Strong measure zero and meager-additive sets through the prism of fractal measures

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In memory of Bohuslav Balcar

Abstract. We develop a theory of sharp measure zero sets that parallels Borel's strong measure zero, and prove a theorem analogous to Galvin–Mycielski–Solovay theorem, namely that a set of reals has sharp measure zero if and only if it is meager-additive. Some consequences: A subset of 2^ω is meager-additive if and only if it is $\mathcal{E}$-additive; if $f\colon 2^\omega \to 2^\omega$ is continuous and X is meager-additive, then so is $f(X)$.

Keywords: meager-additive; $\mathcal{E}$-additive; strong measure zero; sharp measure zero; Hausdorff dimension; Hausdorff measure

Classification: 03E05, 03E20, 28A78

1. Introduction

99 years ago É. Borel in [5] conceived the notion of *strong measure zero*: by his definition, a metric space X has *strong measure zero* (thereinafter Smz) if for any sequence $\langle \varepsilon_n \rangle$ of positive numbers there is a cover $\{U_n\}$ of X such that $\operatorname{diam} U_n \leqslant \varepsilon_n$ for all n. In the same paper, É. Borel conjectured that every Smz set of reals was countable. This statement known as *Borel conjecture* attracted a lot of attention.

Borel conjecture. It is well-known that Borel conjecture is independent of Zermelo–Fraenkel set theory with axiom of choice (ZFC), the usual axioms of set theory. The proof of the consistency of its failure was settled in 1940 by W. Sierpiński, see [30], who proved in 1928 that the continuum hypothesis yields a counterexample, namely the Luzin set, and K. Gödel, who announced in 1938, see [14], and published in 1940, see [15], the proof of the consistency of the continuum hypothesis.

The consistency of the Borel conjecture remained open until 1976 when R. Laver proved in his ground-breaking paper [22] Borel conjecture to be indeed consistent with ZFC.

DOI 10.14712/1213-7243.2015.277

The author was supported by the grant GAČR 15–082185 of the Grant Agency of the Czech Republic. Work on this project was partially conducted during the author's sabbatical stay at the Instituto de Matemáticas, Unidad Morelia, Universidad Nacional Autonóma de México supported by CONACyT grant no. 125108.

To complete the picture, T. J. Carlson in [6] proved in 1993 that the Borel conjecture implies that every separable Smz metric space is countable.

Over time numerous characterizations of strong measure zero were discovered. We are going to focus on (besides the definition itself) three such characterizations.

Hausdorff dimension. One way to characterize Smz is via Hausdorff dimension: It is almost obvious that a Smz space has Hausdorff dimension zero. Since Smz is preserved by uniformly continuous mappings, it follows that any uniformly continuous image of a Smz space has Hausdorff dimension zero. It is not difficult to prove that the latter property actually characterizes Smz. The essence of this characterization can be traced back to A. S. Besicovitch's papers [3], [4].

Galvin's game. A characterization in terms of infinite games was recently published by F. Galvin in [12] who attributes it to F. Galvin, J. Mycielski and R. M. Solovay. Consider the following game: the playground is a subset X of a σ-compact metric space. At the nth inning, Player I chooses $\varepsilon_n > 0$ and Player II responds with a set $U_n \subseteq X$ such that $\operatorname{diam} U_n \leqslant \varepsilon_n$. Player II wins if the sets U_n form a cover of X, otherwise Player I wins. Denote this game $G(X)$.

Theorem 1.1 ([12])**.** *Let X be a subset of a σ-compact metric space. The set X is Smz if and only if Player I does not have a winning strategy in the game $G(X)$.*

Galvin–Mycielski–Solovay theorem. Confirming Prikry's conjecture, F. Galvin, J. Mycielski and R. M. Solovay in [11] proved a rather surprising characterization of Smz subsets of the real line. Recall that for $A, B \subseteq \mathbb{R}$, we define $A + B = \{x + y \colon x \in A,\ y \in B\}$ (and likewise for other groups).

Theorem 1.2 ([11])**.** *A set $X \subseteq \mathbb{R}$ is Smz if and only if $X + M \neq \mathbb{R}$ for each meager set $M \subseteq \mathbb{R}$.*

We will refer to this result as the *Galvin–Mycielski–Solovay theorem.* Recently M. Kysiak in [21], D. H. Fremlin in [9] and W. Wohofsky in [36] showed that an analogous theorem holds for all σ-compact metrizable groups. The theorem was further investigated by M. Hrušák, W. Wohofsky and O. Zindulka in [18] and M. Hrušák and J. Zapletal in [19] who found, roughly speaking, that under the continuum hypothesis the Galvin–Mycielski–Solovay theorem does not extend beyond σ-compact metrizable groups.

In summary, we thus have four strikingly different descriptions of Smz:

- "combinatorial"—Borel's definition;
- "fractal"—by Hausdorff dimension of images;
- "game-theoretic"—by Galvin's game (restriction: subsets of σ-compact spaces);
- "algebraic" — by the Galvin–Mycielski–Solovay theorem (restriction: subsets of σ-compact metrizable groups).

Sharp measure zero. Consider the characterization of Smz by Hausdorff dimension: X is Smz if and only if $\dim_{\mathsf{H}} f(X) = 0$ for every uniformly continuous mapping f. One may, just out of curiosity, ask what happens when the Hausdorff dimension is replaced with some other fractal dimension. Here we will consider the so called *upper Hausdorff dimension* $\overline{\dim}_{\mathsf{H}}$ introduced in [39]. We will say a metric space X has *sharp measure zero* (thereinafter $\mathsf{Smz}^\sharp$) if $\overline{\dim}_{\mathsf{H}} f(X) = 0$ for every uniformly continuous mapping f.

It turns out that $\mathsf{Smz}^\sharp$ sets can be characterized by a property very much like Borel's definition of Smz, and that properties of $\mathsf{Smz}^\sharp$ sets nicely parallel those of Smz sets. In particular, $\mathsf{Smz}^\sharp$ is characterized by a slight modification of Galvin's game.

One of the highlights of Section 3 devoted to $\mathsf{Smz}^\sharp$ is the following improvement of a theorem of M. Scheepers, see [28, Theorem 1]: a product of a Smz set and a $\mathsf{Smz}^\sharp$ set is Smz.

Meager-additive sets. The Cantor set 2^ω with the coordinatewise addition modulo 2 is a second countable compact topological group.

Consider the following strengthening of the algebraic property of the Galvin–Mycielski–Solovay theorem: say that a set $X \subseteq 2^\omega$ is *meager-additive* if $X + M$ is meager for every meager set $M \subseteq 2^\omega$. The notion generalizes to other topological groups, and in particular to finite cartesian powers of 2^ω and $\mathbb{R}$, in an obvious way.

Meager-additive sets in 2^ω have received a lot of attention. They were investigated by many, most notably by T. Bartoszyński and H. Judah in [1], J. Pawlikowski in [26] and S. Shelah in [29]. Combinatorial properties of meager-additive sets described by J. Pawlikowski in [26] and S. Shelah in [29] allow to prove a rather surprising theorem that is one of the summits of the present paper.

Theorem 1.3. *A set $X \subseteq 2^\omega$ is $\mathsf{Smz}^\sharp$ if and only if it is meager-additive.*

In summary, we thus have four descriptions of $\mathsf{Smz}^\sharp$ that perfectly parallel those of Smz:

- ◦ "combinatorial"—a Borel-like definition, cf. Theorem 3.11 (and Definition 3.8);
- ◦ "fractal"—by upper Hausdorff measures, cf. Theorem 3.7;
- ◦ "game-theoretic"—by a Galvin-like game, this time without any restriction, cf. Theorem 4.2;
- ◦ "algebraic"—by meager-additive sets (restriction: subsets of 2^ω or Euclidean spaces and their finite powers); cf. Corollary 5.10 and Theorem 6.4.

Consequences include, for instance:

- ◦ meager-additive sets are preserved by continuous mappings $f\colon 2^\omega \to 2^\omega$;
- ◦ a product of a Smz and a meager-additive set is Smz;
- ◦ meager-additive sets are universally meager (cf. Proposition 6.11).

Besides meager-additive sets, we also consider the following notion: a set $X \subseteq 2^\omega$ is called *$\mathcal{E}$-additive* if for every F_σ-set $E \subseteq 2^\omega$ of (Haar) measure zero the set $X+E$ is contained in an F_σ-set of (Haar) measure zero. We prove the following:

Theorem 1.4. *A set $X \subseteq 2^\omega$ is meager-additive if and only if it is $\mathcal{E}$-additive.*

This theorem answers a question of A. Nowik and T. Weiss, see [25].

Some common notation used throughout the paper includes $|A|$ for the cardinality of a set A, ω for the set of natural numbers, $[\omega]^\omega$ for the collection of infinite subsets of ω, ω^ω for the family of all sequences of natural numbers, and $\omega^{\uparrow\omega}$ for the family of nondecreasing unbounded sequences of natural numbers.

2. Strong measure zero via Hausdorff measure

In this section we establish a few characterizations of strong measure zero in terms of Hausdorff measures and dimensions based on a classical Besicovitch result, see [3], [4], and derive some consequences.

Hausdorff measure. Before getting any further we need to review Hausdorff measure and dimension. We set up the necessary definitions and recall relevant facts.

Let X be a metric space. If $A \subseteq X$, then $\operatorname{diam} A$ denotes the diameter of A. A closed ball of radius r centered at x is denoted by $B(x,r)$.

A nondecreasing, right-continuous function $h\colon [0,\infty) \to [0,\infty)$ such as that $h(0)=0$ and $h(r)>0$ if $r>0$ is called a *gauge*. The following is the common ordering of gauges, cf. [27]:

$$g \prec h \quad \stackrel{\text{def}}{\equiv} \quad \lim_{r\to 0+} \frac{h(r)}{g(r)} = 0.$$

In the case when $h(r)=r^s$ for some $s>0$ we write $g \prec s$ instead of $g \prec h$.

Notice that for any sequence $\langle h_n \rangle$ of gauges there is a gauge h such that $h \prec h_n$ for all n.

If $\delta>0$, a cover $\mathcal{A}$ of a set $E \subseteq X$ is termed a *δ-fine cover* if $\operatorname{diam} A \leqslant \delta$ for all $A \in \mathcal{A}$. If h is a gauge, the *h-dimensional Hausdorff measure* $\mathcal{H}^h(E)$ of a set $E \subseteq X$ is defined thus: For each $\delta>0$ set

$$\mathcal{H}^h_\delta(E) = \inf\left\{ \sum_{n\in\omega} h(\operatorname{diam} E_n) \colon \{E_n\} \text{ is a countable } \delta\text{-fine cover of } E \right\}$$

and put $\mathcal{H}^h(E) = \sup_{\delta>0} \mathcal{H}^h_\delta(E)$.

In the common case when $h(r)=r^s$ for some $s>0$, we write $\mathcal{H}^s$ for $\mathcal{H}^h$, and the same licence is used for other measures and set functions arising from gauges.

Properties of Hausdorff measures are well-known. The following, including the two lemmas, can be found e.g., in [27]. The restriction of $\mathcal{H}^h$ to Borel sets is a G_δ-regular Borel measure. Recall that a sequence of sets $\langle E_n : n \in \omega \rangle$ is termed a *λ-cover* of $E \subseteq X$ if every point of E is contained in infinitely many E_n's.

Lemma 2.1. *The Hausdorff measure $\mathcal{H}^h(E)=0$ if and only if E admits a countable λ-cover $\langle E_n\rangle$ such that $\sum_{n\in\omega} h(\operatorname{diam} E_n)<\infty$.*

Lemma 2.2. (i) *If $\mathcal{H}^h(X)<\infty$ and $h\prec g$, then $\mathcal{H}^g(X)=0$.*
(ii) *If $\mathcal{H}^h(X)=0$, then there is $g\prec h$ such that $\mathcal{H}^g(X)=0$.*

We will also need a cartesian product inequality. Given two metric spaces X and Y with respective metrics d_X and d_Y, provide the cartesian product $X\times Y$ with the maximum metric

$$d((x_1,y_1),(x_2,y_2))=\max(d_X(x_1,x_2),d_Y(y_1,y_2)). \tag{1}$$

A gauge h satisfies the *doubling condition* or h is *doubling* if

$$\varlimsup_{r\to 0}\frac{h(2r)}{h(r)}<\infty.$$

Lemma 2.3 ([20], [17])**.** *Let X,Y be metric spaces, g a gauge and h a doubling gauge. Then $\mathcal{H}^h(X)\,\mathcal{H}^g(Y)\leqslant\mathcal{H}^{hg}(X\times Y)$.*

The following lemma on Lipschitz images and its counterpart for uniformly continuous mappings are well-known, see, e.g., [27, Theorem 29].

Lemma 2.4. *Let $f\colon (X,d_X)\to(Y,d_Y)$ be a mapping.*

(i) *If f is uniformly continuous and a gauge g is its modulus, i.e.,*

$$d_Y(f(x),f(y))\leqslant g(d_X(x,y)),\qquad x,y\in X, \tag{2}$$

then $\mathcal{H}^h(f(X))\leqslant\mathcal{H}^{h\circ g}(X)$ for any gauge h.

(ii) *If f is Lipschitz with Lipschitz constant L, then $\mathcal{H}^s(f(X))\leqslant L^s\mathcal{H}^s(X)$ for any $s>0$.*

Recall that the *Hausdorff dimension* of X is defined by

$$\dim_{\mathsf{H}} X=\sup\{s>0\colon \mathcal{H}^s(X)=\infty\}=\inf\{s>0\colon \mathcal{H}^s(X)=0\}.$$

Properties of Hausdorff dimension are well-known. In particular, $\dim_{\mathsf{H}} X=0$ if X is countable; and it follows from Lemma 2.4 (ii) that if $f\colon X\to Y$ is Lipschitz, then $\dim_{\mathsf{H}} f(X)\leqslant\dim_{\mathsf{H}} X$.

Theorem 2.5. *Let X be a metric space. The following are equivalent.*

(i) *X is* Smz*;*
(ii) *$\mathcal{H}^h(X)=0$ for each gauge h;*
(iii) *$\dim_{\mathsf{H}} f(X)=0$ for each uniformly continuous mapping f on X;*
(iv) *$\dim_{\mathsf{H}}(X,\varrho)=0$ for each uniformly equivalent metric ϱ on X.*

Proof: The equivalence of (i) and (ii) is due to A. S. Besicovitch, see [3], [4]. We include an outline of the argument.

(i) $\Rightarrow$ (ii): Let h be a gauge. For each $\delta > 0$ pick $\langle \varepsilon_n \rangle \in (0,\infty)^\omega$ such that $h(\varepsilon_n) < \delta 2^{-n}$ and let $\langle U_n \rangle$ be an $\langle \varepsilon_n \rangle$-fine cover of X. Then $\sum h(\operatorname{diam} U_n) < \delta$, which is enough.

(ii) $\Rightarrow$ (i): Let $\langle \varepsilon_n \rangle \in (0,\infty)^\omega$. Pick a gauge h such that $h(\varepsilon_n) > 1/n$. There is a countable cover $\{U_n\}$ such that $\sum h(\operatorname{diam} U_n) < 1$. Pick $\delta_n > \operatorname{diam} U_n$ such that $\sum h(\delta_n) < 1$. Since $\delta_n > 0$, rearranging the sequence we may suppose that δ_n decrease. Therefore $nh(\delta_n) \leqslant \sum_{i<n} h(\delta_n) < 1$. It follows that $h(\delta_n) < 1/n < h(\varepsilon_n)$ and consequently $\delta_n < \varepsilon_n$. Hence $\{U_n\}$ is an $\langle \varepsilon_n \rangle$-fine cover of X.

(ii) $\Rightarrow$ (iii): Let $s > 0$ be arbitrary. Let $f\colon X \to Y$ be uniformly continuous and let g be the modulus of f. Define $h(x) = (g(x))^s$. By (ii) $\mathcal{H}^h(X) = 0$ and thus Lemma 2.4 (i) yields $\mathcal{H}^s(f(X)) \leqslant \mathcal{H}^h(X) = 0$. Since this holds for all $s > 0$, we have $\dim_{\mathsf{H}} f(X) = 0$.

(iii) $\Rightarrow$ (iv): This is trivial.

(iv) $\Rightarrow$ (ii): Denote by d the metric of X. Let h be a gauge. Choose a strictly increasing, convex (and in particular subadditive) gauge g such that $g \prec h$. The properties of g ensure that $\varrho(x,y) = g(d(x,y))$ is a uniformly equivalent metric on X. The identity map $\mathrm{id}_X\colon (X,\varrho) \to (X,d)$ is of course uniformly continuous and its modulus is g^{-1}, the inverse of g. Hence by Lemma 2.4 (i) $\mathcal{H}^h(X,d) \leqslant \mathcal{H}^{h\circ g^{-1}}(X,\varrho)$. Since $\mathcal{H}^1(X,\varrho) = 0$ by assumption and $h \circ g^{-1} \succ 1$ by the choice of g, we have $\mathcal{H}^{h\circ g^{-1}}(X,\varrho) = 0$ by Lemma 2.2 (i). Thus $\mathcal{H}^h(X,d) = 0$, as required. $\square$

Our next goal is to characterize Smz by behavior of cartesian products. We need to recall first a few facts about the Cantor set.

Cantor set. The set of all countable binary sequences is denoted by 2^ω. The set of all finite binary sequences is denoted by $2^{<\omega}$, i.e., $2^{<\omega} = \bigcup_{n\in\omega} 2^n = \{f\colon n \to 2\colon n \in \omega\}$. For $p \in 2^{<\omega}$, let $[\![p]\!] = \{x \in 2^\omega\colon p \subseteq x\}$ denote the cone determined by p. The family of all cones forms a basis for the topology of 2^ω and for $T \subseteq 2^{<\omega}$ we let $[\![T]\!] = \bigcup_{p\in T} [\![p]\!]$. It is well-known that this topology is second countable and compact. It also obtains from the so called least difference metric: For $x \neq y \in 2^\omega$, set $n(x,y) = \min\{i \in \omega\colon x(i) \neq y(i)\}$ and define $d(x,y) = 2^{-n(x,y)}$.

The coordinatewise addition modulo 2 makes 2^ω a compact topological group. Routine proofs show that in this metric, $\mathcal{H}^1$ coincides on Borel sets with its Haar measure, i.e., the usual product measure on 2^ω. In particular $\mathcal{H}^1(2^\omega) = 1$.

We consider the important σ-ideal $\mathcal{E}$ on 2^ω generated by closed null sets, i.e., the ideal of all subsets of 2^ω that are contained in an F_σ set of Haar measure zero.

Lemma 2.6. (i) *For each $I \in [\omega]^\omega$, the set $C_I = \{x \in 2^\omega\colon x{\restriction}I \equiv 0\}$ is in $\mathcal{E}$.*

(ii) *For each $h \prec 1$ there is $I \in [\omega]^\omega$ such that $\mathcal{H}^h(C_I) > 0$.*

Proof: (i) Let $I \in [\omega]^\omega$. For each $n \in \omega$, the family $\{[\![p]\!]\colon p \in C_I{\restriction}n\}$ is obviously a 2^{-n}-cover of C_I of cardinality $2^{|n\setminus I|}$. Therefore $\mathcal{H}^1_{2^{-n}}(C_I) \leqslant 2^{|n\setminus I|}2^{-n} = 2^{-|n\cap I|}$. Hence $\mathcal{H}^1(C_I) \leqslant \lim_{n\to\infty} 2^{-|n\cap I|} = 0$.

(ii) $h \prec 1$ yields $h(2^{-n})/2^{-n} \to \infty$. Therefore there is $I \in [\omega]^\omega$ sparse enough to satisfy $2^{|n\cap I|} \leqslant h(2^{-n})/2^{-n}$, i.e., $2^{-|n\setminus I|} \leqslant h(2^{-n})$ for all $n \in \omega$. Consider the product measure λ on C_I given as follows: If $p \in 2^n$ and $[\![p]\!] \cap C_I \neq \emptyset$, put $\lambda([\![p]\!] \cap C_I) = 2^{-|n\setminus I|}$. Straightforward calculation shows that $h(\operatorname{diam} E) \geqslant \lambda(E)$ for each $E \subseteq C_I$. Hence $\sum_n h(\operatorname{diam} E_n) \geqslant \sum_n \lambda(E_n) \geqslant \lambda(C_I) = 1$ for each cover $\{E_n\}$ of C_I and $\mathcal{H}^h(C_I) \geqslant 1$ follows. □

Theorem 2.7. *The following are equivalent.*

(i) X *is* Smz*;*
(ii) $\mathcal{H}^h(X\times Y) = 0$ *for every gauge* h *and every* σ*-compact metric space* Y *such that* $\mathcal{H}^h(Y) = 0$*;*
(iii) $\mathcal{H}^1(X \times E) = 0$ *for every* $E \in \mathcal{E}$*;*
(iv) $\mathcal{H}^1(X \times C_I) = 0$ *for every* $I \in [\omega]^\omega$.

Proof: (i) $\Rightarrow$ (ii): Suppose X is Smz. We may clearly suppose that Y is compact. Fix $\eta > 0$. Since $\mathcal{H}^h(Y) = 0$ for each $j \in \omega$ there is a finite family $\mathcal{U}_j$ of (open) covers such that $\sum_{U\in\mathcal{U}_j} h(\operatorname{diam} U) < 2^{-j}\eta$. We may also assume that $\operatorname{diam} U < \eta$ for all $U \in \mathcal{U}_j$.

Let $\varepsilon_j = \min\{\operatorname{diam} U \colon U \in \mathcal{U}_j\}$. Choose a cover $\{V_j\}$ of a metric space X such that $\operatorname{diam} V_j \leqslant \varepsilon_j$ and define

$$\mathcal{W} = \{V_j \times U \colon j \in \omega,\ U \in \mathcal{U}_j\}.$$

It is obvious that $\mathcal{W}$ is a cover of $X \times Y$. Since $\operatorname{diam}(V_j \times U) = \operatorname{diam} U$ for all j and $U \in \mathcal{U}_j$ by the choice of ε_j, we have

$$\sum_{W\in\mathcal{W}} h(\operatorname{diam} W) = \sum_{j\in\omega}\sum_{U\in\mathcal{U}_j} h(\operatorname{diam} U) < \sum_{j\in\omega} 2^{-j}\eta = 2\eta.$$

Therefore $\mathcal{H}^h_\eta(X \times Y) < 2\eta$, which is enough for $\mathcal{H}^h(X \times Y) = 0$, as η was arbitrary.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv): These are trivial.

(iv) $\Rightarrow$ (i): Suppose X is not Smz. We will show that $\mathcal{H}^1(X \times C_I) > 0$ for some $I \in [\omega]^\omega$. By assumption and Theorem 2.5 there is a gauge h such that $\mathcal{H}^h(X) > 0$. *Mutatis mutandis* we may assume h be concave and $h(r) \geqslant \sqrt{r}$. In particular, by concavity of h the function $g(r) = r/h(r)$ is increasing and $h(r) \geqslant \sqrt{r}$ yields $\lim_{r\to 0} g(r) = 0$, i.e., g is a gauge, and $g \prec 1$. Moreover, $g(2r) = 2r/h(2r) \leqslant 2r/h(r) = 2g(r)$, i.e., g is doubling.

Use Lemma 2.6 (ii) to find $I \in [\omega]^\omega$ such that $\mathcal{H}^g(C_I) > 0$. Since g is doubling, we may apply Lemma 2.3:

$$\mathcal{H}^1(X \times C_I) = \mathcal{H}^{h\cdot g}(X \times C_I) \geqslant \mathcal{H}^h(X) \cdot \mathcal{H}^g(C_I) > 0.$$

□

Corollary 2.8. *If* X *is* Smz, *then* $\dim_{\mathsf{H}} X \times Y = \dim_{\mathsf{H}} Y$ *for every* σ*-compact metric space* Y.

3. Sharp measure zero

In this section we develop the elementary theory of a notion a bit stronger than that of strong measure zero. The following definition is inspired by Theorem 2.5 (iii).

Definition 3.1. A metric space X has *sharp measure zero* ($\mathsf{Smz}^\sharp$) if for every uniformly continuous mapping $f\colon X \to Y$ into a complete metric space Y there is a σ-compact set $K \subseteq Y$ such that $f(X) \subseteq K$ and $\dim_{\mathsf{H}} K = 0$.

It is clear that $\mathsf{Smz}^\sharp$ implies Smz and that every countable set is $\mathsf{Smz}^\sharp$. It is also obvious that $\mathsf{Smz}^\sharp$ is a σ-additive property and that it is preserved by uniformly continuous maps:

Proposition 3.2. (i) *If X is a metric space, then the family of all $\mathsf{Smz}^\sharp$ subsets of X forms a σ-ideal.*

(ii) *If X is $\mathsf{Smz}^\sharp$ and $f\colon X \to Y$ is a uniformly continuous mapping, then $f(X)$ is $\mathsf{Smz}^\sharp$.*

In this section we provide a few characterizations and describe a few properties of sharp measure zero.

Upper Hausdorff measure. It turns out that sharp measure zero can be described in terms of a fractal measure very similar to Hausdorff measure. It is defined thus: Let h be a gauge. For each $\delta > 0$ set

$$\overline{\mathcal{H}}^h_\delta(E) = \inf\left\{\sum_{n=0}^{N} h(\operatorname{diam} E_n)\colon \{E_n : n \leqslant N\} \text{ is a } \textit{finite } \delta\text{-fine cover of } E\right\}.$$

Then put $\overline{\mathcal{H}}^h_0(E) = \sup_{\delta>0} \overline{\mathcal{H}}^h_\delta(E)$. The only difference from $\mathcal{H}^h$ is that only finite covers are taken into account. It is easy to check that $\overline{\mathcal{H}}^h_0$ is finitely subadditive, but unfortunately it is not a measure, since it need not be σ-additive. To overcome this difficulty we apply to $\overline{\mathcal{H}}^h_0$ the operation known as Munroe's *Method I construction* (cf. [23] or [27]):

$$\overline{\mathcal{H}}^h(E) = \inf\left\{\sum_{n\in\omega} \overline{\mathcal{H}}^h_0(E_n)\colon E \subseteq \bigcup_{n\in\omega} E_n\right\}.$$

Thus the defined set function is indeed an outer measure whose restriction to Borel sets is a Borel measure.

Definition 3.3. The measure $\overline{\mathcal{H}}^h$ is called the *h-dimensional upper Hausdorff measure.*

We list some properties of $\overline{\mathcal{H}}^h_0$ and $\overline{\mathcal{H}}^h$. Some of them will be utilized below and some are provided just to shed more light on the notion of upper Hausdorff measure. The straightforward proofs are omitted. Denote $\mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ the family of countable unions of sets E with $\overline{\mathcal{H}}^h_0(E) = 0$. We also write $E_n \nearrow E$ to denote that $\langle E_n \rangle$ is an increasing sequence of sets with union E.

Lemma 3.4. *Let h be a gauge and E a set in a metric space.*

(i) *If $\overline{\mathcal{H}}^h_0(E) < \infty$, then E is totally bounded.*
(ii) *$\overline{\mathcal{H}}^h_0(E) = \overline{\mathcal{H}}^h_0(\overline{E})$.*
(iii) *$\overline{\mathcal{H}}^h_0(E) = \mathcal{H}^h(E)$ if E is compact.*
(iv) *If $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$, then $\overline{\mathcal{H}}^h(E) = 0$.*
(v) *If X is complete, $E \subseteq X$ and $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$, then there is a σ-compact set $K \supseteq E$ such that $\mathcal{H}^h(K) = 0$.*
(vi) *If X is complete and $E \subseteq X$, then $\overline{\mathcal{H}}^h(E) = \inf\{\mathcal{H}^h(K) \colon K \supseteq E$ is σ-compact$\}$.*
(vii) *In particular $\overline{\mathcal{H}}^h(E) = \mathcal{H}^h(E)$ if E is σ-compact.*
(viii) *If $g \prec h$ and $\overline{\mathcal{H}}^g(E) < \infty$, then $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$; in particular, $\overline{\mathcal{H}}^h(E) = 0$.*
(ix) *If $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$, then there is a sequence $E_n \nearrow E$ such that $\overline{\mathcal{H}}^h_0(E_n) = 0$ for all n.*
(x) *If $\overline{\mathcal{H}}^h(E) < s$, then there is a sequence $E_n \nearrow E$ such that $\sup \overline{\mathcal{H}}^h_0(E_n) < s$.*

We will also need lemmas that parallel Lemmas 2.3 and 2.4. As to the proofs, Lemma 3.5 is proved in the Appendix and Lemma 3.6 is proved exactly the same way as Lemma 2.4.

Lemma 3.5. *Let X, Y be metric spaces and g a gauge and h a doubling gauge. Then $\mathcal{H}^h(X)\,\overline{\mathcal{H}}^g(Y) \leqslant \overline{\mathcal{H}}^{hg}(X \times Y)$.*

Lemma 3.6. *Let $f \colon (X, d_X) \to (Y, d_Y)$ be a mapping.*

(i) *If f is uniformly continuous and a gauge g is its modulus, i.e., satisfies (2), then $\overline{\mathcal{H}}^h(f(X)) \leqslant \overline{\mathcal{H}}^{h\circ g}(X)$ for any gauge h.*
(ii) *If f is Lipschitz with Lipschitz constant L, then $\overline{\mathcal{H}}^s(f(X)) \leqslant L^s \overline{\mathcal{H}}^s(X)$ for any $s > 0$.*

The corresponding *upper Hausdorff dimension* of X, introduced in [39], is defined by

$$\overline{\dim}_{\mathsf{H}} X = \sup\{s > 0 \colon \overline{\mathcal{H}}^s(X) = \infty\} = \inf\{s > 0 \colon \overline{\mathcal{H}}^s(X) = 0\}.$$

Since $\mathcal{H}^h \leqslant \overline{\mathcal{H}}^h$ for each gauge h, it is clear that $\dim_{\mathsf{H}} X \leqslant \overline{\dim}_{\mathsf{H}} X$. The inequality may be strict, cf. examples in [39, Section 2] and [40, Example 4.2].

It follows from Lemma 3.4 (vi) that if X is a complete metric space and $E \subseteq X$, then $\overline{\dim}_{\mathsf{H}} E = \inf\{\dim_{\mathsf{H}} K \colon K \supseteq E$ is σ-compact$\}$. In particular, if X is σ-compact, then $\dim_{\mathsf{H}} X = \overline{\dim}_{\mathsf{H}} X$.

It follows from Lemma 3.6 (ii) that if $f \colon X \to Y$ is Lipschitz, then $\overline{\dim}_{\mathsf{H}} f(X) \leqslant \overline{\dim}_{\mathsf{H}} X$.

We now establish the $\mathsf{Smz}^\sharp$ counterpart of Theorem 2.5.

Theorem 3.7. *Let X be a metric space. The following are equivalent.*

(i) *X is $\mathsf{Smz}^\sharp$;*
(ii) *$\overline{\mathcal{H}}^h(X) = 0$ for each gauge h;*

(iii) $\overline{\dim}_{\mathsf{H}} f(X) = 0$ *for each uniformly continuous mapping* f *on* X*;*
(iv) $\overline{\dim}_{\mathsf{H}}(X, \varrho) = 0$ *for each uniformly equivalent metric* ϱ *on* X*.*

Proof: (i) $\Rightarrow$ (iii): This follows at once from Lemma 3.4 (vii).

(iii) $\Rightarrow$ (iv): This is trivial.

(iv) $\Rightarrow$ (ii) $\Rightarrow$ (iii): These go exactly the same way as that in Theorem 2.5, one has to employ Lemma 3.6 instead of Lemma 2.4. We show (ii) $\Rightarrow$ (i). Let $f\colon X \to Y$ be uniformly continuous, g its gauge (i.e., satisfies (2)), Y complete. Let h', h be gauges such that $h' \prec h \prec s$ for all $s > 0$. Since $\overline{\mathcal{H}}^{h' \circ g}(X) = 0$ by (ii), Lemma 3.6 (i) yields $\overline{\mathcal{H}}^{h'}(f(X)) = 0$. By Lemma 3.4 (viii) we have $f(X) \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$. Therefore Lemma 3.4 (v) yields a σ-compact set $K \supseteq f(X)$ such that $\overline{\mathcal{H}}^h(K) = 0$. By Lemma 3.4 (viii) we have $\overline{\mathcal{H}}^s(K) = 0$ for each $s > 0$. ☐

Our next goal is to describe $\mathsf{Smz}^\sharp$ in terms of covers. The characterization parallels Borel's original definition of Smz.

Definition 3.8. Let $\langle U_n \rangle$ be a sequence of sets in X. Recall that $\langle U_n \rangle$ is called a γ-*cover* if each $x \in X$ belongs to all but finitely many U_n.

Recall that $\langle U_n \rangle$ is called γ-*groupable cover* if there is a partition $\omega = I_0 \cup I_1 \cup I_2 \cup \cdots$ into consecutive finite intervals (i.e. I_{j+1} is on the right of I_j for all j) such that the sequence $\left\langle \bigcup_{n \in I_j} U_n \colon j \in \omega \right\rangle$ is a γ-cover. The partition $\langle I_j \rangle$ will be occasionally called *witnessing* and the finite families $\{U_n \colon n \in I_j\}$ will be occasionally called *witnessing families.*

The following is a counterpart of Lemma 2.1.

Lemma 3.9. *The space* $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ *if and only if* E *has a* γ*-groupable cover* $\langle U_n \rangle$ *such that* $\sum_{n \in \omega} h(\operatorname{diam} U_n) < \infty$*.*

Proof: $\Rightarrow$: Let $E_n \nearrow E$, $\overline{\mathcal{H}}^h_0(E_n) = 0$. For each n let $\mathcal{G}_n$ be a finite cover of E_n such that $\sum_{G \in \mathcal{G}_n} h(\operatorname{diam} G) < 2^{-n}$. The required cover is $\mathcal{G} = \bigcup_n \mathcal{G}_n$, with $\mathcal{G}_n$ the witnessing families.

$\Leftarrow$: Let $\mathcal{G}_j$ be the witnessing families. Put $E_k = \bigcap_{j \geqslant k} \bigcup \mathcal{G}_j$. Then $E = \bigcup_{k \in \omega} E_k$. Fix k. The set E_k is covered by each $\mathcal{G}_j$, $j \geqslant k$, and $\sum_{G \in \mathcal{G}_j} h(\operatorname{diam} G)$ is as small as needed if j is large enough. Hence $\overline{\mathcal{H}}^h_0(E_k) = 0$. ☐

We often deal with sequences of positive real numbers. Instead of writing always "let $\langle \varepsilon_n \rangle$ be a sequence of positive numbers" we briefly write "let $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$".

Let X be a metric space and let $\langle U_n \colon n \in \omega \rangle$ be a sequence of subsets of X. Say that $\langle U_n \colon n \in \omega \rangle$ is $\langle \varepsilon_n \rangle$-*fine* if $\operatorname{diam} U_n \leqslant \varepsilon_n$ holds for all n.

Lemma 3.10. (i) *For each* $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ *there exists a gauge* h *such that if* X *is a metric space and* $\overline{\mathcal{H}}^h(X) = 0$*, then* X *admits an* $\langle \varepsilon_n \rangle$*-fine* γ*-groupable cover.*

(ii) *For each gauge* h *there exists* $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ *such that if* X *is a metric space that admits an* $\langle \varepsilon_n \rangle$*-fine* γ*-groupable cover, then* $\overline{\mathcal{H}}^h(X) = 0$*.*

Proof: (i) Let $\langle\varepsilon_n\rangle \in (0,\infty)^\omega$. Choose a gauge g such that $g(\varepsilon_n) > 1/n$ for all $n \geqslant 1$ and then a gauge $h \prec g$. Suppose X is a metric space such that $\overline{\mathcal{H}}^h(X) = 0$. The goal is to find an $\langle\varepsilon_n\rangle$-fine γ-groupable cover of X.

By Lemma 3.4 (viii) we have $X \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^g_0)$. By Lemma 3.9 there is a γ-groupable cover $\langle G_n\rangle$ such that $\sum_n g(\operatorname{diam} G_n) < \infty$. Let $\{I_j : j \in \omega\}$ be the witnessing partition and $\mathcal{G}_j = \{G_n : n \in I_j\}$ the witnessing families.

We plan to permute the cover so that diameters decrease. One obstacle is that some of them may be 0. Another one is that permutation may break down the witnessing families. We have to work around these difficulties.

For each n choose $\delta_n > \operatorname{diam} G_n$ so that $\sum_n g(\delta_n) < \infty$. Then recursively choose an increasing sequence $\langle j_k\rangle$ such that for all $k \in \omega$

(a) $\sum\{g(\delta_n) : n \in I_{j_k}\} < 2^{-k-1}$;

(b) $\max\{\delta_n : n \in I_{j_{k+1}}\} < \min\{\delta_n : n \in I_{j_k}\}$ (this is possible since δ_n's are positive).

Let $I = \bigcup_{k\in\omega} I_{j_k}$. Permute G_n's within each $\mathcal{G}_{j_k}$ so that δ_n does not increase as n increases. Together with (b) this ensures that the sequence $\langle\delta_n : n \in I\rangle$ is nonincreasing. For each $i \in \omega$ let $i^* \in I$ be the unique index such that $i = |I \cap i^*|$ and define $H_i = G_{i^*}$. It follows, with the aid of (a) and the definition of g, that for all $i \in \omega$

$$\begin{aligned} g(\operatorname{diam} H_i) = g(\operatorname{diam} G_{i^*}) \leqslant g(\delta_{i^*}) &\leqslant \frac{1}{i}\sum\{g(\delta_m) : m \in I,\ m \leqslant i^*\} \\ &\leqslant \frac{1}{i}\sum\{g(\delta_m) : m \in I\} \leqslant \frac{1}{i} < g(\varepsilon_i) \end{aligned}$$

and thus $\operatorname{diam} H_i \leqslant \varepsilon_i$, i.e., $\langle H_i\rangle$ is an $\langle\varepsilon_i\rangle$-fine sequence. Moreover, the families $\mathcal{G}_{j_k}$, $k \in \omega$, witness that $\langle H_i\rangle$ is a γ-groupable cover.

(ii) Let h be a gauge. Choose $\varepsilon_n > 0$ to satisfy $\sum_n h(\varepsilon_n) < \infty$. If X is a metric space admitting an $\langle\varepsilon_n\rangle$-fine γ-groupable cover $\langle G_n\rangle$, then $\sum_n h(\operatorname{diam} G_n) \leqslant \sum_n h(\varepsilon_n) < \infty$. By Lemma 3.9 and Lemma 3.4 (iv) $\overline{\mathcal{H}}^h(X) = 0$. □

The Borel-like definition of $\mathsf{Smz}^\sharp$ now follows at once from the above lemma and Theorem 3.7.

Theorem 3.11. *Let X be a metric space. The space X is $\mathsf{Smz}^\sharp$ if and only if for each $\langle\varepsilon_n\rangle \in (0,\infty)^\omega$, X has an $\langle\varepsilon_n\rangle$-fine γ-groupable cover.*

Our next goal is to set up a counterpart to Theorem 2.7.

Theorem 3.12. *Let X be a metric space. The following are equivalent.*

(i) *X is $\mathsf{Smz}^\sharp$;*

(ii) *for each gauge h, $Y \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ and each complete space $Z \supseteq X$ there is a σ-compact F, $X \subseteq F \subseteq Z$, such that $\overline{\mathcal{H}}^h(F \times Y) = 0$;*

(iii) *$\overline{\mathcal{H}}^h(X \times Y) = 0$ for each gauge h and $Y \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$;*

(iv) *$\overline{\mathcal{H}}^1(X \times E) = 0$ for each $E \in \mathcal{E}$;*

(v) *$\overline{\mathcal{H}}^1(X \times C_I) = 0$ for each $I \in [\omega]^\omega$.*

Proof: The proof is similar to that of Theorem 2.7.

(iii) $\Rightarrow$ (iv): This follows from $\mathcal{E} = \mathcal{N}_\sigma(\overline{\mathcal{H}}^1_0)$ which in turn follows from Lemma 3.4 (iii) and (iv). The only nontrivial implications are (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Let $Z \supseteq X$ be a complete metric space. Suppose X is $\mathsf{Smz}^\sharp$. By Lemma 3.4 (vi), X is contained in a σ-compact set $K \subseteq Z$. Let h be a gauge and $Y \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$. Lemma 3.9 yields a γ-groupable cover $\mathcal{U}$ of Y such that $\sum_{U\in\mathcal{U}} h(\operatorname{diam} U) < \infty$. For each $U \in \mathcal{U}$ there is $\delta_U > \operatorname{diam} U$ such that $\sum_{U\in\mathcal{U}} h(\delta_U) < \infty$. Denote by $\mathcal{U}_j$ the witnessing families. Let $\varepsilon_j = \min\{\delta_U : U \in \mathcal{U}_j\}$. Using Theorem 3.11 choose a γ-groupable cover $\langle V_j\rangle$ of X such that $\operatorname{diam} V_j \leqslant \varepsilon_j$. We may assume that each V_j is a closed subset of Z. Denote by $\mathcal{V}_k$ the witnessing families. Define

$$\mathcal{W} = \{V_j \times U : j \in \omega,\, U \in \mathcal{U}_j\},$$
$$F = K \cap \bigcup_{i\in\omega} \bigcap_{k\geqslant i} \bigcup \mathcal{V}_k.$$

The set $F \subseteq Z$ is clearly an F_σ subset of K and is thus σ-compact. Also $X \subseteq F$. It is easy to check that $\mathcal{W}$ is a γ-groupable cover of $F \times Y$. Since $\operatorname{diam}(V_j \times U) \leqslant \delta_U$ for all j and $U \in \mathcal{U}_j$ by the choice of ε_j, we have $\sum_{W\in\mathcal{W}} h(\operatorname{diam} W) \leqslant \sum_{U\in\mathcal{U}} h(\delta_U) < \infty$. Using Lemma 3.9 it follows that $F \times Y \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ and in particular $\overline{\mathcal{H}}^h(F \times Y) = 0$.

(v) $\Rightarrow$ (i): Suppose X is not $\mathsf{Smz}^\sharp$. We will show that $\overline{\mathcal{H}}^1(X \times C_I) > 0$ for some $I \in [\omega]^\omega$. By assumption and Theorem 3.7 there is a gauge h such that $\overline{\mathcal{H}}^h(X) > 0$. As well as in the proof of Theorem 2.7 suppose h is concave, and find a doubling gauge $g \prec 1$ such that $g(r)h(r) = r$. Then use Lemma 2.6 (ii) to find $I \in [\omega]^\omega$ such that $\mathcal{H}^g(C_I) > 0$ and apply Lemma 3.5:

$$\overline{\mathcal{H}}^1(X \times C_I) = \overline{\mathcal{H}}^{h\cdot g}(X \times C_I) \geqslant \overline{\mathcal{H}}^h(X) \cdot \mathcal{H}^g(C_I) > 0.$$

□

Corollary 3.13. *If X is $\mathsf{Smz}^\sharp$ then $\overline{\dim}_{\mathsf{H}} X \times Y = \overline{\dim}_{\mathsf{H}} Y$ for every metric space Y. In particular, $\overline{\dim}_{\mathsf{H}} X \times Y = \dim_{\mathsf{H}} Y$ if Y is σ-compact.*

Products of Smz and $\mathsf{Smz}^\sharp$ sets. It is well known that a product of two Smz sets need not be Smz (cf. [9, 534P]). But if one of the factors is $\mathsf{Smz}^\sharp$, the product is Smz:

Theorem 3.14. (i) *If X and Y are $\mathsf{Smz}^\sharp$, then $X \times Y$ is $\mathsf{Smz}^\sharp$.*
(ii) *If X is Smz and Y is $\mathsf{Smz}^\sharp$, then $X \times Y$ is Smz.*

Proof: Suppose Y is $\mathsf{Smz}^\sharp$. By Theorem 3.7 (ii) and Lemma 3.4 (viii), $Y \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^h_0)$ for all gauges h.

(i) If X is $\mathsf{Smz}^\sharp$, then Theorem 3.12 (iii) yields $\overline{\mathcal{H}}^h(X \times Y) = 0$ for all gauges h, which is by Theorem 3.7 (ii) enough.

(ii) If X is Smz, then Lemma 3.4 (v) and Theorem 2.7 (ii) yield $\mathcal{H}^h(X\times Y)=0$ for all gauges h, which is by Theorem 2.5 (ii) enough. $\square$

4. Galvin's game

As already discussed in the introduction, F. Galvin in [12] succeeded to characterize Smz sets in σ-compact metric spaces in terms of a game, cf. Theorem 1.1.

We consider a similar game and prove a counterpart of Galvin's theorem for $\mathsf{Smz}^\sharp$ sets. Note the striking similarity with Galvin's game.

Definition 4.1. Let X be a metric space. The game $G^\sharp(X)$ is played as follows: At the nth inning, Player I chooses $\varepsilon_n > 0$ and Player II responds with a set $U_n \subseteq X$ such that $\operatorname{diam} U_n \leqslant \varepsilon_n$. Player II wins if the sequence of sets $\langle U_n\rangle$ forms a γ-groupable cover of X, otherwise Player I wins.

Theorem 4.2. *A metric space X is $\mathsf{Smz}^\sharp$ if and only if Player I does not have a winning strategy in $G^\sharp(X)$.*

Note that, unlike in Galvin's theorem, X is not a priori supposed to be a subset of a σ-compact space.

Proof: The backwards implication is trivial: if X is not $\mathsf{Smz}^\sharp$, then by Theorem 3.11 there is $\langle\varepsilon_n\rangle \in (0,\infty)^\omega$ such that X has no $\langle\varepsilon_n\rangle$-fine γ-groupable cover. The winning strategy for Player I is of course to play ε_n at the nth inning.

For the forward implication we modify Galvin's proof. Suppose that X is $\mathsf{Smz}^\sharp$. By Theorem 3.7 (ii) $\overline{\mathcal{H}}^h(X)=0$ for any gauge h and thus Lemma 3.4 (i) yields an increasing sequence $F_n\nearrow X$ of totally bounded sets.

Let σ be a strategy for Player I. We will show that σ is not winning.

Recall [12, Lemma 1]: *If F is totally bounded, then for every $\delta>0$ there is a finite collection δ-fine $\mathcal{B}$ of sets, such that every subset of F of diameter at most $\delta/3$ is contained in some $B\in\mathcal{B}$.*

Using this fact, build recursively δ_n and $\mathcal{B}_n$ as follows:

(a) $\delta_n=\min\{\sigma(B_0,B_1,\dots,B_{n-1})\colon B_i\in\mathcal{B}_i,\ i<n\}$;

(b) $\mathcal{B}_n$ is a finite δ_n-fine collection of subsets of F_n;

(c) if $A\subseteq F_n$ and $\operatorname{diam} A\leqslant\delta_n/3$, then $A\subseteq B$ for some $B\in\mathcal{B}_n$.

Since X is $\mathsf{Smz}^\sharp$, by Theorem 3.11 there is a $\langle\delta_n/3\rangle$-fine γ-groupable cover $\langle A_n\rangle$ of X. By (c), for each $n\in\omega$ we may choose $B_n\in\mathcal{B}_n$ such that $A_n\cap F_n\subseteq B_n$. Put $\varepsilon_n=\sigma(B_0,B_1,\dots,B_{n-1})$. Since $\operatorname{diam} B_n\leqslant\delta_n\leqslant\varepsilon_n$, the sequence $\langle\varepsilon_0,B_1,\varepsilon_2,B_2,\dots\rangle$ is played according to the strategy σ.

The sequence $\langle B_n\rangle$ is clearly $\langle\varepsilon_n\rangle$-fine. We claim that it is also a γ-groupable cover of X. We know that $\langle A_n\rangle$ is a γ-groupable cover. Let $\langle I_j\rangle$ be the witnessing partition of ω. Fix $x\in X$. We have $\forall^\infty j\ \exists k\in I_j\ x\in A_k$ and since $F_n\nearrow X$, also $\forall^\infty j\ \forall k\in I_j\ x\in F_k$. Therefore $\forall^\infty j\ \exists k\in I_j\ x\in A_k\cap F_k\subseteq B_k$. Thus the partition $\langle I_j\rangle$ is also witnessing that $\langle B_n\rangle$ is a γ-groupable cover of X.

Consequently, σ is not a winning strategy. $\square$

5. $\mathsf{Smz}^\sharp$-sets versus $\mathcal{M}$-additive and $\mathcal{E}$-additive sets

In this section we look closer at $\mathsf{Smz}^\sharp$ subsets of the Cantor set 2^ω. Inspired by the Galvin–Mycielski–Solovay theorem, Theorem 1.2, we prove that $\mathsf{Smz}^\sharp$ sets in 2^ω are meager-additive and vice versa. Recall that $\mathcal{M}$ denotes the ideal of meager sets.

A set $X \subseteq 2^\omega$ is called *$\mathcal{M}$-additive* (or *meager-additive*) if $\forall M \in \mathcal{M} \quad X + M \in \mathcal{M}$. We also define a seemingly stronger notion: call X *sharply $\mathcal{M}$-additive* if $\forall M \in \mathcal{M}\ \exists F \supseteq X$ σ-compact $\ F + M \in \mathcal{M}$.

Theorem 5.1. *For any set $X \subseteq 2^\omega$, the following are equivalent.*
 (i) *X is $\mathsf{Smz}^\sharp$;*
 (ii) *X is $\mathcal{M}$-additive;*
 (iii) *X is sharply $\mathcal{M}$-additive;*
 (iv) *$\forall M \in \mathcal{M}\ \exists F \supseteq X$ σ-compact $\ F + M \neq 2^\omega$.*

Recall that $\mathcal{E}$ is the ideal of Haar null F_σ-sets in 2^ω. We consider also $\mathcal{E}$-additive and sharply $\mathcal{E}$-additive sets. A set $X \subseteq 2^\omega$ is called *$\mathcal{E}$-additive* if $\forall M \in \mathcal{E}\ X + M \in \mathcal{E}$ and *sharply $\mathcal{E}$-additive* if $\forall M \in \mathcal{E}\ \exists F \supseteq X$ σ-compact $F + M \in \mathcal{E}$.

Theorem 5.2. *For any set $X \subseteq 2^\omega$, the following are equivalent.*
 (i) *X is $\mathsf{Smz}^\sharp$;*
 (ii) *X is $\mathcal{E}$-additive;*
 (iii) *X is sharply $\mathcal{E}$-additive.*

Proof: We shall prove now 5.1 (i) $\Rightarrow$ 5.2 (ii) and 5.2 (iii) $\Rightarrow$ 5.1 (iv) $\Rightarrow$ 5.1 (iii) $\Rightarrow$ 5.1 (ii). The remaining implications 5.1 (ii) $\Rightarrow$ 5.1 (i) and 5.2 (ii) $\Rightarrow$ 5.2 (iii) are subject to standalone Propositions 5.4 and 5.7.

5.1 (i) $\Rightarrow$ 5.2 (ii): Assume that X is $\mathsf{Smz}^\sharp$. Let $E \in \mathcal{E}$, i.e., $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}^1_0)$. We may suppose E is σ-compact. By Theorem 3.12 (ii) there is a σ-compact set $F \supseteq X$ such that $\overline{\mathcal{H}}^1(F \times E) = 0$. Since the mapping $(x,y) \mapsto x + y$ is Lipschitz, Lemma 3.6 (ii) yields $\overline{\mathcal{H}}^1(F + E) = 0$, which is enough, because $F + E$ is σ-compact.

5.2 (iii) $\Rightarrow$ 5.1 (iv): Denote by $\mathcal{N}$ the ideal of Haar null sets in 2^ω. We employ a theorem of J. Pawlikowski, see [26] (or see also [1, Theorem 8.1.19]): *For each $M \in \mathcal{M}$ there exists $E \in \mathcal{E}$ such that for each $Y \subseteq 2^\omega$, if $Y + E \in \mathcal{N}$, then $Y + M \neq 2^\omega$.*

Suppose X is sharply $\mathcal{E}$-additive. Let $M \in \mathcal{M}$. Let $E \in \mathcal{E}$ be the set guaranteed by Pawlikowski's theorem. Since X is sharply $\mathcal{E}$-additive, there is $F \supseteq X$ σ-compact such that $F + E \in \mathcal{E} \subseteq \mathcal{N}$. Therefore $F + M \neq 2^\omega$. Thus 5.1 (iv) holds.

5.1 (iv) $\Rightarrow$ 5.1 (iii): Suppose that 5.1 (iv) holds and let $M \in \mathcal{M}$. We may assume that M is σ-compact. Let $Q \subseteq 2^\omega$ be a countable dense set. Clearly $Q+M$ is meager. Therefore there is $F \supseteq X$ σ-compact such that $Q+M+F \neq 2^\omega$. Choose $z \notin Q + M + F$. Then, for all $q \in Q$, $z \notin q + M + F$, i.e., $z + q \notin M + F$. Therefore $(M + F) \cap (z + Q) = \emptyset$. Since Q is dense, so is $z + Q$. Therefore the complement of $F + M$ is dense.

Since $F+M$ is a continuous image of the σ-compact set $F\times M$, it is σ-compact as well. Since it has a dense complement, it is meager.

5.1 (iii) $\Rightarrow$ 5.1 (ii): This is obvious. $\square$

In order to prove that every $\mathcal{M}$-additive set is $\mathsf{Smz}^\sharp$ we need Shelah's characterization of $\mathcal{M}$-additive sets in [29] (or see [1, Theorem 2.7.17]):

Lemma 5.3 ([29]). *The space $X\subseteq 2^\omega$ is $\mathcal{M}$-additive if and only if*

$$\begin{aligned}&\forall\, f\in\omega^{\uparrow\omega}\ \exists\, g\in\omega^\omega\ \exists\, y\in 2^\omega\ \forall\, x\in X\ \forall^\infty n\ \exists\, k\\ &\qquad g(n)\leqslant f(k)<f(k+1)\leqslant g(n+1)\ \ \&\ \ x{\restriction}[f(k),f(k+1))=y{\restriction}[f(k),f(k+1)).\end{aligned}$$

Proposition 5.4. *If $X\subseteq 2^\omega$ is $\mathcal{M}$-additive, then X is $\mathsf{Smz}^\sharp$.*

Proof: Let $X\subseteq 2^\omega$ be $\mathcal{M}$-additive. Let h be a gauge. By Theorem 3.7, it is enough to show that $\overline{\mathcal{H}}^h(X)=0$. Define recursively $f\in\omega^{\uparrow\omega}$ to satisfy

$$2^{f(k)}\cdot h\big(2^{-f(k+1)}\big)\leqslant 2^{-k},\qquad k\in\omega.$$

By Lemma 5.3 there is $g\in\omega^\omega$ and $y\in 2^\omega$ such that

$$\begin{aligned}&\forall\, x\in X\ \forall^\infty n\ \exists\, k\\ &\qquad g(n)\leqslant f(k)<g(n+1)\ \ \&\ \ x{\restriction}[f(k),f(k+1))=y{\restriction}[f(k),f(k+1)).\end{aligned}\tag{3}$$

Recall that if $p\in 2^{<\omega}$ then $[\![p]\!]$ denotes the cone $\{f\in 2^\omega : p\subseteq f\}$. Define

$$\begin{aligned}\mathcal{B}_k&=\{[\![p^\frown y{\restriction}[f(k),f(k+1))]\!] : p\in 2^{f(k)}\}, && k\in\omega,\\ \mathcal{G}_n&=\bigcup\{\mathcal{B}_k : g(n)\leqslant f(k)<g(n+1)\}, && n\in\omega,\\ \mathcal{B}&=\bigcup_{k\in\omega}\mathcal{B}_k=\bigcup_{n\in\omega}\mathcal{G}_n.\end{aligned}$$

With this notation (3) reads

$$\forall\, x\in X\ \forall^\infty n\ \exists\, G\in\mathcal{G}_n\qquad x\in G.\tag{4}$$

Since each of the families $\mathcal{G}_n$ is finite, it follows that $\mathcal{G}_n$'s witness that $\mathcal{B}$ is a γ-groupable cover of X. Using Lemma 3.9 (and Lemma 3.4 (iv)) it remains to show that the Hausdorff sum $\sum_{B\in\mathcal{B}}h(\operatorname{diam}B)$ is finite. Since $|\mathcal{B}_k|=2^{f(k)}$ and $\operatorname{diam}B=2^{-f(k+1)}$ for all k and all $B\in\mathcal{B}_k$, we have

$$\sum_{B\in\mathcal{B}}h(\operatorname{diam}B)=\sum_{k\in\omega}\sum_{B\in\mathcal{B}_k}h(\operatorname{diam}B)=\sum_{k\in\omega}2^{f(k)}\cdot h(2^{-f(k+1)})\leqslant\sum_{k\in\omega}2^{-k}<\infty.$$

$\square$

In order to prove that every $\mathcal{E}$-additive set is sharply $\mathcal{E}$-additive, we employ a combinatorial description of closed null sets given by T. Bartoszyński and S. Shelah in [2], see also [1, 2.6.A]. For $f \in \omega^{\uparrow\omega}$ let

$$\boldsymbol{\mathcal{C}}_f = \left\{\langle F_n\rangle \colon \forall\, n \in \omega \ \left(F_n \subseteq 2^{[f(n),f(n+1))} \ \& \ \frac{|F_n|}{2^{f(n+1)-f(n)}} \leqslant \frac{1}{2^n}\right)\right\}$$

and for $f \in \omega^{\uparrow\omega}$ and $F = \langle F_n\rangle \in \boldsymbol{\mathcal{C}}_f$ define

$$S(f,F) = \{z \in 2^\omega \colon \forall^\infty n \in \omega \ \ z{\restriction}[f(n), f(n+1)) \in F_n\}.$$

It is easy to check that $S(f,F) \in \mathcal{E}$ for all $f \in \omega^{\uparrow\omega}$ and $F \in \boldsymbol{\mathcal{C}}_f$. By [2, Theorem 4.2] (or see [1, Lemma 2.6.3]), these sets actually form a base of $\mathcal{E}$. We need a little more:

Lemma 5.5. $\forall E \in \mathcal{E}\ \forall f \in \omega^{\uparrow\omega}\ \exists\, g \in \omega^{\uparrow\omega}\ \exists\, G \in \boldsymbol{\mathcal{C}}_{f\circ g} \quad E \subseteq S(f\circ g, G)$.

Proof: Let $E \in \mathcal{E}$. We may suppose that $E_n \nearrow E$ with E_n's compact. Recall that for $T \subseteq 2^{<\omega}$ we let $[\![T]\!] = \bigcup_{p\in T}[\![p]\!]$. It is easy to show that if $C \subseteq 2^\omega$ is a compact null set, then

$$\forall\, \varepsilon > 0\ \forall^\infty n\ \exists\, T \subseteq 2^n \qquad C \subseteq [\![T]\!] \ \& \ \frac{|T|}{2^n} < \varepsilon.$$

Therefore we may recursively define $g \in \omega^{\uparrow\omega}$ in such a way that $g(n+1) > g(n)$ and

$$\exists\, T_n \subseteq 2^{f\circ g(n+1)} \qquad E_n \subseteq [\![T_n]\!] \ \& \ \frac{|T_n|}{2^{f\circ g(n+1)}} < \frac{1}{4^{f\circ g(n)}}. \tag{5}$$

Write $h = f \circ g$. For $n \in \omega$ define $G_n = \{s{\restriction}[h(n), h(n+1)) \colon s \in T_n\}$. Obviously $|G_n| \leqslant |T_n|$. Therefore (5) yields

$$\frac{|G_n|}{2^{h(n+1)-h(n)}} \leqslant \frac{|T_n|}{2^{h(n+1)}}\, 2^{h(n)} \leqslant \frac{1}{4^{h(n)}}\, 2^{h(n)} \leqslant \frac{1}{2^{h(n)}} \leqslant \frac{1}{2^n}.$$

Thus $\langle G_n\rangle \in \boldsymbol{\mathcal{C}}_h$ and since $E_n \nearrow E$, we also have $E \subseteq S(h,G)$, as desired. $\square$

Lemma 5.6. *Let $f, g \in \omega^{\uparrow\omega}$, $F \in \boldsymbol{\mathcal{C}}_f$ and $G \in \boldsymbol{\mathcal{C}}_{f\circ g}$. Then $S(f,F) \subseteq S(f\circ g, G)$ if and only if*

$$\forall^\infty n \in \omega\ \forall\, k \in [g(n), g(n+1)) \qquad F_k \subseteq \{z{\restriction}[f(k), f(k+1)) \colon z \in G_n\}. \tag{6}$$

Proof: Suppose condition (6) fails. Then there is $I \in [\omega]^\omega$ such that

$$\forall\, n \in I\ \ \exists\, k_n \in [g(n), g(n+1))\ \exists\, z_{k_n} \in F_{k_n}\ \forall\, z \in G_n \qquad z_{k_n} \not\subseteq z. \tag{7}$$

For each $k \notin \{k_n \colon n \in I\}$ choose $z_k \in F_k$ and let $z \in 2^\omega$ be a sequence that extends simultaneously all z_k's (including those defined in (7)). Then obviously $z \in S(f,F)$. On the other hand, condition (7) ensures that $z \notin S(f\circ g, G)$. Thus $S(f,F) \subseteq S(f\circ g, G)$ yields (6). The reverse implication is straightforward. $\square$

Proposition 5.7. *If $X \subseteq 2^\omega$ is $\mathcal{E}$-additive, then X is sharply $\mathcal{E}$-additive.*

Proof: Suppose X is $\mathcal{E}$-additive. Let $E \in \mathcal{E}$. We are looking for a σ-compact set $\widetilde{X} \supseteq X$ such that $\widetilde{X} + E \in \mathcal{E}$.

There are $f \in \omega^{\uparrow\omega}$ and $F \in \boldsymbol{\mathcal{C}}_f$ such that $E \subseteq S(f,F)$. Since $S(f,F) \in \mathcal{E}$, we have $X + S(f,F) \in \mathcal{E}$. By Lemma 5.5 there are g and $G \in \boldsymbol{\mathcal{C}}_{f\circ g}$ such that $X + S(f,F) \subseteq S(f\circ g, G)$, i.e., $x + S(f,F) \subseteq S(f\circ g, G)$ for all $x \in X$.

The set $\widetilde{X}$ we are looking for is

$$\widetilde{X} = \{x \in 2^\omega \colon x + S(f,F) \subseteq S(f\circ g, G)\}.$$

Obviously $X \subseteq \widetilde{X}$. It is also obvious that $\widetilde{X} + E \subseteq \widetilde{X} + S(f,F) \subseteq S(f\circ g, G) \in \mathcal{E}$. Thus it remains to show that $\widetilde{X}$ is F_σ.

For any $x \in 2^\omega$ and $k \in \omega$ set

$$F_k^x = \{z + x{\restriction}[f(k), f(k+1)) \colon z \in F_k\}$$

and consider the sequence $F^x = \langle F_k^x \rangle$. Clearly $F^x \in \boldsymbol{\mathcal{C}}_f$ and $S(f, F^x) = x + S(f,F)$. Therefore $\widetilde{X} = \{x \in 2^\omega \colon S(f,F^x) \subseteq S(f\circ g, G)\}$. Use Lemma 5.6 to conclude that

$$x \in \widetilde{X} \;\Leftrightarrow\; \forall^\infty n \in \omega \;\forall\, k \in [g(n), g(n+1)) \;\; F_k^x \subseteq \{z{\restriction}[f(k), f(k+1)) \colon z \in G_n\}.$$

It follows that $\widetilde{X}$ is F_σ as long as the sets

$$A_{n,k} = \{x \in 2^\omega \colon F_k^x \subseteq \{z{\restriction}[f(k), f(k+1)) \colon z \in G_n\}\}$$

are closed for all n and all $k \in [g(n), g(n+1))$. Fix $n \in \omega$ and $k \in [g(n), g(n+1))$. Decoding the definitions yields

$$x \in A_{n,k} \;\Leftrightarrow\; \exists\, y \in 2^{[f(k), f(k+1))} \;\; y \subseteq x \;\;\&\;\; \forall\, z \in F_k \;\exists\, t \in G_n \;\; z + y \subseteq t.$$

Since the set $\{y \in 2^{[f(k),f(k+1))} \colon \forall\, z \in F_k \;\exists\, t \in G_n \;\; z + y \subseteq t\}$ is finite, the set $A_{n,k}$ is closed, as required. We are done. $\square$

The proof of Theorems 5.1 and 5.2 is now complete. Here are a few consequences. First of them is Theorem 1.4: $X \subseteq 2^\omega$ is $\mathcal{M}$-additive if and only if it is $\mathcal{E}$-additive.

The next one follows easily from the equivalence of $\mathsf{Smz}^\sharp$ and $\mathcal{M}$-additivity and (taking in account that a continuous map on 2^ω is uniformly continuous) Proposition 3.2 (ii).

Corollary 5.8. *Let $f\colon 2^\omega \to 2^\omega$ be a continuous mapping. If $X \subseteq 2^\omega$ is $\mathcal{M}$-additive, then so is $f(X)$.*

The following is a little surprising.

Corollary 5.9. *If $X \subseteq 2^\omega$ is $\mathcal{E}$-additive, then $\varphi(X \times E) \in \mathcal{E}$ for each $E \in \mathcal{E}$ and every Lipschitz mapping $\varphi\colon 2^\omega \times 2^\omega \to 2^\omega$.*

Proof: Let $E \in \mathcal{E}$ be σ-compact. Since X is $\mathcal{E}$-additive, it is $\mathsf{Smz}^\sharp$. By Theorem 3.12 (ii), there is a σ-compact set $K \supseteq X$ such that $\overline{\mathcal{H}}^1(K \times E) = 0$. By Lemma 3.6 (ii), $\overline{\mathcal{H}}^1(\varphi(K \times E)) = 0$, i.e., $\varphi(X \times E) \in \mathcal{E}$. □

The notion of $\mathcal{M}$-additive sets extends to finite cartesian powers of 2^ω in the obvious manner.

Corollary 5.10. *A set $X \subseteq (2^\omega)^n$ is $\mathcal{M}$-additive if and only if it is $\mathsf{Smz}^\sharp$.*

Proof: We provide the argument for $n = 2$. Let $X \subseteq (2^\omega)^2$. Denote by X_1, X_2 the two projections of X. An easy application of Kuratowski–Ulam theorem proves that if X is $\mathcal{M}$-additive in $(2^\omega)^2$, then both X_1 and X_2 are $\mathcal{M}$-additive in 2^ω. (*Hint*: If $M \subseteq 2^\omega$ is meager, then $X + (M \times 2^\omega)$ is meager in $(2^\omega)^2$. Since $(X_1 + M) \times 2^\omega = X + (M \times 2^\omega)$, the set $X_1 + M$ is meager, as required.) Therefore they are $\mathsf{Smz}^\sharp$ and by Theorem 3.14 (i) $X_1 \times X_2$ is $\mathsf{Smz}^\sharp$. A fortiori, X is $\mathsf{Smz}^\sharp$.

Now suppose X is $\mathsf{Smz}^\sharp$. Then both X_1 and X_2, being Lipschitz images of X, are by Proposition 3.2 (ii) also $\mathsf{Smz}^\sharp$ and thus $\mathcal{M}$-additive in 2^ω. By [32, Theorem 1] a product of two $\mathcal{M}$-additive sets in 2^ω is $\mathcal{M}$-additive. Therefore $X_1 \times X_2$ and a fortiori X is $\mathcal{M}$-additive. □

6. Remarks

$\mathsf{Smz}^\sharp$ sets on the line. It is clear how the notion of $\mathcal{M}$-additive set extends to other topological groups. The addition operations on 2^ω and on the real line $\mathbb{R}$ are so different that it was not understood for a long time if $\mathcal{M}$-additive sets on $\mathbb{R}$ behave the same way as those on 2^ω. Finally T. Weiss in [33] found the following solution. Let $T\colon 2^\omega \to [0,1]$ be the standard mapping defined by $T(x) = \sum_{n\in\omega} 2^{-n-1}x(n)$.

Theorem 6.1 ([33, 1.10])**.** *A set $X \subseteq [0,1]$ is $\mathcal{M}$-additive in $\mathbb{R}$ if and only if $T^{-1}(X)$ is $\mathcal{M}$-additive in 2^ω.*

A similar (and much easier) result holds for $\mathsf{Smz}^\sharp$ sets:

Proposition 6.2. *A set $X \subseteq [0,1]$ is $\mathsf{Smz}^\sharp$ if and only if $T^{-1}(X)$ is $\mathsf{Smz}^\sharp$.*

Proof: Lemma 3.5 of [31] asserts that for any set $U \subseteq [0,1]$ there are sets $U_0, U_1 \subseteq 2^\omega$ such that $U_0 \cup U_1 = T^{-1}(U)$ and $\operatorname{diam} U_i \leqslant \operatorname{diam} U$ for both $i = 0, 1$. It follows that $\overline{\mathcal{H}}^h(T^{-1}(X)) \leqslant 2\overline{\mathcal{H}}^h(X)$ for every gauge h. On the other hand, since T is 1-Lipschitz, $\overline{\mathcal{H}}^h(X) \leqslant \overline{\mathcal{H}}^h(T^{-1}(X))$ by Lemma 3.6 (i).

Use the two inequalities and Theorem 3.7 (ii) to conclude the proof. □

Theorem 6.3. *A set $X \subseteq \mathbb{R}$ is $\mathcal{M}$-additive if and only if it is $\mathsf{Smz}^\sharp$.*

Proof: Since both properties are σ-additive, we may clearly suppose $X \subseteq [0,1]$. The proof is a straightforward application of Theorem 5.1 and the above theorem and proposition. □

This theorem extends to $\mathbb{R}^n$: By [33, 11] a product of two $\mathcal{M}$-additive sets on $\mathbb{R}$ is $\mathcal{M}$-additive on $\mathbb{R}^2$. Using this fact and the above Theorem 6.3 one can repeat the proof of Corollary 5.10 to show:

Theorem 6.4. *A set $X \subseteq \mathbb{R}^n$ is $\mathcal{M}$-additive if and only if it is $\mathsf{Smz}^\sharp$.*

Corollary 6.5. *Let $X \subseteq \mathbb{R}^2$. The following are equivalent.*

(i) *X is $\mathcal{M}$-additive;*
(ii) *all orthogonal projections of X on lines are $\mathcal{M}$-additive;*
(iii) *at least two orthogonal projections of X on lines are $\mathcal{M}$-additive.*

Proof in outline: Since orthogonal projections are uniformly continuous, (i) $\Rightarrow$ (ii) follows at once from Theorem 6.4 and Proposition 3.2 (ii).

(iii) $\Rightarrow$ (i): Let L_1, L_2 be two nonparallel lines and π_1, π_2 the corresponding orthogonal projections. *Mutatis mutandis* we may suppose that L_1 is the x-axis and L_2 is the y-axis. Thus $X \subseteq \pi_1 X \times \pi_2 X$. $\square$

γ-sets. A. Nowik and T. Weiss in [25, Proposition 3.7] proved that every γ-set of reals is $\mathcal{M}$-additive. We will show a generalization of this result, namely that all γ-sets are $\mathsf{Smz}^\sharp$.

Recall the notion of γ-set, as introduced by J. Gerlits and Z. Nagy in [13]. A family $\mathcal{U}$ of open sets in a separable metric space X is called an *ω-cover* of X if every finite subset of X is contained in some $U \in \mathcal{U}$. A separable metric space X is a *γ-set* if every ω-cover of X contains a γ-cover.

Proposition 6.6. *Every γ-set is $\mathsf{Smz}^\sharp$.*

Proof: Let X be an infinite γ-set. We use Theorem 3.11 to show that X is $\mathsf{Smz}^\sharp$. Let $\langle \varepsilon_n \rangle \in (0,\infty)^\omega$. We may suppose $\langle \varepsilon_n \rangle$ is decreasing. We are looking for an $\langle \varepsilon_n \rangle$-fine γ-groupable cover. For $n \in \omega$ define $\delta_n = \varepsilon_{0+1+2+\cdots+n}$. Let $Q \subseteq X$ be a countable dense set in X. Fix an infinite set $\{z_n : n \in \omega\} \subseteq X \setminus Q$. For $n \in \omega$ and $F \in [Q]^n$ put

$$F^\circ = \bigcup_{x \in F} B\Big(x, \frac{1}{2}\delta_n\Big) \setminus \{z_n\}.$$

The family $\{F^\circ : F \in [Q]^{<\omega}\}$ is obviously an ω-cover. Therefore there is a sequence $\langle F_k \rangle$ of finite sets such that $\langle F_k^\circ \rangle$ is a γ-cover. If $|F| = n$, then F° misses z_n. It follows that the cardinalities of the F_k's are unbounded. Thus we may choose a subsequence $\langle k_m \rangle$ such that $|F_{k_0}| < |F_{k_1}| < |F_{k_2}| < \dots$. The sequence $\langle F_{k_m}^\circ : m \in \omega \rangle$ is still a γ-cover.

Write $j_m = |F_{k_m}|$. Form a sequence $\langle x_i \rangle$ as follows: First enumerate all points in F_{k_0}, then continue with points of F_{k_1} and so on. Note that if $x_i \in F_{k_m}$, then $i \leqslant j_0 + j_1 + \cdots + j_m \leqslant 0 + 1 + \cdots + j_m$ and thus $\varepsilon_i \geqslant \varepsilon_{0+1+\cdots+j_m} = \delta_{j_m}$. Consequently $F_{k_m}^\circ \subseteq \bigcup \{B(x_i, \varepsilon_i/2) : x_i \in F_{k_m}\}$ and it follows that the families $\mathcal{G}_m = \{B(x_i, \varepsilon_i/2) : x_i \in F_{k_m}\}$ are witnessing that $\langle B(x_i, \varepsilon_i/2) \rangle$ is an $\langle \varepsilon_i \rangle$-fine γ-groupable cover. $\square$

Scheepers' theorem. A metric space X has the *Hurewicz property* if for any sequence $\langle \mathcal{U}_n \rangle$ of open covers there are finite families $\mathcal{F}_n \subseteq \mathcal{U}_n$ such that, letting $F_n = \bigcup \mathcal{F}_n$, the sequence $\langle F_n \rangle$ is a γ-cover of X.

M. Scheepers in [28, Theorem 1, Lemma 3] proved that a product of a Smz set and a Smz set with the Hurewicz property is Smz. It is straightforward that a Smz space with the Hurewicz property is $\mathsf{Smz}^\sharp$. Therefore Theorem 3.14 (ii) improves Scheepers' result. We claim that it is a proper extension: there is a CH example of a $\mathsf{Smz}^\sharp$ set that lacks the Hurewicz property.

Proposition 6.7. *Assuming the continuum hypothesis, there is a* $\mathsf{Smz}^\sharp$ *set that does not have the Hurewicz property.*

PROOF: It follows from [10, Theorem 1] and its proof that under the continuum hypothesis there is a γ-set $X \subseteq 2^\omega$ that is concentrated on a countable set D. By Proposition 6.6, X is $\mathsf{Smz}^\sharp$. On the other hand, as proved in [24, Theorem 20], the set $X \setminus D$ does not have the Hurewicz property and since it is a subset of X, it is $\mathsf{Smz}^\sharp$. □

Corazza's model. Theorem 3.14 (ii) also raises the question whether a space whose product with any Smz set of reals is Smz has to be $\mathsf{Smz}^\sharp$. The answer is consistently no. A similar observation was noted without proof in [24] and also in [34].

We choose "reals" to refer to 2^ω, since by Proposition 6.2 it makes no difference if we work in 2^ω or $\mathbb{R}$. The following argument came out from a discussion with T. Weiss. P. Corazza in [7] constructs a forcing extension with the following properties. Denote by X the set of ground model reals.

(a) X is not a meager set in the Corazza extension;
(b) $|X| = \omega_1$;
(c) a set of reals Y in the extension is Smz if and only if $|Y| \leqslant \omega_1$.

By (a), X is not $\mathcal{M}$-additive and hence not $\mathsf{Smz}^\sharp$. By (b) and (c), if Y is any Smz set of reals, then $|X \times Y| \leqslant \omega_1$. Since $2^\omega \times 2^\omega$ is uniformly homeomorphic to 2^ω, $X \times Y$ is a uniformly continuous image of a set of cardinality at most ω_1. Such a set is Smz by (c) and thus $X \times Y$ is Smz as well. We proved the following:

Proposition 6.8. *In the Corazza model there is a set $X \subseteq 2^\omega$ that is not* $\mathsf{Smz}^\sharp$ *and yet $X \times Y$ is* Smz *for each* Smz *set $Y \subseteq 2^\omega$.*

However, the following question remains unanswered:

Question 6.9. Is it consistent that there is a metric space X such that $X \times Y$ is Smz for every Smz metric space Y and yet X is not $\mathsf{Smz}^\sharp$?

Dimension inequalities. By Corollaries 2.8 and 3.13, if Y is a σ-compact metric space, then $\underline{\dim}_{\mathsf{H}}\, X \times Y = \dim_{\mathsf{H}} Y$ if X is Smz, and $\overline{\dim}_{\mathsf{H}}\, X \times Y = \dim_{\mathsf{H}} Y$ if X is $\mathsf{Smz}^\sharp$.

We note that the assumption of σ-compactness imposed upon Y cannot be dropped:

B. Tsaban and T. Weiss in [35, Theorem 4] construct, under $\mathfrak{p} = \mathfrak{c}$, a γ-set X (that is by Proposition 6.6 $\mathsf{Smz}^\sharp$) and a set $Y \subseteq \mathbb{R}$ such that $\dim_{\mathsf{H}} Y = 0$ and yet $\dim_{\mathsf{H}} X \times Y = 1$.

Universally meager sets. Recall that a separable metric space E is termed *universally meager*, see [37], [38], if for any perfect Polish spaces Y, X such that $E \subseteq X$ and every continuous one-to-one mapping $f\colon Y \to X$ the set $f^{-1}(E)$ is meager in Y. We show that $\mathsf{Smz}^\sharp$ sets are universally meager.

Lemma 6.10. *Let X, Y, Z be perfect Polish spaces and $\varphi\colon Y \to X$ a continuous one-to-one mapping. Let $\mathcal{F}$ be an equicontinuous family of uniformly continuous mappings of Z into X. If $E \subseteq Z$ is $\mathsf{Smz}^\sharp$, then there is a σ-compact set $F \supseteq E$ such that $\varphi^{-1} f(F)$ is meager in Y for all $f \in \mathcal{F}$.*

Proof: Let $\{U_n\}$ be a countable base for Y. As φ is one-to-one the set $\varphi(U_n)$ is analytic and uncountable for each n. Therefore it contains a perfect set and thus is not $\mathsf{Smz}^\sharp$, i.e., by Theorem 3.7 there is a gauge h_n such that $\overline{\mathcal{H}}^{h_n}(\varphi(U_n)) > 0$. Choose a gauge h such that $h \prec h_n$ for all n, so that, by Lemma 3.4 (viii), $\overline{\mathcal{H}}^h(\varphi(U_n)) > 0$ for all n. Therefore $\overline{\mathcal{H}}^h(\varphi(U)) > 0$ for each nonempty set U open in Y.

Since $\mathcal{F}$ is equicontinuous, there is a gauge g such that (2) is satisfied by each $f \in \mathcal{F}$. By Theorem 3.7 and Lemma 3.4 (viii), $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^{h \circ g})$. Therefore Lemma 3.4 (v) and (vii) yield a σ-compact set $F \supseteq E$ such that $\overline{\mathcal{H}}^{h \circ g}(F) = 0$. Hence Lemma 3.6 (i) guarantees that $\overline{\mathcal{H}}^h(f(F)) = 0$ for all $f \in \mathcal{F}$. Therefore the F_σ-set $\varphi^{-1} f(F)$ is meager in Y: for otherwise it would contain an open set witnessing $\overline{\mathcal{H}}^h(f(F)) > 0$. $\square$

Apply this lemma with $Z = X$ and $\mathcal{F} = \{\mathrm{id}_X\}$ to get

Proposition 6.11. *Every $\mathsf{Smz}^\sharp$ set is universally meager.*

Meager-additive sets in topological groups. There is an obvious question: how far beyond 2^ω and $\mathbb{R}$ we can extend the equivalence of $\mathcal{M}$-additive and $\mathsf{Smz}^\sharp$.

Question 6.12. For what Polish groups are the notions of $\mathcal{M}$-additive and $\mathsf{Smz}^\sharp$ equivalent?

Null-additive sets. A set $X \subseteq 2^\omega$ is termed *null-additive* if for every Haar null set N the set $X + N$ is Haar null. In a follow-up of the present paper we will show that null-additive sets in 2^ω can be described in terms of packing measures and dimensions.

7. Appendix: Hausdorff measures on cartesian products

In this appendix, we prove a few integral inequalities needed for the proof of Lemma 3.5. They generalize those proved by J. D. Howroyd in his famous thesis in [16] and J. D. Kelly, see [20].

We need a notion of a weighted Hausdorff measure. Let X be a metric space. Say that a countable collection of pairs $\{(c_i, E_i) \colon i \in I\}$ is a *weighted cover* of $E \subseteq X$ if $c_i > 0$ and $E_i \subseteq X$ for all $i \in I$ and $\sum\{c_i \colon x \in E_i\} \geqslant 1$ for all $x \in E$. We say it is δ-fine if the cover $\{E_i \colon i \in I\}$ is δ-fine, i.e., if $\operatorname{diam} E_i \leqslant \delta$ for all $i \in I$.

Let g be a gauge and $E \subseteq X$. For each $\delta > 0$ set

$$\boldsymbol{\lambda}^g_\delta(E) = \inf\Bigl\{\sum_{i\in\omega} c_i g(\operatorname{diam} E_i) \colon \{(c_i, E_i)\} \text{ is a } \delta\text{-fine weighted cover of } E\Bigr\}$$

and put $\boldsymbol{\lambda}^g(E) = \sup_{\delta>0} \boldsymbol{\lambda}^g_\delta(E)$.

Properties of the weighted Hausdorff measures are discussed, e.g., in the two mentioned papers [16], [20]. Trivially $\boldsymbol{\lambda}^g \leqslant \mathcal{H}^g$. The converse inequality holds if g satisfies the doubling condition. It was proved in [8, 2.10.24] for compact sets and in [16, 9.8] in full generality.

Theorem 7.1 ([16, 9.8])**.** *If g is a doubling gauge, then $\boldsymbol{\lambda}^g = \mathcal{H}^g$.*

The integrals in the following inequalities are the usual upper Lebesgue integrals: If μ is a Borel measure on a metric space X and $f \colon X \to [-\infty, \infty]$ a function, then

$$\int^* f \,\mathrm{d}\mu = \inf\Bigl\{\int \varphi \,\mathrm{d}\mu \colon \varphi \geqslant f \text{ Borel measurable}\Bigr\}.$$

Let X, Y be metric spaces. Denote their respective metrics by d_X and d_Y. Recall that the cartesian product $X \times Y$ is equipped by the maximum metric (1). For a set $E \subseteq X \times Y$ and $x \in X$, write E_x for the vertical section $\{y \in Y \colon (x, y) \in E\}$.

Theorem 7.2. *Let X, Y be metric spaces and $E \subseteq X \times Y$. Let g, h be gauges. Then*

(i) $\int^* \boldsymbol{\lambda}^g(E_x) \,\mathrm{d}\boldsymbol{\lambda}^h(x) \leqslant \boldsymbol{\lambda}^{gh}(E)$;

(ii) $\int^* \mathcal{H}^g(E_x) \,\mathrm{d}\boldsymbol{\lambda}^h(x) \leqslant \mathcal{H}^{gh}(E)$.

Proof: First of all, we may assume that E is Borel, since both Hausdorff and weighted Hausdorff measures are Borel regular. It is also routine to check that if E is Borel, then the integrands $x \mapsto \boldsymbol{\lambda}^g(E_x)$ and $x \mapsto \mathcal{H}^g(E_x)$ are Borel measurable. Therefore both integrals are standard Lebesgue integrals.

We prove (i) first and then indicate how to get (ii) by the same proof. Approximating the integrand from below by a simple function and replacing X with the projection of E onto the x-axis reduces (i) to the following:

$$\text{If } \boldsymbol{\lambda}^g(E_x) > \gamma \text{ for all } x \in X, \text{ then } \boldsymbol{\lambda}^{gh}(E) \geqslant \gamma \boldsymbol{\lambda}^h(X). \tag{8}$$

Fix $\varepsilon > 0$. For every $x \in X$ there is $\delta_x > 0$ such that $\boldsymbol{\lambda}^g_{\delta_x}(E_x) > \gamma$. Therefore there is $\hat\delta > 0$ and a set $\widehat X \subseteq X$ such that $\boldsymbol{\lambda}^g_{\hat\delta}(E_x) > \gamma$ for all $x \in \widehat X$ and

$\boldsymbol{\lambda}^h(\widehat{X}) > \boldsymbol{\lambda}^h(X) - \varepsilon$. Since $\boldsymbol{\lambda}^h$ can be approximated by $\boldsymbol{\lambda}^h_\delta$, there is $\delta^* > 0$ such that $\boldsymbol{\lambda}^h_{\delta^*}(\widehat{X}) > \boldsymbol{\lambda}^h(X) - \varepsilon$. So if we let $\delta = \min\{\hat{\delta}, \delta^*\}$, we have

(a) $\boldsymbol{\lambda}^g_\delta(E_x) > \gamma$ for all $x \in \widehat{X}$;

(b) $\boldsymbol{\lambda}^h_\delta(\widehat{X}) > \boldsymbol{\lambda}^h(X) - \varepsilon$.

Let $\mathcal{C} = \{(c_i, E_i) \colon i \in I\}$ be a δ-fine weighted cover of E. Denote by p_X and p_Y the respective projections. For each $i \in I$ let $d_i = (c_i/\gamma) g(\operatorname{diam} p_Y(E_i))$ and consider the family $\mathcal{D} = \{(d_i, p_X(E_i)) \colon i \in I\}$.

For each $x \in \widehat{X}$ we have

$$\sum_{x \in p_X(E_i)} d_i = \frac{1}{\gamma} \sum_{x \in p_X(E_i)} c_i g(\operatorname{diam} p_Y(E_i)) \geqslant \frac{1}{\gamma} \sum_{(E_i)_x \neq \emptyset} c_i g(\operatorname{diam} (E_i)_x)$$

and since the family $\{(c_i, (E_i)_x) \colon (E_i)_x \neq \emptyset\}$ is obviously a δ-fine weighted cover of E_x, the latter sum is estimated from below by $\boldsymbol{\lambda}^g_\delta(E_x)$. Therefore (a) and the above calculation shows that $\sum_{x \in p_X(E_i)} d_i \geqslant 1$ for all $x \in \widehat{X}$, i.e., that $\mathcal{D}$ is a δ-fine weighted cover of $\widehat{X}$. Therefore

$$\begin{aligned}\boldsymbol{\lambda}^h_\delta(\widehat{X}) &\leqslant \sum d_i h(\operatorname{diam} p_X(E_i)) \leqslant \sum \frac{c_i}{\gamma} g(\operatorname{diam} p_Y(E_i)) h(\operatorname{diam} p_X(E_i)) \\ &\leqslant \frac{1}{\gamma} \sum c_i g(\operatorname{diam} E_i) h(\operatorname{diam} E_i) = \frac{1}{\gamma} \sum c_i (gh)(\operatorname{diam} E_i).\end{aligned}$$

Multiplying with γ and taking the infimum over all δ-fine weighted covers of E yields $\gamma \boldsymbol{\lambda}^h_\delta(\widehat{X}) \leqslant \boldsymbol{\lambda}^{gh}_\delta(E)$. It thus follows from (b) that

$$\gamma(\boldsymbol{\lambda}^h(X) - \varepsilon) \leqslant \gamma \boldsymbol{\lambda}^h_\delta(\widehat{X}) \leqslant \boldsymbol{\lambda}^{gh}_\delta(E) \leqslant \boldsymbol{\lambda}^{gh}(E)$$

and (8) obtains on letting $\varepsilon \to 0$.

(ii) follows from this proof simply by imposing an extra condition: require that $c_i = 1$ for all $i \in I$. □

Combining this theorem with Theorem 7.1 yields

Corollary 7.3. *If h is a doubling gauge, then $\int^* \mathcal{H}^g(E_x)\,\mathrm{d}\mathcal{H}^h(x) \leqslant \mathcal{H}^{gh}(E)$.*

Theorem 7.4. *Let X, Y be metric spaces and $E \subseteq X \times Y$. If g, h are gauges, then*

(i) $\int^* \overline{\mathcal{H}}^g(E_x)\,\mathrm{d}\boldsymbol{\lambda}^h(x) \leqslant \overline{\mathcal{H}}^{gh}(E)$;

(ii) *and if h is doubling, then* $\int^* \overline{\mathcal{H}}^g(E_x)\,\mathrm{d}\mathcal{H}^h(x) \leqslant \overline{\mathcal{H}}^{gh}(E)$.

Proof: The second inequality follows at once from the first one and Theorem 7.1. The first inequality obtains from Theorem 7.2 (ii) as follows: We may suppose that X and Y are both complete metric spaces. Let $K \supseteq E$ be a σ-compact set. By Lemma 3.4 (vii) $\overline{\mathcal{H}}^g(E_x) \leqslant \overline{\mathcal{H}}^g(K_x) = \mathcal{H}^g(K_x)$ for all x, hence Theorem 7.2 (ii) yields

$$\int^* \overline{\mathcal{H}}^g(E_x)\,\mathrm{d}\boldsymbol{\lambda}^h(x) \leqslant \int^* \mathcal{H}^g(K_x)\,\mathrm{d}\boldsymbol{\lambda}^h(x) \leqslant \mathcal{H}^{gh}(K).$$

Apply Lemma 3.4 (vi) to conclude the proof. □

A particular choice of $E = X \times Y$ yields Lemmas 2.3 and 3.5.

Acknowledgment. I would like to thank Michael Hrušák for his kind support and never-ending enlightening discussions while I was visiting Instituto de Matemáticas, Unidad Morelia, Universidad Nacional Autonóma de México, Peter Eliaš for numerous valuable comments, Tomasz Weiss for help with Proposition 6.8, Marion Scheepers and Boaz Tsaban who made me write the paper and last but not least, to the anonymous referee for his very thorough job.

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(*Received* December 29, 2017*, revised* September 29, 2018)