\forall\, y \in \mathcal{Y})\, \neg f \,\mathrm{R}\, y$.)
- We call $\mathcal{X}$ an R-unbounded family, if $\neg(\exists\, g)\, (\forall\, x \in \mathcal{X}) x \,\mathrm{R}\, g$, and an R-dominating family if $(\forall\, f)\, (\exists\, x \in \mathcal{X})\, f \,\mathrm{R}\, x$.

[7] They use the notation $(*^{<\lambda}_{c,h})$, cf. [15, Definition 6].

- Let $\mathfrak{b}_i$ be the minimal size of an R_i-unbounded family,
- and let $\mathfrak{d}_i$ be the minimal size of an R_i-dominating family.

We only need the following connection between R_i and the cardinal characteristics:

Lemma 2.5. (1) $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_1$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}_1$.
(2) $\mathfrak{b} = \mathfrak{b}_2$ *and* $\mathfrak{d} = \mathfrak{d}_2$.
(3) $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_3$ *and* $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_3$.
(4) $\mathrm{non}(\mathcal{M}) = \mathfrak{b}_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_4$.

Proof: (2) holds by definition. (1) can be found in [2, 6.5.B]. (4) is a result of [14] and [1], cf. [2, 2.4.1 and 2.4.7].

To see (3), we work in the space $\Omega = \prod_{h\in\omega} b(h)$, with the b defined in Definition 2.3 and the usual (uniform) measure. It is well known that we get the same values for the characteristics $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$ whether we define them using Ω or, as usual, 2^ω (or $[0,1]$ for that matter, etc). Given $\psi \in \mathcal{S}$, note that

$$N_\psi = \{\eta \in \Omega \colon (\exists^\infty h)\ \eta(h) \in \psi(h)\}$$

is a Null set, as $\{\eta \in \Omega \colon (\forall h > k)\ \eta(h) \notin \psi(h)\}$ has measure $\prod_{h>k}(1 - |\psi(h)|/b(h)) \geq \prod_{h>k}(1-(h+1)^{-3})$, which converges to 1 for $k \to \infty$.

Let $\mathcal{A} \subseteq \mathcal{S}$ be an R_3-unbounded family. So for every $\varphi \in \widehat{\mathcal{S}}$ there is some $\psi \in A$ such that $(\exists^\infty h)\ \psi(h) \supseteq \varphi(h)$. In particular, for each $\eta \in \Omega$ there is a $\psi \in A$ with $\eta \in N_\psi$; i.e., $\mathrm{cov}(\mathcal{N}) \leq |\mathcal{A}|$.

Analogously, let X be a non-null set (in Ω). For each ψ there is an $x \in X \setminus N_\psi$, so $\varphi_x(n) = \{x(n)\}$ satisfies $\psi \,\mathrm{R}_3\, \varphi_x$. $\square$

Remark 2.6. As shown implicitly in [15], and explicitly in [5, 4.22], we actually get $\mathrm{cov}(\mathcal{N}) \leq c^{\exists}_{b,\varrho^{\mathrm{Id}}} \leq \mathfrak{b}_3$.

Definition 2.7. Let P be a ccc forcing, λ an uncountable regular cardinal, and $\mathrm{R}_i \subseteq X \times Y$ one of the relations above (so for $i = 1,2,4$, $Y = X$, and for $i = 3$ $Y = \widehat{\mathcal{S}}_e$). The forcing P is (R_i, λ)-good, if for each P-name r for an element of Y there is (in V) a nonempty set $\mathcal{Y} \subseteq Y$ of size less than λ such that every $f \in X$ (in V) that is R_i-unbounded by $\mathcal{Y}$ is forced to be R_i-unbounded by r as well.

Note that λ-good trivially implies μ-good if $\mu \geq \lambda$ are regular.

Lemma 2.8. *Let λ be uncountable regular.*

(a) *Forcings of size less than λ are (R_i, λ)-good. In particular, Cohen forcing is $(\mathrm{R}_i, \aleph_1)$-good.*
(b) *A FS ccc iteration of (R_i, λ)-good forcings (and in particular, a composition of two such forcings) is (R_i, λ)-good.*
(1) *A sub-Boolean-algebra of the random algebra is $(\mathrm{R}_1, \aleph_1)$-good. Any σ-centered forcing notion is $(\mathrm{R}_1, \aleph_1)$-good.*
(3) *A (ϱ, π)-linked forcing is $(\mathrm{R}_3, \aleph_1)$-good (for the ϱ, π of Definition 1.12).*

Proof: (a) & (b) For $i = 1, 2, 4$ this is proven in [10], cf. [2, 6.4]. The same proof works for $i = 3$, as shown in [15, Lemmas 12, 13]. The proof for the uniform framework can be found in [5, 4.10, 4.14].

(1) follows from [10] and [11], cf. [2, 6.5.17–18].

(3) is shown in [15, Lemma 10], cf. [5, Lemma 4.24]; as our choice of π, ϱ and b (see Definition 2.3) satisfies $\pi(h) \geq b(h)^{\varrho(h)^h} = ((h+1)^2 \varrho(h)^{h+1})^{\varrho(h)^h}$. □

Each relation R_i is a subset of some $X \times Y$, where X is either 2^ω, ω^ω (or homeomorphic to it) or $\mathcal{S}$, and Y is the range of R_i.

Lemma 2.9. *For each i and each $g \in Y$, the set $\{f \in X \colon f \,\mathrm{R}_i\, g\} \subseteq X$ is meager.*

Proof: We have explicitly defined each $f \,\mathrm{R}_i\, g$ as $\forall^* n \; R_i^n(f, g)$ for some R_i^n. The lemma follows easily from the fact that for each $n \in \omega$, the set $\{f \in X \colon R_i^n(f, g)\}$ is closed nowhere dense. □

Lemma 2.10. *Let $\lambda \leq \kappa \leq \mu$ be uncountable regular cardinals. Force with μ many Cohen reals $(c_\alpha)_{\alpha \in \mu}$, followed by an (R_i, λ)-good forcing. Note that each Cohen real c_β can be interpreted as element of the Polish space X where $\mathrm{R}_i \subseteq X \times Y$. Then we get: For every real r in the final extension Y, the set $\{\alpha \in \kappa \colon c_\alpha$ is R_i-unbounded by $r\}$ is cobounded in κ. I.e., $(\exists\, \alpha \in \kappa)\, (\forall\, \beta \in \kappa \setminus \alpha)$ $\neg c_\alpha \,\mathrm{R}_i\, r$.*

Proof: Work in the intermediate extension after κ many Cohen reals, let us call it V_κ. The remaining forcing (i.e., $\mu \setminus \kappa$ many Cohens composed with the good forcing) is good; so applying the definition we get (in V_κ) a set $\mathcal{Y} \subseteq Y$ of size less than λ.

As the initial Cohen extension is ccc, and $\kappa \geq \lambda$ is regular, we get some $\alpha \in \kappa$ such that each element y of $\mathcal{Y}$ already exists in the extension by the first α many Cohens, call it V_α.

Fix some $\beta \in \kappa \setminus \alpha$ and $y \in Y$. As $\{x \in X \colon x \,\mathrm{R}_i\, y\}$ is a meager set already defined in V_α, we get $\neg c_\beta \,\mathrm{R}_i\, y$. Accordingly, c_β is unbounded by $\mathcal{Y}$; and, by the definition of good, unbounded by r as well. □

In the light of this result, let us revisit Lemma 2.5 with some new notation, the "linearly cofinally unbounded" property LCU:

Definition 2.11. For $i = 1, 2, 3, 4$, γ a limit ordinal, and P a ccc forcing notion, let $\mathsf{LCU}_i(P, \gamma)$ stand for:

> There is a sequence $(x_\alpha)_{\alpha \in \gamma}$ of P-names such that for every P-name y $\;(\exists\, \alpha \in \gamma)\, (\forall\, \beta \in \gamma \setminus \alpha)\; P \Vdash \neg x_\beta \,\mathrm{R}_i\, y)$.

Lemma 2.12. ◦ *The $\mathsf{LCU}_i(P, \delta)$ property is equivalent to $\mathsf{LCU}_i(P, \mathrm{cf}(\delta))$.*

◦ *If λ is regular, then $\mathsf{LCU}_i(P, \lambda)$ implies $\mathfrak{b}_i \leq \lambda$ and $\mathfrak{d}_i \geq \lambda$.*

In particular:

1. *The $\mathsf{LCU}_1(P, \lambda)$ property implies $P \Vdash (\,\mathrm{add}(\mathcal{N}) \leq \lambda \;\&\; \mathrm{cof}(\mathcal{N}) \geq \lambda\,)$.*
2. *The $\mathsf{LCU}_2(P, \lambda)$ property implies $P \Vdash (\,\mathfrak{b} \leq \lambda \;\&\; \mathfrak{d} \geq \lambda\,)$.*

(3) *The* $\mathsf{LCU}_3(P,\lambda)$ *property implies* $P \Vdash (\,\mathrm{cov}(\mathcal{N}) \le \lambda \ \&\ \mathrm{non}(\mathcal{N}) \ge \lambda\,)$.
(4) *The* $\mathsf{LCU}_4(P,\lambda)$ *property implies* $P \Vdash (\,\mathrm{non}(\mathcal{M}) \le \lambda \ \&\ \mathrm{cov}(\mathcal{M}) \ge \lambda\,)$.

Proof: Assume that $(\alpha_\beta)_{\beta\in\mathrm{cf}(\delta)}$ is increasing continuous and cofinal in δ. If $(x_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P,\delta)$, then $(x_{\alpha_\beta})_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$. And if $(x_\beta)_{\beta\in\mathrm{cf}(\delta)}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$, then $(y_\alpha)_{\alpha\in\delta}$ witnesses $\mathsf{LCU}_i(P,\mathrm{cf}(\delta))$, where $y_\alpha := x_\beta$ for $\alpha \in [\alpha_\beta, \alpha_{\beta+1})$.

The set $\{x_\alpha\colon \alpha\in\lambda\}$ is certainly forced to be R_i-unbounded; and given a set $Y = \{y_j\colon j<\theta\}$ of $\theta<\lambda$ many P-names, each has a bound $\alpha_j\in\lambda$ so that $(\forall\,\beta\in\lambda\setminus\alpha_j)\ P\Vdash \neg x_\beta\,\mathrm{R}_i\,y_j)$, so for any $\beta\in\lambda$ above all α_j we get $P\Vdash\neg x_\beta\,\mathrm{R}_i\,y_j$ for all j; i.e., Y cannot be dominating. □

2.2 The initial forcing $\mathbb{P}^5$ and the COB property. We will assume the following throughout the paper:

Assumption 2.13.
- $\lambda_1<\lambda_2<\lambda_3<\lambda_4<\lambda_5$ are regular uncountable cardinals such that $\mu<\lambda_i$ implies $\mu^{\aleph_0}<\lambda_i$.
- We set $\delta_5=\lambda_5+\lambda_5$, and partition $\delta_5\setminus\lambda_5$ into unbounded sets S^i for $i=1,\dots,4$. Fix for each $\alpha\in\delta_5\setminus\lambda_5$ a $w_\alpha\subseteq\alpha$ such that $\{w_\alpha\colon\alpha\in S^i\}$ is cofinal[8] in $[\delta_5]^{<\lambda_i}$ for each $i=1,\dots,4$.

The reader can assume that $(\lambda_i)_{i=1,\dots,5}$ and $(S^i)_{i=1,\dots,4}$ have been fixed once and for all (let us call them "fixed parameters"), whereas we will investigate various possibilities for $\overline{w}=(w_\alpha)_{\alpha\in\delta_5\setminus\lambda_5}$ in the following. (We will call a $\overline{w}$ which satisfies the assumption a "cofinal parameter".)

We define by induction:

Definition 2.14. We define the FS iteration $(P_\alpha,Q_\alpha)_{\alpha\in\delta_5}$ and for $\alpha>\lambda_5$, P'_α as follows: If $\alpha\in\lambda_5$, then Q_α is Cohen forcing. In particular, the generic at α is determined by the Cohen real η_α. For $\alpha\in\delta_5\setminus\lambda_5$:

(1) $$Q^{\mathrm{full}}_\alpha := \left\{\begin{array}{c}\text{Amoeba}\\ \text{Hechler}\\ \text{Random}\\ \widetilde{\mathbb{E}}\end{array}\right\} \qquad \text{for } \alpha \text{ in} \left\{\begin{array}{c}S^1\\ S^2\\ S^3\\ S^4\end{array}\right. .$$

So Q^{full}_α is a Borel definable subset of the reals, and the Q^{full}_α-generic is determined, in a Borel way, by the canonical generic real η_α.

(2) The set P'_α is the set of conditions $p\in P_\alpha$ satisfying the following for each $\beta\in\mathrm{supp}(p)$: $\beta\in w_\alpha$ and there is (in the ground model) a countable $u\subseteq w_\alpha\cap\beta$ and a Borel function $B\colon(\omega^\omega)^u\to Q^{\mathrm{full}}_\beta$ such that $p\restriction\beta$ forces that $p(\beta)=B((\eta_\gamma)_{\gamma\in u})$. We *assume* that

$$P'_\alpha \text{ is a complete subforcing of } P_\alpha. \tag{2.15}$$

(3) In the P_α-extension, let M_α be the induced P'_α-extension of V. Then Q_α is the M_α-evaluation of Q^{full}_α. Or equivalently (by absoluteness): $Q_\alpha =$

[8] i.e., if $\alpha\in S^i$ then $|w_\alpha|<\lambda_i$, and for all $u\subseteq\delta_5$, $|u|<\lambda_i$ there is some $\alpha\in S^i$ with $w_\alpha\supseteq u$.

$Q^{\text{full}}_\alpha \cap M_\alpha$. We call Q_α a "partial Q^{full}_α forcing" (e.g.: a "partial random forcing").

Some notes:

- For item (3) of Definition 2.14 to make sense, (2.15) is required.
- We do not require any "transitivity" of the w_α, i.e., $\beta \in w_\alpha$ does generally not imply $w_\beta \subseteq w_\alpha$.
- We do not require (and it will generally not be true) that P_α forces that Q_α is a *complete* subforcing of Q^{full}_α.

A simple absoluteness argument (between M_α and $V[G_\alpha]$) shows:

Lemma 2.16. *P_α forces:*

(a) *The forcing Q_α is an incompatibility preserving subforcing of Q^{full}_α and in particular ccc. (And so, P_α itself is ccc for all α.)*
(b) *For $\alpha \in S^i$, $|Q_\alpha| < \lambda_i$.*
(c) *The forcing Q_α forces that its generic filter $G(\alpha)$ is also generic over M_α. So from the point of view of M_α, $M_\alpha[G(\alpha)]$ is a Q^{full}_α-extention.*
(2) *For $\alpha \in S^2$, the partial Hechler forcing Q_α is σ-centered.*
(3) *For $\alpha \in S^3$, the partial random forcing Q_α is equivalent to a subalgebra of the random algebra.*
(4) *For $\alpha \in S^4$, a partial $\widetilde{\mathbb{E}}$ forcing is (ϱ, π)-linked and basically equivalent to a subalgebra of the random algebra (as in Lemma* 1.19 (e)*).*

Proof: (b) $|P'_\alpha| \le |w_\alpha|^{\aleph_0} \times 2^{\aleph_0} < \lambda_i$ by Assumption 2.13. There is a set of nice P'_α-names of size less than λ_i such that every P'_α-name for a real has an equivalent name in this set. Accordingly, the size of the reals in M_α is forced to be less than λ_i.

(c) is trivial, as Q_α is element of the transitive class M_α.

(4) By Lemma 1.19 (b) we know that M_α thinks that $\widetilde{\mathbb{E}}$ is (ϱ, π)-linked; i.e., that there is a family[9] Q^i_j as in Definition 1.18. Being l-linked is obviously absolute between M_α and $V[G_\alpha]$ for any $l < \omega$, and $M_\alpha \vDash \bigcup_{h\in\omega, i<\varrho(h)} Q^h_i = Q^{\text{full}}_\alpha$ translates to $V[G_\alpha] \vDash \bigcup_{h\in\omega, i<\varrho(h)} Q^h_i = Q_\alpha$.

Similarly, M_α thinks that $\widetilde{\mathbb{E}}$ satisfies 1.19 (e), i.e., that there is some dense $Q' \subseteq \widetilde{\mathbb{E}}$ and a dense embedding from Q' to a subalgebra B' of the random algebra.

So from the point of view of $V[G_\alpha]$, there is a Q' dense in $\widetilde{\mathbb{E}} \cap M_\alpha$ and a dense embedding of Q' into some B', which is a subalgebra of the random algebra in M_α and therefore of the random algebra in $V[G_\alpha]$. $\square$

It is easy to see that (2.15) is a "closure property" of w_α:

Lemma 2.17. *Assume we have constructed (in the ground model) $(P_\beta, Q_\beta)_{\beta<\alpha}$ and w_α according to Definition* 2.14 *for some $\alpha \in S^i$, $i = 1, \ldots, 4$. This determines the (limit or composition) P_α.*

---

[9]Actually there is even a Borel definable family Q^i_j, see the proof of Lemma 1.19 (a), but this is not required here.

(a) *For every P_α-name τ of a real, there is (in V) a countable $u \subseteq \alpha$ and a Borel function $B\colon (\omega^\omega)^u \to \omega^\omega$ such that P_α forces $\tau = B((\eta_\gamma)_{\gamma\in u})$. (So if $w_\alpha \supseteq u$ satisfies* (2.15)*, then P_α forces that $\tau \in M_\alpha$.)*

(b) *The set of w_α satisfying* (2.15) *is an ω_1-club in $[\alpha]^{<\lambda_i}$ (in the ground model).*

(A set $A \subseteq [\alpha]^{<\lambda_i}$ is an ω_1-club, if for each $a \in [\alpha]^{<\lambda_i}$ there is a $b \supseteq a$ in A, and if $(a^i)_{i\in\omega_1}$ is an increasing sequence of sets in A, then the limit $b :- \bigcup_{i\in\omega_1} a^i$ is in A as well.)

Proof: The first item follows easily from the fact that we are dealing with a forsing set (FS) ccc iteration where the generics of all iterands Q_β are Borel-determined by some generic real η_β. (See, e.g., [12, 1.2] for more details.)

Any $w \in [\alpha]^{<\lambda_i}$ defines some P^w_α. We first define w' for such a w:

Set $X = [P^w_\alpha]^{\leq\aleph_0}$, as set of size at most $(2^{\aleph_0} \times |w|^{\aleph_0})^{\aleph_0} < \lambda_i$. For $x \in X$, pick some $p \in P_\alpha$ stronger than all conditions in x (if such a condition exists), and some $q \in P_\alpha$ incompatible to each element of x (again, if possible). There is a countable $w_x \subseteq \alpha$ such that $p, q \in P^{w_x}$. Set $w' := w \cup \bigcup_{x\in X} w_x$.

Start with any $w_0 \in [\alpha]^{<\lambda_i}$. Construct an increasing continuous chain in $[\alpha]^{<\lambda_i}$ with $w^{k+1} = (w^k)'$. Then $w^{\omega_1} \supseteq w_0$ is in the set of w satisfying (2.15); which shows that this set is unbounded. It is equally easy to see that it is closed under increasing sequences of length ω_1. □

For later reference, we explicitly state the assumption we used (for every $\alpha \in \delta_5 \setminus \lambda_5$):

Assumption 2.18. The set w_α is sufficiently closed so that (2.15) is satisfied.

Let us also restate Lemma 2.17 (a):

Lemma 2.19. *For each $\mathbb{P}^5$-name f of a real, there is a countable set $u \subseteq \delta_5$ such that $w_\alpha \supseteq u$ implies that ($\mathbb{P}^5$ forces that) $f \in M_\alpha$.*

Lemma 2.20. *The $\mathsf{LCU}_i(\mathbb{P}^5, \kappa)$ property holds for $i = 1, 3, 4$ and each regular cardinal κ in $[\lambda_i, \lambda_5]$.*

Proof: This follows from Lemma 2.16:

For $i = 1$, partial random and partial $\widetilde{\mathbb{E}}$ forcings are basically equivalent to a sub-Boolean-algebra of the random algebra; and partial Hechler forcings are σ-centered. The partial amoeba forcings are small, i.e., have size less than λ_1. So according to Lemma 2.8, all iterands Q_α (and therefore the limits as well) are $(\mathrm{R}_1, \lambda_1)$-good.

For $i = 3$, note that partial $\widetilde{\mathbb{E}}$ forcings are (ϱ, π)-linked. All other iterands have size less than λ_3, so the forcing is $(\mathrm{R}_3, \lambda_3)$-good.

For $i = 4$ it is enough to note that *all* iterands are small, i.e., of size less than λ_4.

We can now apply Lemma 2.10. □

So in particular, $\mathbb{P}^5$ forces $\mathrm{add}(\mathcal{N}) \leq \lambda_1$, $\mathrm{cov}(\mathcal{N}) \leq \lambda_3$, $\mathrm{non}(\mathcal{M}) \leq \lambda_4$ and $\mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \lambda_5 = 2^{\aleph_0}$; i.e., the respective left hand characteristics are small. We now show that they are also large, using the "cone of bounds" property COB:

Definition 2.21. For a ccc forcing notion P, regular uncountable cardinals λ, μ and $i = 1, 2, 4$, let $\mathsf{COB}_i(P, \lambda, \mu)$ stand for:

> There is a $<\!\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a real $(\exists\, s \in S)\,(\forall\, t \succ s)\, P \Vdash f\, \mathrm{R}_i\, g_t$.

For $i = 3$, let $\mathsf{COB}_3(P, \lambda, \mu)$ stand for:

> There is a $<\!\lambda$-directed partial order $(S, \prec)$ of size μ and a sequence $(g_s)_{s\in S}$ of P-names for reals such that for each P-name f of a null-set $\quad (\exists\, s \in S)\,(\forall\, t \succ s)\, P \Vdash g_t \notin f$.

So s is the tip of a cone that consists of elements bounding f, where in case $i = 3$ we implicitly use an additional relation $N\, \mathrm{R}_3{}'\, r$ expressing that the null-set N does not contain the real r. Note that $\mathrm{cov}(\mathcal{N})$ is the bounding number $\mathfrak{b}_3'$ of $\mathrm{R}_3{}'$, and $\mathrm{non}(\mathcal{N})$ the dominating number $\mathfrak{d}_3'$. So $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_3' \leq \mathfrak{b}_3$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{d}_3' \geq \mathfrak{d}_3$ (as defined in Lemma 2.5).

The $\mathsf{COB}_i(P, \lambda, \mu)$ property implies that P forces that $\mathfrak{b}_i \geq \lambda$ and that $\mathfrak{d}_i \leq \mu$ for $i = 1, 2, 4$, and the same for $i = 3$ and $\mathfrak{b}_3'$, $\mathfrak{d}_3'$: Clearly P forces that $\{g_s \colon s \in \mathcal{S}\}$ is dominating. And if A is set of names of size $\kappa < \lambda$, then for each $f \in A$ the definition gives a bound $s(f)$ and directedness some $t \succ s(f)$ for all f, i.e., g_t bounds all elements of A. So we get:

Lemma 2.22. (1) *The* $\mathsf{COB}_1(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathrm{add}(\mathcal{N}) \geq \lambda\ \&\ \mathrm{cof}(\mathcal{N}) \leq \mu)$.
(2) *The* $\mathsf{COB}_2(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathfrak{b} \geq \lambda\ \&\ \mathfrak{d} \leq \mu)$.
(3) *The* $\mathsf{COB}_3(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathrm{cov}(\mathcal{N}) \geq \lambda\ \&\ \mathrm{non}(\mathcal{N}) \leq \mu)$.
(4) *The* $\mathsf{COB}_4(P, \lambda, \mu)$ *property implies* $P \Vdash (\mathrm{non}(\mathcal{M}) \geq \lambda\ \&\ \mathrm{cov}(\mathcal{M}) \leq \mu)$.

Lemma 2.23. *The* $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$ *property holds for* $i = 1, 2, 3, 4$.

Proof: We use the following facts (provable in ZFC, or true in the P_α-extention, respectively):

1. Amoeba forcing adds a sequence $\bar{b}$ which R_1-dominates the old elements of $\mathcal{C}$.
   (The simple proof can be found in [7, Lemma 1.4], a slight variation in [2].)
   Accordingly (by absoluteness), the generic real η_α for partial amoeba forcing Q_α R_1-dominates $\mathcal{C} \cap M_\alpha$.
2. Hechler forcing adds a real which R_2-dominates all old reals.
   Accordingly, the generic real η_α for partial Hechler forcing Q_α R_2-dominates all reals in M_α.
3. Random forcing adds a random real.

Accordingly, the generic real η_α for partial random forcing Q_α is not in any null set whose Borel-code is in M_α.

(4) The generic branch $\eta \in \lim(T^*)$ added by $\widetilde{\mathbb{E}}$ is eventually different to each old real, i.e., R_4-dominates the old reals.
(This was shown in Lemma 1.19 (c).)
Accordingly, the generic branch η_α for partial $\widetilde{\mathbb{E}}$ forcing Q_α R_4-dominates the reals in M_α.

Fix $i \in \{1,2,3,4\}$, and set $S = S^i$ and $s \prec t$ if $w_s \subsetneq w_t$, and let g_s be η_s, i.e., the generic added at s (e.g., the partial random real in case of $i = 3$, etc.).

Fix a $\mathbb{P}^5$-name f for a real. It depends (in a Borel way) on a countable index set $w^* \subseteq \delta_5$. Fix some $s \in S^i$ such that $w_s \supseteq w^*$. Pick any $t \succ s$. Then $w_t \supseteq w_s \supseteq w^*$, so ($\mathbb{P}^5$ forces that) $f \in M_t$, so, as just argued, $\mathbb{P}^5 \Vdash f\,\mathrm{R}_i\,g_t$ (or: $\mathbb{P}^5 \Vdash f\,\mathrm{R}'_3\,g_t$ for $i = 3$). $\square$

So to summarize what we know so far about $\mathbb{P}^5$: Whenever we choose (in addition to the "fixed" λ_i, S^i) a cofinal parameter $\overline{w}$ satisfying Assumptions 2.13 and 2.18, we get

Fact 2.24.
- ∘ The COB_i property holds for $i = 1,2,3,4$. So the left hand side characteristics are large.
- ∘ The LCU_i property holds for $i = 1,3,4$. So the left hand side characteristics other than $\mathfrak{b}$ are small.

What is missing is "$\mathfrak{b}$ small". We do not claim that this will be forced for every $\overline{w}$ as above; but we will show in the rest of Section 2 that we can choose such a $\overline{w}$.

2.3 FAMs in the P_α-extension compatible with M_α, explicit conditions.

We first investigate sequences $\bar{q} = (q_l)_{l\in\omega}$ of Q_α-conditions that are in M_α, i.e., the (evaluations of) P'_α-names for ω-sequences in Q^{full}_α. For $\alpha \in S^3 \cup S^4$, M_α thinks that Q_α (i.e., Q^{full}_α) has FAM-limits. So if M_α thinks that Ξ_0 is a FAM, then for any sequence $\bar{q}$ in M_α there is a condition $\lim_{\Xi_0}(\bar{q})$ in M_α (and thus in Q_α). We can relativize Lemma 1.8 to sequences in M_α:

Lemma 2.25. *Assume that $\alpha \in S^3 \cup S^4$, that Ξ is a P_α-name for a FAM and that Ξ_0, the restriction of Ξ to M_α, is forced to be in M_α. Then there is a $P_{\alpha+1}$-name Ξ^+ for a FAM such that for all* $(\mathrm{trunk}^*, \mathrm{loss}^*)$*-sequences $\bar{q}$ in M_α,*

$$\lim_{\Xi_0}(\bar{q}) \in G(\alpha) \ \ \textit{implies} \ \ \Xi^+(A_{\bar{q}}) \geq 1 - \sqrt{\mathrm{loss}^*}.$$

$A_{\bar{q}}$ was defined in (1.9) (here we use $G(\alpha)$ instead of G, of course).

Proof: This Lemma is implicitly used in [16]. Note that P'_α is a complete subforcing of P_α, and so there is a quotient R such that $P_\alpha = P'_\alpha * R$. We consider the following (commuting) diagram:

$$\begin{tikzcd} V \arrow[r, "P_\alpha"] \arrow[dr, "P'_\alpha"'] & V_\alpha \arrow[r, "Q_\alpha"] & V_{\alpha+1} \\ & M_\alpha \arrow[u, "R"'] \arrow[r, "Q_\alpha"'] & \boxed{} \arrow[u] \end{tikzcd}$$

Note that (P'_α forces that) $R * Q_\alpha = R \times Q_\alpha$. So from the point of view of M_α:

- $Q_\alpha = Q_\alpha^{\text{full}}$ has FAM limits, and Ξ_0 is a FAM. So there is a Q_α-name for a FAM Ξ_0^+ satisfying Lemma 1.8.
- R is a ccc forcing, and there is an R-name[10] Ξ for a FAM extending Ξ_0.
- So there is $R \times Q_\alpha$-name Ξ^+ for a FAM extending both Ξ_0^+ and Ξ (cf. [16, Claim 1.6]).

Back in V, this defines the $P_{\alpha+1}$-name Ξ^+. Let $\bar{q} = (q_l)_{l\in\omega}$ be a sequence in M_α. Then $M_\alpha[G(\alpha)]$ thinks: If $\lim_{\Xi_0}(\bar{q}) \in G(\alpha)$, then $\Xi_0^+(A_{\bar{q}})$ is large enough. This is upwards absolute to $V[G_{\alpha+1}]$ (as $A_{\bar{q}}$ is absolute). $\square$

For later reference, we will reformulate the lemma for a specific instance of "sequence in M_α". Recall that a sequence in M_α corresponds to a "P'_α-name of a sequence in Q_α^{full}". This is not equivalent to a "P_α-name for a sequence in Q_α", which would correspond to an arbitrary sequence in Q_α (of which there are $|\alpha + \aleph_0|^{\aleph_0}$ many, while there are only less than λ_i many sequences in M_α). However, we can define the following:

Definition 2.26.
- An explicit Q_α-condition (in V) is a P'_α-name for a Q_α^{full} condition.
- A condition $p \in \mathbb{P}^5$ is explicit, if for all $\alpha \in \operatorname{supp}(p) \cap (S^4 \cup S^5)$, $p(\alpha)$ is an explicit Q_α-condition.

Here we mean that for $p(\alpha)$ there is a P'_α-name q_α such that $p \restriction \alpha \Vdash p(\alpha) = q_\alpha$ (and the map $\alpha \mapsto q_\alpha$ exists in the ground model, i.e., we do not just have a P_α-name for a P'_α-condition q_α).

Lemma 2.27. *The set of explicit conditions is dense.*

Proof: We show by induction that the set D_α of explicit conditions in P_α is dense in P_α. As we are dealing with FS iterations, limits are clear. Assume that $(p, q) \in P_{\alpha+1}$. Then p forces that there is a P'_α-name q' such that $q' = q$. Strengthen p to some $p' \in D_\alpha$ deciding q'. Then $(p', q') \leq (p, q)$ is explicit. $\square$

Note that any sequence in V of explicit Q_α-conditions defines a sequence of conditions in M_α (as $V \subseteq M_\alpha$). So we get:

Lemma 2.28. *Let α, Ξ, and Ξ^+ be as in Lemma 2.25, and let $(p_l)_{l\in\omega}$ be (in V) a sequence of explicit conditions in $\mathbb{P}^5$ such that $\alpha \in \operatorname{supp}(p_l)$ for all $l \in \omega$. Set*

---

[10]We identify the P_α-name Ξ in V and the induced R-name in $M_\alpha = V[G'_\alpha]$.

$q_l := p_l(\alpha)$ and $\bar{q} := (q_l)_{l\in\omega}$, and assume that $(\mathrm{trunk}(q_l), \mathrm{loss}(q_l))$ *is forced to be equal to some constant* $(\mathrm{trunk}^*, \mathrm{loss}^*)$.

Then there is a P'_α-name for a Q^{full}_α-condition (and thus a P_α-name for a Q_α-condition) $\lim_{\Xi_0}(\bar{q})$ *such that* $\lim_{\Xi_0}(\bar{q})$ *forces that* $\Xi^+(A_{\bar{q}}) \le 1 - \sqrt{\mathrm{loss}^*}$.

2.4 Dealing with $\mathfrak{b}$ (without generalized continuum hypothesis (GCH)). In this section, we follow [7, 1.3], additionally using techniques inspired by [16].

We assume the following (in addition to Assumption 2.13):

Assumption 2.29. (This section only.) Let $\chi < \lambda_3$ is regular such that $\chi^{\aleph_0} = \chi$, $\chi^+ \ge \lambda_2$ and $2^\chi = |\delta_5| = \lambda_5$.

Set $S^0 = \lambda_5 \cup S^1 \cup S^2$. So $\delta_5 = S^0 \cup S^3 \cup S^4$, and $\mathbb{P}^5$ is a FS ccc iteration along δ_5 such that $\alpha \in S^0$ implies $|Q_\alpha| < \lambda_2$, i.e., $|Q_\alpha| \le \chi$ (and Q_α is a partial random forcing for $\alpha \in S^3$ and a partial $\widetilde{\mathbb{E}}$-forcing for $\alpha \in S^4$).

Let us fix for each $\alpha \in S^0$ a P_α-name

$$i_\alpha \colon Q_\alpha \to \chi \ \text{ injective.} \tag{2.30}$$

Definition 2.31.
- A "partial guardrail" is a function h defined on a subset of δ_5 such that for $\alpha \in \mathrm{dom}(h)$: $h(\alpha) \in \chi$ if $\alpha \in S^0$; and $h(\alpha)$ is a pair (x, y) with $x \in H(\aleph_0)$ and y a rational number otherwise. (Any $(\mathrm{trunk}, \mathrm{loss})$-pair is of this form.)
- A "countable guardrail" is a partial guardrail with countable domain. A "full guardrail" is a partial guardrail with domain δ_5.

We will use the following lemma, which is a consequence of the Engelking–Karlowicz theorem, see [6], on the density of box products (cf. [8, 5.1]):

Lemma 2.32 (as $|\delta_5| \le 2^\chi$)**.** *There is a family H^* of full guardrails of cardinality χ such that each countable guardrail is extended by some $h \in H^*$. We will fix such an H^*.*

Note that the notion of guardrail (and the density property required in Lemma 2.32) only depends on the "fixed" parameters χ, δ_5, S^0, S^3 and S^4; so we can fix an H^* that will work for all these fixed parameters and all choices of the cofinal parameter $\overline{w}$.

Once we have decided on $\overline{w}$, and thus have defined $\mathbb{P}^5$, we can define the following:

Definition 2.33. The set $D^* \subseteq \mathbb{P}^5$ consists of p such that there is a partial guardrail h (and we say: "p follows h") with $\mathrm{dom}(h) \supseteq \mathrm{supp}(p)$ and for all $\alpha \in \mathrm{supp}(p)$ applies:
- If $\alpha \in S^0$, then $p \restriction \alpha \Vdash i_\alpha(p(\alpha)) = h_\alpha$.
- If $\alpha \in S^3 \cup S^4$, the empty condition of P_α forces

$$p(\alpha) \in Q_\alpha \qquad \text{and} \qquad (\mathrm{trunk}(p(\alpha)), \mathrm{loss}(p(\alpha))) = h(\alpha).$$

- Furthermore, $\sum_{\alpha\in\mathrm{supp}(p)\cap(S^3\cup S^4)}\sqrt{\mathrm{loss}(p(\alpha))}<1/2$.
- A condition p is explicit (as in Definition 2.26).

Lemma 2.34. *The set $D^*\subseteq\mathbb{P}^5$ is dense.*

Proof: By induction we show that for any sequence $(\varepsilon_i)_{i\in\omega}$ of positive numbers the following set of p is dense: If $\mathrm{supp}(p)=\{\alpha_0,\dots,\alpha_m\}$, where $\alpha_0>\alpha_1>\dots$ (i.e., we enumerate downwards), $\mathrm{loss}^p_{\alpha_n}<\varepsilon_n$ whenever $\alpha_n\in S^3\cup S^4$. For the successor step, we use that the set of $q\in Q_\alpha$ such that $\mathrm{loss}(q)<\varepsilon_0$ is forced to be dense. □

Remark 2.35. So the set of conditions following *some* guardrail is dense. For each *fixed* guardrail h, the set of all conditions p following h is n-linked, provided that each loss in the domain of h is less than $1/n$ (cf. Assumption 1.5).

Definition 2.36. A "Δ-system with heart ∇ following the guardrail h" is a family $\bar p=(p_i)_{i\in I}$ of conditions such that:

- all p_i are in D^* and follow h;
- $(\mathrm{supp}(p_i))_{i\in I}$ is a Δ system with heart ∇ in the usual sense (so $\nabla\subseteq\delta_5$ is finite);
- the following is independent of $i\in I$:
  - $|\mathrm{supp}(p_i)|$, which we call $m^{\bar p}$.
    Let $(\alpha_i^{\bar p,n})_{n<m^{\bar p}}$ increasingly enumerate $\mathrm{supp}(p_i)$.
  - Whether $\alpha_i^{\bar p,n}$ is less than, equal to or bigger than the kth element of ∇.
    In particular it is independent of i whether $\alpha_i^{\bar p,n}\in\nabla$, in which case we call n a "heart position".
  - Whether $\alpha_i^{\bar p,n}$ is in S^0, in S^3 or in S^4.
    If $\alpha_i^{\bar p,n}\in S^j$, we call n an "S^j-position".
  - If n is not an S^0-position,[11] the value of $h(\alpha_i^{\bar p,n})=:(\mathrm{trunk}^{\bar p,n},\mathrm{loss}^{\bar p,n})$.
    If n is an S^0-position, we set $\mathrm{loss}^{\bar p,n}:=0$.

A "countable Δ-system" $\bar p=(p_l\colon l\in\omega)$ is a Δ system that additionally satisfies:

- For each non-heart position[12] $n<m^{\bar p}$, the sequence $(\alpha_l^{\bar p,n})_{l\in\omega}$ is strictly increasing.

Fact 2.37.
- Each infinite Δ-system $(p_i)_{i\in I}$ contains a countable Δ-system. I.e., there is a sequence i_l in I such that $(p_{i_l})_{l\in\omega}$ is a countable Δ-system.
- If $\bar p$ is a Δ-system (or: a countable Δ-system) following h with heart ∇, and $\beta\in\nabla\cup(\max(\nabla+1))$, then $\bar p\restriction\beta:=(p_i\restriction\beta)_{i\in I}$ is again a Δ-system (or: a countable Δ-system, respectively) following h, now with heart $\nabla\cap\beta$.

[11] If n is a S^0-position, $h(\alpha_i^{\bar p,n})$ will generally not be independent of i; unless of course n is a heart position.

[12] For a heart position n, $(\alpha_l^{\bar p,n})_{l\in\omega}$ is of course constant.

Definition 2.38. Let $\bar{p}$ be a countable Δ-system, and assume that a sequence $\bar{\Xi} = (\Xi_\alpha)_{\alpha\in\nabla\cap(S^3\cup S^4)}$ is such that each Ξ_α is a P_α-name for a FAM and P_α forces that Ξ_α restricted to M_α is in M_α. Then we can define $q = \lim_{\bar{\Xi}}(\bar{p})$ to be the following $\mathbb{P}^5$-condition with support ∇:

- If $\alpha \in \nabla \cap S^0$, then $q(\alpha)$ is the common value of all $p_n(\alpha)$. (Recall that this value is already determined by the guardrail h.)
- If $\alpha \in \nabla \cap (S^3 \cup S^4)$, then $q(\alpha)$ is (forced by $\mathbb{P}^5_\alpha$ to be) $\lim_{\Xi_\alpha}(p_l(\alpha))_{l\in\omega}$, see Lemma 2.28.

We now give a specific way to construct such $\overline{w}$, which allows to keep $\mathfrak{b}$ small.

Lemma/Construction 2.39. *We can construct by induction on $\alpha \in \delta_5$ for each $h \in H^*$ some Ξ^h_α, and if $\alpha > \kappa_5$, also w_α, such that:*

(a) *Each Ξ^h_α is a P_α-name of a FAM extending $\bigcup_{\beta<\alpha}\Xi^h_\beta$.*

(b) *Let α be a limit of countable cofinality: Assume $\bar{p}$ is a countable Δ-system in P_α following h, and $n < m^{\bar{p}}$ such that $(\alpha^{\bar{p},n}_l)_{l\in\omega}$ has supremum α. Then $A_{\bar{p},n}$ is forced to have Ξ^h_α-measure 1, where*

$$A_{\bar{p},n} := \{k \in \omega \colon |\{l \in I_k : p_l(\alpha^{\bar{p},n}_l) \in G(\alpha^{\bar{p},n}_l)\}| \geq |I_k|(1 - \sqrt{\mathrm{loss}^{\bar{p},n}}\,)\}.$$

(c) *For each countable Δ-system $\bar{p}$ in P_α following h, the P_α-condition $\lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar{p})$ is well-defined and forces*

$$\Xi^h_\alpha(A_{\bar{p}}) \geq 1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}, \text{ where}$$

$$A_{\bar{p}} := \Big\{k \in \omega \colon |\{l \in I_k : p_l \in G_\alpha\}| \geq |I_k|\Big(1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}\Big)\Big\}.$$

(d) *For $\alpha > \kappa_5$, w_α is "sufficiently closed". More specifically: It satisfies Assumptions 2.13 and 2.18, and if $\alpha \in S^3 \cup S^4$ then P_α forces that Ξ^h_α restricted to M_α is in M_α.*

Actually, the set of w_α satisfying this is an ω_1-club set.

Proof: (a&c) for $\mathrm{cf}(\alpha) > \omega$: We set $\Xi^h_\alpha = \bigcup_{\beta<\alpha}\Xi^h_\beta$. As there are no new reals at uncountable confinalities, this is a FAM. Each countable Δ-system is bounded by some $\beta < \alpha$, and, by induction, (c) holds for β; so (c) holds for α as well.

(a&b) for $\mathrm{cf}(\alpha) = \omega$: Fix h. We will show that P_α forces $A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} \neq \emptyset$, where A is a Ξ^h_β-positive set for some $\beta < \alpha$, and each $(\bar{p}^j, n^j)$ is as in (b).

Then we can work in the P_α-extension and apply Fact 1.3 (a), using $\bigcup_{\beta<\alpha}\Xi^h_\beta$ as the partial FAM Ξ'. This gives an extension of Ξ' to a FAM Ξ^h_α that assigns measure one to all $A_{\bar{p},n}$, showing that (a) and (b) are satisfied.

So assume towards a contradiction that some $p \in P_\alpha$ forces

$$A \cap \bigcap_{j<j^*} A_{\bar{p}^j,n^j} = \emptyset.$$

We can assume that p decides the β such that $A \in V_\beta$, that β is above the hearts of all Δ-sequences $\bar{p}^j$ involved, and that $\mathrm{supp}(p) \subseteq \beta$. We can extend p to some $p^* \in P_\beta$ to decide $k \in A$ for some "large" k: By large, we mean:

- Let $F(l; n, p)$ (the cumulative binomial probability distribution) be the probability that n independent experiments, each with success probability p, will have at most l successful outcomes. As $\lim_{n\to\infty} F(np'; n, p) = 0$ for all $p' < p$, and as $\lim_{k\to\infty} |I_k| = \infty$, we can find some k such that

$$F(|I_k|p'_j; |I_k|, p_j) < \frac{1}{2j^*} \tag{2.40}$$

  for all $j < j^*$, where we set $p'_j := 1 - \sqrt{\mathrm{loss}^{\bar{p}^j, n^j}}$ and $p_j := 1 - (1 + \sqrt{2}/2) \times \mathrm{loss}^{\bar{p}^j, n^j}$. (Note that $p'_j < p_j$, as $\mathrm{loss}^{\bar{p}^j, n^j} \leq 1/2$.)
- All elements of $Y = \{\alpha_l^{\bar{p}^j, n^j} : j < j^* \text{ and } l \in I_k\}$ are larger than β. (This is possible as each sequence $(\alpha_l^{\bar{p}^j, n^j})_{l<\omega}$ has supremum α.) We enumerate Y by the increasing sequence $(\beta_i)_{i\in M}$, and set $\beta_{-1} = \beta$.

We will find $q \leq p^*$ forcing that $k \in \bigcap_{j<j^*} A_{\bar{p}^j, n^j}$.

To this end, we define a finite tree $\mathcal{T}$ of height M, and assign to each $s \in \mathcal{T}$ of height i a condition $q_s \in P_{\beta_{i-1}+1}$ (decreasing along each branch) and a probability $\mathrm{pr}_s \in [0, 1]$, such that $\sum_{t \rhd s} \mathrm{pr}_t = 1$ for all non-terminal nodes $s \in \mathcal{T}$. For s the root of $\mathcal{T}$, i.e., for the unique s of height 0, we set $q_s = p^* \in P_{\beta_{-1}}$ and $\mathrm{pr}_s = 1$.

So assume we have already constructed $q_s \in P_{\beta_{i-1}+1}$ for some s of height $i < M$. We will now take care of index β_i and construct the set of successors of s, and for each successor t, a $q_t \leq q_s$ in P_{β_i+1}.

- If $\beta_i \in S^0$, the guardrail guarantees that $\beta_i \in \mathrm{supp}(p_l^j)$ implies $p_l^j \restriction \beta_i \Vdash i_{\beta_i}(p_l^j(\beta_i)) = h(\beta_i)$. In that case we use a unique $\mathcal{T}$-successor t of s, and we set $q_t = q_s{}^\frown(\beta_i, i_{\beta_i}^{-1} h(\beta_i))$, and $\mathrm{pr}_t = 1$.
  In the following we assume $\beta_i \notin S^0$.
- Let J_i be the set of $j < j^*$ such that there is an $l \in I_k$ with $\alpha_l^{\bar{p}^j, n^j} = \beta_i$ (there is at most one such l). For $j \in J_i$ set $r_i^j = p_l^j(\beta_i)$ for the according l. So each r_i^j is a P_{β_i}-name for an element of Q_{β_i}.
  The guardrail gives us the constant value $(\mathrm{trunk}_i^*, \mathrm{loss}_i^*) := h(\beta_i)$ (which is equal to $(\mathrm{trunk}^{\bar{p}^j, n^j}, \mathrm{loss}^{\bar{p}^j, n^j})$ for all $j \in J_i$).
- The case $\beta_i \in S^3$, i.e., the case of random forcing, is basically [16, 2.14]: For $x \subseteq [\mathrm{trunk}_i^*]$, set $\mathrm{Leb}^{\mathrm{rel}}(x) = \mathrm{Leb}(x)/\mathrm{Leb}([\mathrm{trunk}_i^*])$. Note that the r_i^j are closed subsets of $[\mathrm{trunk}_i^*]$ and $\mathrm{Leb}^{\mathrm{rel}}(r_i^j) \geq 1 - \mathrm{loss}_i^*$.

  Let $\mathcal{B}^*$ be the power set of $[\mathrm{trunk}_i^*]$; and let $\mathcal{B}$ be the sub-Boolean-algebra generated by r_i^j, $j \in J_i$, let $\mathcal{X}$ be the set of atoms and $\mathcal{X}' = \{x \in \mathcal{X} \colon \mathrm{Leb}^{\mathrm{rel}}(x) > 0\}$. So $|\mathcal{X}'| \leq 2^{J_i} \leq 2^{j^*}$, $\sum_{x\in\mathcal{X}'} \mathrm{Leb}^{\mathrm{rel}}(x) = 1$, and $\sum_{x\in\mathcal{X}', x\subseteq r_i^j} \mathrm{Leb}^{\mathrm{rel}}(x) = \mathrm{Leb}^{\mathrm{rel}}(r_i^j)$.

So far, $\mathcal{X}'$ is a P_{β_i}-name. Now we increase q_s inside P_{β_i} to some q^+ deciding which of the (finitely many) Boolean combinations result in elements of $\mathcal{X}'$, and also deciding rational numbers y_x, $x \in \mathcal{X}'$, with sum 1 such that $|\operatorname{Leb}^{\mathrm{rel}}(x) - y_x| < ((\sqrt{2}-1)/2)\operatorname{loss}_i^* \cdot 2^{-j^*}$.

We can now define the immediate successors of s in $\mathcal{T}$: For each $x \in \mathcal{X}'$, add an immediate successor t_x and assign to it the probability $\mathrm{pr}_{t_x} = y_x$ and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where r_x is a (name for a) partial random condition below x (such a condition exists, as the Lebesgue positive intersection of finitely many partial random condition contains a partial random condition).

Note that when we choose a successor t randomly (according to the assigned probabilities pr_t), then for each $j \in J$ the probability of $q^+ \Vdash q_t(\beta_i) \leq r_i^j$ is at least

$$
\begin{aligned}
\sum_{x\in\mathcal{X}', x\subseteq r_i^j} \mathrm{pr}_x &\geq \sum_{x\in\mathcal{X}', x\subseteq r_i^j} \Big(\operatorname{Leb}^{\mathrm{rel}}(x) - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^* \cdot 2^{-j^*}\Big) \\
&\geq \Big(\sum_{x\in\mathcal{X}', x\subseteq r_i^j} \operatorname{Leb}^{\mathrm{rel}}(x)\Big) - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^* \\
&= \operatorname{Leb}^{\mathrm{rel}}(r_i^j) - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^* \\
&\geq 1 - \operatorname{loss}_i^* - \frac{\sqrt{2}-1}{2}\operatorname{loss}_i^* \\
&= 1 - \frac{1+\sqrt{2}}{2}\operatorname{loss}_i^* .
\end{aligned}
$$

- The case $\beta_i \in S^4$, i.e., the case of $\widetilde{\mathbb{E}}$:

  Recall that $\widetilde{\mathbb{E}}$-conditions are subtrees of some basic compact tree T^*, and there is a h such that: if $\max\{|I_k|, j^*\}$ many conditions share a common node (above their stems) at height h, then they are compatible.

  All conditions r_i^j have the same stem $s^* = \operatorname{trunk}_i^*$. For each $j \in J_i$, set $d(j) = r_i^j \cap \omega^h$. Note that ($P_{\beta_i}$ forces that) $d(j)$ is a subset of $T^* \cap [s^*] \cap \omega^h$ of relative size greater than or equal to $1 - \operatorname{loss}_i^*/2$ (according to Lemma 1.19 (d)). First find $q^+ \leq q_s$ in P_{β_i} deciding all $d(j)$.

  We can now define the immediate successors of s in $\mathcal{T}$: For each $x \in T^* \cap [s^*] \cap \omega^h$ add an immediate successor t_x, and assign to it the uniform probability (i.e., $\mathrm{pr}_{t_x} = |T^* \cap [s^*] \cap \omega^h|^{-1}$) and the condition $q_{t_x} = q^{+\frown}(\beta_i, r_x)$, where r_x is a partial $\widetilde{\mathbb{E}}$-condition stronger than all r_i^j that satisfy $x \in d(j)$. (Such a condition exists, as we can intersect less than or equal to j^* many conditions of height h.)

  If we choose t randomly, then for each $j \in J$ the probability of $q^+ \Vdash q_t \leq r_i^j$ is at least $1 - \operatorname{loss}_i^*/2 \geq 1 - ((1+\sqrt{2})/2)\operatorname{loss}_i^*$.

In the end, we get a tree $\mathcal{T}$ of height M, and we can choose a random branch through $\mathcal{T}$, according to the assigned probabilities. We can identify the branch with its terminal node t^*, so in this notation the branch t^* has probability $\prod_{n\leq M} \mathrm{pr}_{t^*\restriction n}$.

Fix $j < j^*$. There are $|I_k|$ many levels $i < M$ such that at β_i we deal with the $(\bar{p}^j, n^j)$-case. Let M^j be the set of these levels. For each $i \in M^j$, we perform an experiment, by asking whether the next step $t \in \mathcal{T}$ (from the current s at level i) will satisfy $q_t \restriction \beta_i \Vdash q_t(\beta_i) \leq r^j_i$. While the exact probability for success will depend on which s at level i we start from, a lower bound is given by $1 - ((1+\sqrt{2})/2)\,\mathrm{loss}^*_i$. Recall that $\mathrm{loss}^*_i = \mathrm{loss}^{\bar{p}^j,n^j}$, and that we set $p_j := 1 - (1+\sqrt{2}/2)\,\mathrm{loss}^*_i$ and $p'_j := 1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}}$ in (2.40). So the chance of our branch t^* having success fewer than $|I_k|(1-\sqrt{\mathrm{loss}^{\bar{p}^j,n^j}})$ many times, out of the $|I_k|$ many tries, (let us call such a t^* "bad for j") is at most $F(|I_k|p'; |I_k|, p) \leq 1/(2j^*)$.

Accordingly, the measure of branches that are not bad for *any* $j < j^*$ is at least $1/2$. Fix such a branch t^*. Then for each $j < j^*$,

$$|\{i \in M^j : q_{t^*} \restriction \beta_i \Vdash q_{t^*}(\beta_i) \leq r^j_i\}| \geq |I_k|\big(1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}}\,\big),$$

and thus q_{t^*} forces that

$$|\{l \in I_k : p_l(\alpha^{\bar{p}^j,n^j}_l) \in G(\alpha^{\bar{p}^j,n^j}_l)\}| \geq |I_k|\big(1 - \sqrt{\mathrm{loss}^{\bar{p}^j,n^j}}\,\big).$$

(c) for $\mathrm{cf}(\alpha) = \omega$: Fix $\bar{p}$ as in the assumption of (c). To simplify notation, let us assume that $\nabla \neq \emptyset$ and that $\sup(\nabla) < \sup(\mathrm{supp}(p_l))$ (for some, or equivalently: all $l \in \omega$). Let $0 < n_0 < m^{\bar{p}}$ be such that $\sup(\nabla)$ is at position $n_0 - 1$ in $\mathrm{supp}(p_l)$, i.e., $\sup(\nabla) = \alpha^{\bar{p},n_0-1}_l$ (independent of l), and set $\beta := \sup(\nabla) + 1$.

The system $\bar{p} \restriction \beta$ is again a countable Δ-system following the same h, and $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar{p})$ is by definition identical to $\lim_{(\Xi^h_\gamma)_{\gamma<\beta}}(\bar{p} \restriction \beta)$, which by induction is a valid condition and forces (c) for $\bar{p} \restriction \beta$. This gives us the set $A_{\bar{p}\restriction\beta}$ of measure at least $1 - \sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}}$.

For the positions $n_0 \leq n < m^{\bar{p}}$, all $(\alpha^{\bar{p},n}_l)_{l\in\omega}$ are strictly increasing sequences above β with some limit $\alpha_n \leq \alpha$. Then (b) (applied to α_n) gives us an according measure-1-set $A_{\bar{p},n}$.

So $\lim_{(\Xi^h_\gamma)_{\gamma<\alpha}}(\bar{p})$ forces that $A' = A_{\bar{p}\restriction\beta} \cap \bigcap_{n_0\leq n<m^{\bar{p}}} A_{\bar{p},n}$ has measure $\Xi^h_\alpha(A') \geq 1 - \sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}} \geq 1 - \sum_{n<m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}$.

Note that $p_l \in G$ if and only if $p_l \restriction \beta \in G_\beta$ and $p_l(\alpha^{\bar{p},n}) \in G(\alpha^{\bar{p},n})$ for all $n_0 \leq n < m^{\bar{p}}$.

Fix $k \in A'$. As $k \in A_{\bar{p}\restriction\beta}$, the relative frequency for $l \in I_k$ *not* to satisfy $p_l \restriction \beta \in G_\beta$ is at most $\sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}}$. For any $n_0 \leq n < m^{\bar{p}}$, as $k \in A_{\bar{p},n}$, the relative frequency for *not* $p_l(\alpha^{\bar{p},n}) \in G(\alpha^{\bar{p},n})$ is at most $\sqrt{\mathrm{loss}^{\bar{p},n}}$. So the relative frequency for $p_l \in G$ to fail is at most $\sum_{n<n_0} \sqrt{\mathrm{loss}^{\bar{p},n}} + \sum_{n_0 \le n < m^{\bar{p}}} \sqrt{\mathrm{loss}^{\bar{p},n}}$, as required.

(a&c) for $\alpha = \gamma + 1$ successor: For $\gamma \in S^0$ this is clear: Let Ξ^h_α be the name of some FAM extending Ξ^h_γ. Let $\bar{p}$ be as in (c), without loss of generality $\gamma \in \nabla$. Then $q^+ := \lim_{(\Xi^h_\beta)_{\beta<\alpha}}(\bar{p}) = q^\frown(\gamma, r)$, where $q := \lim_{(\Xi^h_\beta)_{\beta<\gamma}}(\bar{p} \restriction \gamma)$ and r is the condition determined by $h(\gamma)$, i.e., each $p_l \restriction \gamma$ forces $p_l(\gamma) = r$. In particular, q^+ forces that $p_l \in G_\alpha$ if and only if $p_l \restriction \gamma \in G_\alpha$. By induction, (c) holds for γ, and therefore we get (c) for α.

Assume $\gamma \in S^3 \cup S^4$. By induction we know that (d) holds for γ, i.e., that Ξ^h_γ restricted to M_γ (call it Ξ_0) is in M_γ. So the requirement in the Definition 2.38 of the limit is satisfied, and thus the limit $q^+ := \lim_{\Xi^h}(\bar{p})$ is well defined for any countable Δ-system $\bar{p}$ as in (c): q^+ has the form $q^\frown(\gamma, r)$ with q and r such that $q = \lim_{(\Xi^h_\beta)_{\beta<\gamma}}(\bar{p} \restriction \gamma)$ and $r = \lim_{\Xi_0}((p_l(\gamma))_{l\in\omega})$. Now Lemma 2.28 gives us the P_α-name Ξ^+, which will be our new Ξ^h_α.

This works as required: Again without loss of generality we can assume $\gamma \in \nabla$. By induction, q forces that $\Xi^h_\gamma(A_{\bar{p}\restriction\gamma}) \ge 1 - \sum_{n<m^p-1} \sqrt{\mathrm{loss}^{\bar{p},n}}$. According to Lemma 2.28 r forces that $\Xi^+(A_{(p_l(\gamma))_{l\in\omega}}) \ge 1 - \sqrt{\mathrm{loss}^{\bar{p},m^p-1}}$. So $q^+ = q^\frown r$ forces that $\Xi^h_\alpha(A_{\bar{p}}) \ge 1 - \sum_{n<m^p} \sqrt{\mathrm{loss}^{\bar{p},n}}$.

(d): So we have (in V) the P_α-name Ξ^h_α. We already know that there is (in V) an ω_1-club set X_0 in $[\alpha]^{<\lambda_i}$ (for the appropriate $i \in \{3, 4\}$) such that $w \in X_0$ implies that w satisfies Assumptions 2.13 and 2.18. So each such $w \in X_0$ defines a complete subforcing P_w of P_α and the P_α-name for the according P_w-extention M_w.

Fix some $w \in X_0$. We will define $w' \supseteq w$ as follows: For a P_w-name (and thus a P_α-name) $r \in 2^\omega$, let s be the name of $\Xi_\alpha(r) \in [0, 1]$. As in Lemma 2.17 (a), we can find a countable w_r determining s. (I.e., there is a Borel function that calculates the real s from the generics at w_r; moreover we know this Borel function in the ground model.) Let $w' \supseteq w$ be in X_0 and contain all these w_r, for a (small representative set of) all P_w-names for reals.

Iterating this construction ω_1 many steps gives us a suitable w_α: Note that the assignment of a name r to the Ξ_α-value s can be done in V, and thus is known to M_α. In addition, M_α sees that for each "actual real" (i.e., element of M_α), the value s is already determined (by P'_α). So the assignment $r \mapsto s$, which is Ξ_α restricted to M_α, is in M_α. □

Note that in (c), when we deal with a countable Δ-system $\bar{p}$ following the guardrail $h \in H^*$, the condition $\lim_{\Xi^h} \bar{p}$ forces in particular that infinitely many p_l are in G. So after carrying out the construction as above, we get a forcing notion $\mathbb{P}^5$ satisfying the following (which is actually the only thing we need from the previous construction, in addition to the fact that we can choose each w_α in an ω_1-club):

Lemma 2.41. *For every countable Δ-system $\bar{p}$ there is some q forcing that infinitely many p_l are in the generic filter.*

PROOF: According to Lemma 2.32, $\bar{p}$ follows some $h \in H^*$; so $q = \lim_{\Xi^h}(\bar{p})$ will work. □

Lemma 2.42. *The property $\mathsf{LCU}_2(\mathbb{P}^5, \kappa)$ is for $\kappa \in [\lambda_2, \lambda_5]$ regular, witnessed by the sequence $(c_\alpha)_{\alpha<\kappa}$ of the first κ many Cohen reals.*

PROOF: Fix a $\mathbb{P}^5$-name $y \in \omega^\omega$. We have to show that $(\exists\, \alpha \in \kappa)\, (\forall\, \beta \in \kappa \setminus \alpha)\, \mathbb{P}^5 \Vdash \neg c_\beta \leq^* y)$.

Assume towards a contradiction that p^* forces that there are unboundedly many $\alpha \in \kappa$ with $c_\alpha \leq^* y$, and enumerate them as $(\alpha_i)_{i\in\kappa}$. Pick $p^i \leq p^*$ deciding α_i to be some β^i, and also deciding n_i such that $(\forall\, m \geq n_i)\, c_{\alpha_i}(m) \leq y(m)$. We can assume that $\beta^i \in \operatorname{supp}(p^i)$. Note that β^i is a Cohen position (as $\beta^i < \kappa \leq \lambda_5$), and we can assume that $p^i(\beta^i)$ is a Cohen condition in V (and not just a P_{β_i}-name for such a condition). By strengthening and thinning out, we may assume:

- The sequence $(p^i)_{i\in\kappa}$ forms a Δ system with heart ∇.
- All n_i are equal to some n^*.
- The condition $p^i(\beta_i)$ is always the same Cohen condition $s \in \omega^{<\omega}$, without loss of generality of length $|s| = n^{**} \geq n^*$.
- For some position $n < m^{\bar{p}}$, β^i is the nth element of $\operatorname{supp}(p^i)$.

Note that this n cannot be a heart condition: For any $\beta \in \kappa$, at most $|\beta|$ many p^i can force $\alpha_i = \beta$, as p^i forces that $\alpha_i \geq i$ for all i.

Pick a countable subset of this Δ-system which forms a countable Δ-system $\bar{p} := (p_l)_{l\in\omega}$. So $p_l = p^{i_l}$ for some $i_l \in \kappa$, and we set $\beta_l = \beta^{i_l}$. In particular all β_l are distinct. Now extend each p_l to p'_l by extending the Cohen condition $p_l(\beta_l) = s$ to $s^\frown l$ (i.e., forcing $c_{\beta_l}(n^{**}) = l$). Note that $\bar{p}' := (p'_i)_{i\in\omega}$ is still a countable Δ-system[13], and by Lemma 2.41 some q forces that infinitely many of the p'_l are in the generic filter. But each such p'_l forces that $c_{\beta_l}(n^{**}) = l \leq y(n^{**})$, a contradiction. □

2.5 The left hand side. We have now finished the consistency proof for the left hand side:

Theorem 2.43. *Assume GCH and let λ_i be an increasing sequence of regular cardinals, none of which is a successor of a cardinal of countable cofinality for $i = 1, \ldots, 5$. Then there is a cofinalities-preserving forcing P resulting in*

$$\operatorname{add}(\mathcal{N}) = \lambda_1 < \operatorname{add}(\mathcal{M}) = \mathfrak{b} = \lambda_2 < \operatorname{cov}(\mathcal{N}) = \lambda_3$$
$$< \operatorname{non}(\mathcal{M}) = \lambda_4 < \operatorname{cov}(\mathcal{M}) = 2^{\aleph_0} = \lambda_5.$$

PROOF: Set $\chi = \lambda_2$, and let R be the set of partial functions $f\colon \chi \times \lambda_5 \to 2$ with $|\operatorname{dom}(f)| < \chi$ (ordered by inclusion). The set R is $<\chi$-closed, χ^+-cc, and adds λ_5 many new elements to 2^χ. So in the R-extension, Assumption 2.29 is satisfied,

[13] Note that $\bar{p}'$ will not follow the same guardrail as $\bar{p}$.

and we can construct $\mathbb{P}^5$ according to Assumption 2.13 and Construction 2.39. Fact 2.24 gives us all inequalities for the left hand side, apart from $\mathfrak{b} \leq \lambda_2$, which we get from 2.42.

In the R-extension, CH holds and P is a FS ccc iteration of length δ_5, $|\delta_5| = \lambda_5$, and each iterated is a set of reals; so $2^{\aleph_0} \leq \lambda_5$ is forced. Also, any FS ccc iteration of length δ (of nontrivial iterands) forces $\mathrm{cov}(\mathcal{M}) \geq \mathrm{cf}(\delta)$: Without loss of generality $\mathrm{cf}(\delta) = \lambda$ is uncountable. Any set A of (Borel codes for) meager sets that has size less than λ already appears at some stage $\alpha < \delta$, and the iteration at state $\alpha + \omega$ adds a Cohen real over the V_α, so A will not cover all reals. $\square$

Remark 2.44. So this consistency result is reasonably general, we can, e.g., use the values $\lambda_i = \aleph_{i+1}$. This is in contrast to the result for the whole diagram, where in particular the small λ_i have to be separated by strongly compact cardinals.

3. Ten different values in Cichoń's diagram

We can now apply, with hardly any change, the technique of [7] to get the following:

Theorem 3.1. *Assume GCH and that $\aleph_1 < \kappa_9 < \lambda_1 < \kappa_8 < \lambda_2 < \kappa_7 < \lambda_3 < \kappa_6 < \lambda_4 < \lambda_5 < \lambda_6 < \lambda_7 < \lambda_8 < \lambda_9$ are regular, λ_i is not a successor of a cardinal of countable cofinality for $i = 1, \ldots, 5$, $\lambda_2 = \chi^+$, with χ regular, and κ_i strongly compact for $i = 6, 7, 8, 9$. Then there is a ccc forcing notion $\mathbb{P}^9$ resulting in:*

$$\begin{aligned}\mathrm{add}(\mathcal{N}) = \lambda_1 < \mathfrak{b} = \mathrm{add}(\mathcal{M}) = \lambda_2 < \mathrm{cov}(\mathcal{N}) = \lambda_3 < \mathrm{non}(\mathcal{M}) = \lambda_4 < \mathrm{cov}(\mathcal{M}) \\ = \lambda_5 < \mathrm{non}(\mathcal{N}) = \lambda_6 < \mathfrak{d} = \mathrm{cof}(\mathcal{M}) = \lambda_7 < \mathrm{cof}(\mathcal{N}) = \lambda_8 < 2^{\aleph_0} = \lambda_9.\end{aligned}$$

To do this, we first have to show that we can achieve the order for the left hand side, i.e., Theorem 2.43, starting with GCH and using a FS ccc iteration $\mathbb{P}^5$ alone (instead of using $P = R * \mathbb{P}^5$, where R is not ccc). This is the only argument that requires $\lambda_2 = \chi^+$. We will just briefly sketch it here, as it can be found with all details in [7, 1.4]:

- We already know that in the R-extension, (where R is $<\chi$-closed, χ^+-cc and forces $2^\chi = \lambda_5$) we can find by the inductive Construction 2.39 suitable w_α such that $R * \mathbb{P}^5$ works.
- We now perform a similar inductive construction in the ground model: At stage α, we know that there is an R-name for a suitable w^1_α of size less than λ_i (where i is 3 in the random and 4 in the $\widetilde{\mathbb{E}}$-case). This name can be covered by some set $\widetilde{w}^1_\alpha$ in V, still of size less than λ_i, as R is χ^+-cc. Moreover, in the R-extension, the suitable parameters form an ω_1-club; so there is a suitable $w^2_\alpha \supseteq \widetilde{w}^1_\alpha$, etc. Iterating ω_1 many times and taking the union at the end leads to w_α in V which is forced by R to be suitable.
- Not only w_α is in V, but the construction for w_α is performed in V, so we can construct the whole sequence $\overline{w} = (w_\alpha)_{\alpha \in \delta_5}$ in V.

- We now know that in the R-extension, the forcing $\mathbb{P}^5$ defined from $\overline{w}$ will satisfy $\mathsf{LCU}_2(\mathbb{P}^5, \kappa)$ in the form of Lemma 2.42.
- By an absoluteness argument, we can show that actually in V the forcing $\mathbb{P}^5$ defined form $\overline{w}$ will satisfy Lemma 2.42 as well.

The rest of the proof is the same as in [7, Section 2], where we interchange $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$ as well as $\mathfrak{d}$ and $\mathrm{non}(\mathcal{N})$.

We cite the following facts from [7, 2.2–2.5]:

Facts 3.2. (a) If κ is a strongly compact cardinal and $\theta > \kappa$ regular, then there is an elementary embedding $j_{\kappa,\theta}\colon V \to M$ (in the following just called j) such that
- the critical point of j is κ, $\mathrm{cf}(j(\kappa)) = |j(\kappa)| = \theta$,
- $\max(\theta, \lambda) \leq j(\lambda) < \max(\theta, \lambda)^+$ for all $\lambda \geq \kappa$ regular, and
- $\mathrm{cf}(j(\lambda)) = \lambda$ for $\lambda \neq \kappa$ regular,

and such that the following is satisfied:

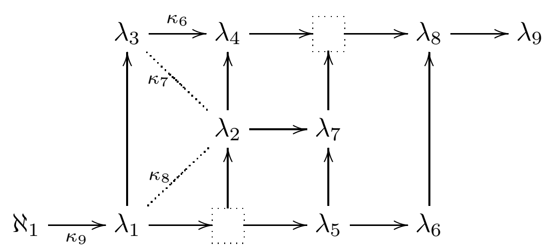
(b) If P is a FS ccc iteration along δ, then $j(P)$ is a FS ccc iteration along $j(\delta)$.

(c) The $\mathsf{LCU}_i(P, \lambda)$ property implies the $\mathsf{LCU}_i(j(P), \mathrm{cf}(j(\lambda)))$ property, and thus $\mathsf{LCU}_i(j(P), \lambda)$ if $\lambda \neq \kappa$ regular.[14]

(d) If $\mathsf{COB}_i(P, \lambda, \mu)$, then $\mathsf{COB}_i(j(P), \lambda, \mu')$ for $\mu' = \begin{cases} |j(\mu)| & \text{if } \kappa > \lambda, \\ \mu & \text{if } \kappa < \lambda. \end{cases}$

Using these facts, it is easy to finish the proof[15]:

Proof of Theorem 3.1: Recall that we want to force the following values to the characteristics of Figure 2 (where we indicate the positions of the κ_i as well):



Step 5: Our first step, called "Step 5" for notational reasons, just uses $\mathbb{P}^5$. This is an iteration of length δ_5 with $\mathrm{cf}(\delta_5) = |\delta_5| = \lambda_5$, satisfying:

(3.3) For all i: $\mathsf{LCU}_i(\mathbb{P}^5, \mu)$ for all $\mu \in [\lambda_i, \lambda_5]$ regular, and $\mathsf{COB}_i(\mathbb{P}^5, \lambda_i, \lambda_5)$.

[14] In [7], we only used "classical" relations R_3 that are defined on a Polish space in an absolute way. In this paper, we use the relation R_3 which is not of this kind. However, the proof still works without any change: The parameter $\mathcal{E}$ used to define the relation R_3, cf. Definition 2.2, is a set of reals. So $j(\mathcal{E}) = \mathcal{E}$, and we can still use the usual absoluteness arguments between M and V. (A parameter not element of $H(\kappa_9)$ might be a problem.)

[15] This is identical to the argument in [7], with the roles of $\mathfrak{b}$ and $\mathrm{cov}(\mathcal{N})$, as well as their duals, switched.

As a consequence, the characteristics are forced by $\mathbb{P}^5$ to have the following values[16] (we also mark the position of κ_6, which we are going to use in the following step):

$$\begin{tikzcd}
 & \lambda_3 \arrow[r, "\kappa_6"] & \lambda_4 \arrow[r] & {} \arrow[r] & \lambda_5 \arrow[r] & \lambda_5 \\
 & & \lambda_2 \arrow[u] \arrow[r] & \lambda_5 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & {} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_5 \arrow[uu] &
\end{tikzcd}$$

Step 6: Consider the embedding $j_6 := j_{\kappa_6,\lambda_6}$. According to Fact 3.2 (b), $\mathbb{P}^6 := j_6(\mathbb{P}^5)$ is a FS ccc iteration of length $\delta_6 := j_6(\delta_5)$. As $|\delta_6| = \lambda_6$, the continuum is forced to have size λ_6.

For $i = 1$, we have $\mathsf{LCU}_1(\mathbb{P}^5, \mu)$ for all regular $\mu \in [\lambda_1, \lambda_5]$, so using Fact 3.2 (c) we get $\mathsf{LCU}_1(\mathbb{P}^6, \mu)$ for all regular size $\mu \in [\lambda_1, \lambda_5]$ different to κ_6; as well as $\mathsf{LCU}_1(\mathbb{P}^6, \lambda_6)$ (as $\mathrm{cf}(j(\kappa_6)) = \lambda_6$). For $\mu = \lambda_1$ the former implies for the iteration $\mathbb{P}^6 \Vdash \mathrm{add}(\mathcal{N}) \leq \lambda_1$, and the latter $\mathbb{P}^6 \Vdash \mathrm{cof}(\mathcal{N}) \geq \lambda_6 = 2^{\aleph_0}$.

More generally, we get from (3.3) and Fact 3.2 (c):

$$\begin{aligned} &\text{For all } i\text{: } \mathsf{LCU}_i(\mathbb{P}^6, \mu) \text{ for all regular } \mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6\}.\\ &\text{For } i < 4\text{: } \mathsf{LCU}_i(\mathbb{P}^6, \lambda_6). \end{aligned} \tag{3.4}$$

So in particular for $\mu = \lambda_i$, we see that the characteristics on the left do not increase; for $\mu = \lambda_5$ that the ones on the right are still at least λ_5; and for $i < 4$ and $\mu = \lambda_6$ that the according characteristics on the right will have size continuum. (But not for $i = 4$, as $\kappa_4 < \lambda_4$. And we will see that $\mathrm{cov}(\mathcal{M})$ is at most λ_5.)

Dually, because $\lambda_3 < \kappa_6 < \lambda_4$, we get from (3.3) and Fact 3.2 (d):

$$\begin{aligned} &\text{For } i < 4\text{: } \mathsf{COB}_i(\mathbb{P}^6, \lambda_i, \lambda_6).\\ &\text{For } i = 4\text{: } \mathsf{COB}_4(\mathbb{P}^6, \lambda_4, \lambda_5). \end{aligned} \tag{3.5}$$

(The former because $|j_6(\lambda_5)| = \max(\lambda_6, \lambda_5) = \lambda_6$.) So the characteristics on the left do not decrease, and $\mathbb{P}^6 \Vdash \mathrm{cov}(\mathcal{M}) \leq \lambda_5$.

[16] These values, and the ones forced by the "intermediate forcings" $\mathbb{P}^6$ to $\mathbb{P}^8$, are not required for the argument; they should just illustrate what is going on.

Accordingly, $\mathbb{P}^6$ forces the following values:

$$\begin{tikzcd}
 & \lambda_3 \arrow[r] \arrow[dr, dotted, no head, "\kappa_7"'] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_6 \arrow[r] & \lambda_6 \\
 & & \lambda_2 \arrow[u] \arrow[r] & \lambda_6 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Step 7: We now apply a new embedding, $j_7 := j_{\kappa_7,\lambda_7}$, to the forcing $\mathbb{P}^6$ that we just constructed. (We always work in V, not in any inner model M or any forcing extention.) As before, set $\mathbb{P}^7 := j_7(\mathbb{P}^6)$, a FS ccc iteration of length $\delta_7 = j_7(\delta_6)$, forcing the continuum to have size λ_7.

Now $\kappa_7 \in (\lambda_2, \lambda_3)$, so arguing as before, we get from (3.4):

(3.6)
For all i: $\mathsf{LCU}_i(\mathbb{P}^7, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7\}$.
For $i < 4$: $\mathsf{LCU}_i(\mathbb{P}^7, \lambda_6)$.
For $i < 3$: $\mathsf{LCU}_i(\mathbb{P}^7, \lambda_7)$.

And from (3.5):

(3.7)
For $i < 3$: $\mathsf{COB}_i(\mathbb{P}^7, \lambda_i, \lambda_7)$.
For $i = 3$: $\mathsf{COB}_3(\mathbb{P}^7, \lambda_3, \lambda_6)$.
For $i = 4$: $\mathsf{COB}_4(\mathbb{P}^7, \lambda_4, \lambda_5)$.

Accordingly, $\mathbb{P}^7$ forces the following values:

$$\begin{tikzcd}
 & \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_7 \arrow[r] & \lambda_7 \\
 & & \lambda_2 \arrow[u] \arrow[r] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r] & \lambda_1 \arrow[uu] \arrow[r] \arrow[ur, dotted, no head, "\kappa_8"] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}$$

Step 8: Now we set $\mathbb{P}^8 := j_{\kappa_8,\lambda_8}(\mathbb{P}^7)$, a FS ccc iteration of length δ_8. Now $\kappa_8 \in (\lambda_1, \lambda_2)$, and as before, we get from (3.6):

(3.8)
For all i: $\mathsf{LCU}_i(\mathbb{P}^8, \mu)$ for all regular $\mu \in [\lambda_i, \lambda_5] \setminus \{\kappa_6, \kappa_7, \kappa_8\}$.
For $i < 4$: $\mathsf{LCU}_i(\mathbb{P}^8, \lambda_6)$.
For $i < 3$: $\mathsf{LCU}_i(\mathbb{P}^8, \lambda_7)$.
For $i < 2$ (i.e., $i = 1$): $\mathsf{LCU}_1(\mathbb{P}^8, \lambda_8)$.

And from (3.7):

$$
\text{(3.9)}\qquad
\begin{aligned}
&\text{For } i = 1\colon\ \mathsf{COB}_1(\mathbb{P}^8, \lambda_1, \lambda_8).\\
&\text{For } i = 2\colon\ \mathsf{COB}_2(\mathbb{P}^8, \lambda_2, \lambda_7).\\
&\text{For } i = 3\colon\ \mathsf{COB}_3(\mathbb{P}^8, \lambda_3, \lambda_6).\\
&\text{For } i = 4\colon\ \mathsf{COB}_4(\mathbb{P}^8, \lambda_4, \lambda_5).
\end{aligned}
$$

Accordingly, $\mathbb{P}^8$ forces the following values:

$$
\begin{tikzcd}
 & \lambda_3 \arrow[r] & \lambda_4 \arrow[r] & \boxed{} \arrow[r] & \lambda_8 \arrow[r] & \lambda_8 \\
 & & \lambda_2 \arrow[u] \arrow[r] & \lambda_7 \arrow[u] & & \\
\aleph_1 \arrow[r, "\kappa_9"'] & \lambda_1 \arrow[uu] \arrow[r] & \boxed{} \arrow[u] \arrow[r] & \lambda_5 \arrow[u] \arrow[r] & \lambda_6 \arrow[uu] &
\end{tikzcd}
$$

$\mathsf{Step\ 9}$: Finally we set $\mathbb{P}^9 := j_{\kappa_9,\lambda_9}(\mathbb{P}^8)$, a FS ccc iteration of length δ_9 with $|\delta_9| = \lambda_9$, i.e., the continuum will have size λ_9. As $\kappa_9 < \lambda_1$, (3.8) and (3.9) also hold for $\mathbb{P}^9$ instead of $\mathbb{P}^8$. Accordingly, we get the same values for the diagram as for $\mathbb{P}^8$, apart from the value for the continuum, λ_9. □

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(*Received* January 22, 2018, *revised* September 29, 2018)