Comment.Math.Univ.Carolin. 60,2 (2019) 231–267

Regularity problem for one class of nonlinear parabolic systems with non-smooth in time principal matrices

Arina A. Arkhipova, Jana Stará

Abstract. Partial regularity of solutions to a class of second order nonlinear parabolic systems with non-smooth in time principal matrices is proved in the paper. The coefficients are assumed to be measurable and bounded in the time variable and VMO-smooth in the space variables uniformly with respect to time. To prove the result, we apply the so-called $A(t)$-caloric approximation method. The method was applied by the authors earlier to study regularity of quasilinear systems.

Keywords: nonlinear parabolic systems; regularity problem

Classification: 35B65, 35D30, 35K99

1. Introduction

In this paper we study partial regularity of weak solutions to the following class of parabolic systems:

$$u_t(z) - \operatorname{div} a(z, \nabla u(z)) = 0, \qquad z = (x,t) \in Q, \tag{1}$$

where $Q = \Omega \times (-T, 0)$, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and a constant T is positive. By u_t we denote the time derivative of a function $u\colon Q \to \mathbb{R}^N$, $N > 1$, and by $\nabla u = (u_{x_1}, \dots, u_{x_n})$, $u_{x_i} = \frac{\partial u}{\partial x_i}$, its space gradient.

We assume that the Carathéodory functions $a(z,p) = \{a^k_\alpha(z,p)\}^{k\leq N}_{\alpha\leq n}$ satisfy the following conditions:
[H1] there exists a number $L > 0$ such that

$$|a(z,p)| \leq L(1+|p|) \qquad \text{a.a. } z \in Q, \quad \forall\, p \in \mathbb{R}^{nN}; \tag{2}$$

[H2] there is $\beta \in (0,1]$ such that

$$|a(x,t,p) - a(y,t,p)| \leq L \min\{1, |x-y|^\beta\}(1+|p|) \qquad \text{a.a. } x, y \in \Omega, \quad t \in (-T,0), \quad \forall\, p \in \mathbb{R}^{nN}; \tag{3}$$

DOI 10.14712/1213-7243.2019.010
First author was supported by RFFI grant 18-01-00472a.

[H3] the coefficients $a(z,p)$ are differentiable with respect to p, the matrix $\frac{\partial a}{\partial p} = \left\{\frac{\partial a^k_\alpha(z,p)}{\partial p^l_\beta}\right\}^{k,l\le N}_{\alpha,\beta\le n}$ has bounded measurable entries, and the ellipticity conditions

$$(4)\quad \nu|\xi|^2 \le \Big(\frac{\partial a(z,p)}{\partial p}\xi\cdot\xi\Big), \qquad \Big|\frac{\partial a(z,p)}{\partial p}\Big| \le \mu, \qquad \text{a.a. } z\in Q,\ \ \forall\, p,\,\xi\in\mathbb{R}^{nN},$$

hold with positive numbers $\nu \le \mu$;
[H4] the matrix $\frac{\partial a}{\partial p}$ is uniformly continuous in $p\in\mathbb{R}^{nN}$ for almost all $z\in Q$, i.e. there exists a nonnegative bounded nondecreasing and concave function $\omega(s)$, $s\in[0,\infty)$, such that $\omega(s)\to 0$, $s\to +0$, and

$$(5)\qquad \Big|\frac{\partial a(z,p)}{\partial p} - \frac{\partial a(z,p^0)}{\partial p}\Big| \le \omega(|p-p^0|^2), \qquad p,\,p^0\in\mathbb{R}^{nN},\ \ \text{a.a. } z\in Q;$$

[H5] the entries of the matrix $\frac{\partial a(z,p)}{\partial p}$ belong to the class $\mathrm{VMO}(\Omega)$ for almost all $t\in\Lambda=(-T,0)$ and all $p\in\mathbb{R}^{nN}$; moreover the following condition holds

$$(6)\quad \sup_{\substack{Q_\varrho(z^0)\subset Q,\\ \varrho\le r,\ p\in\mathbb{R}^{nN}}} ⨍_{Q_\varrho(z^0)} \Big|\frac{\partial a(x,t,p)}{\partial p} - \Big(\frac{\partial a}{\partial p}\Big)_{\varrho,x^0}(t;p)\Big|^2\,\mathrm{d}z =: q(r)\to 0, \qquad r\to 0,$$

where $Q_\varrho(z^0)$ is a parabolic cylinder (see the notation below) and

$$\Big(\frac{\partial a}{\partial p}\Big)_{\varrho,x^0}(t;p) = ⨍_{B_\varrho(x^0)} \frac{\partial a(x,t,p)}{\partial p}\,\mathrm{d}x \qquad \text{a.a. } t\in\Lambda,\ \ \forall\, p\in\mathbb{R}^{nN}.$$

Here $B_\varrho(x^0)$ is a ball in $\mathbb{R}^n$ centered in x^0 with the radius ϱ.

As we study only *interior* partial regularity of weak solutions to system (1) we can assume that conditions [H1]–[H5] are satisfied locally in Q.

We consider weak solutions u of system (1) defined as follows:

Definition 1.1. A function $u\in V(Q) := L^2((-T,0);W^1_2(\Omega))$ is a weak solution to system (1) if it satisfies the identity

$$(7)\qquad \int_Q [-u(z)\cdot\varphi_t(z) + a(z,\nabla u(z))\cdot\nabla\varphi(z)]\,\mathrm{d}z = 0$$

for all $\varphi\in \overset{o}{W}{}^1_2(Q) = \overline{[C_0^\infty(Q)]}_{W^1_2(Q)}$.

In this paper we continue to study optimal conditions on the principal parts of different classes of parabolic systems to relax the known assumptions on the data which guarantee partial regularity of weak solutions.

In our previous papers [8] and [5] we considered *quasilinear* systems

$$(8)\qquad u_t - \operatorname{div}(A(z,u)\nabla u) = 0, \qquad z\in Q,$$

and proved partial regularity of weak solutions under relaxed smoothness assumptions on the principal matrix $A(x, t, u)$ in the arguments x and t. We assumed in these papers only boundedness in the time variable and integral VMO-smoothness in the space variables $x = (x_1, \ldots, x_n)$ of the matrix $A(x, t, u)$. We studied regularity of solutions inside Q in [8] and up to the parabolic boundary of Q under the Cauchy–Dirichlet conditions in [5]. (Regularity of weak solutions to the Venttsel boundary problem for linear and quasilinear parabolic systems under the same assumptions on the data was proved by the $A(t)$-caloric approximation method in [3], [4].)

Further we proved partial regularity of weak solutions to a class of nondivergence type quasilinear systems

$$u_t - A(x, t, u, \nabla u)\nabla^2 u = 0, \qquad z \in Q, \tag{9}$$

under relaxed assumptions on the matrix $A(x, t, u, p)$ in [6].

We also proved partial regularity of weak solutions to a class of $2m$-order quasilinear parabolic systems under relaxed smoothness conditions on the principal matrix in [7].

To relax known regularity assumptions on the data, we applied in our works the so-called "A(t)-caloric approximation" method. This approach is a modification of the A-caloric approximation method suggested and successfully applied by F. Duzaar and G. Mingione in [18] (see also [9]) to study regularity to a wide class of nonlinear parabolic systems:

$$u_t - \operatorname{div} a(z, u, \nabla u) = 0, \qquad z = (x, t) \in Q. \tag{10}$$

We denote by A an elliptic constant $[nN \times nN]$-matrix and by $A(t)$ an elliptic matrix with bounded measurable entries depending on t .

Using properties of the fractional Sobolev spaces, the authors of [18] proved by A-caloric approximation method new results on the partial regularity and obtained estimates of the singular sets of solutions. Note that in the elliptic setting the possibility of using the correspondent "A-harmonic approximation method" was exploited in [17] and [19] (for the origin of the method see [18]). Another approach to study partial regularity for elliptic problems one can find in [12], [13].

We do not consider here systems (10) where functions $a(z, u, p)$ depend on the argument u explicitly. As it is known, to study systems (10) we need additional considerations and we are able to obtain only more rough estimate of the singular sets.

In [18] the class of systems (1) was studied separately. In this case the authors assumed that the functions $a(x, t, p)$ satisfy the Hölder continuity condition in the variables $z = (x, t)$ with an exponent $\beta \in (0, 1)$ (in the parabolic metric δ). We recall that the parabolic metric δ is defined as follows

$$\delta(z^1; z^2) = \max\{|x^1 - x^2|, |t^1 - t^2|^{1/2}\}, \qquad z^1 = (x^1, t^1), \quad z^2 = (x^2, t^2) \in \mathbb{R}^{n+1}.$$

In particular it means that the functions $a(x,t,p)$ are assumed to be Hölder continuous in t with the exponent $\beta/2$ in [18].

Moreover, it was supposed in that paper that the entries of the matrix $\frac{\partial a(z,p)}{\partial p}$ were continuous functions in z. Under natural assumptions on the behavior of a and $\frac{\partial a(z,p)}{\partial p}$ in the argument p it was proved that

$$\dim_P \Sigma := \inf\{\lambda > 0\colon \mathcal{H}_\lambda(\Sigma;\delta) = 0\} \le n + 2 - 2\beta - \varepsilon_0 \tag{11}$$

for the closed singular set Σ of a weak solution $u \in V(Q)$ to system (1) where $\mathcal{H}_\lambda(\Sigma;\delta)$ is the canonical Hausdorff measure of the set Σ constructed in $\mathbb{R}^{n+1}$ with respect to the parabolic metric δ. Here $\varepsilon_0 > 0$ is a number depending on the data of the problem. On the open set $Q_0 = Q \setminus \Sigma$ the gradient ∇u was proved to be the Hölder continuous function with the exponent β in the parabolic metric.

To study dependence between smoothness assumptions on the data of elliptic systems and estimates of the singular sets of weak solutions, G. Mingione used the properties of fractional Sobolev spaces (in particular, Poincaré type inequalities) in [27]. The approach was later modified by F. Duzaar, G. Mingione in [18] to improve known before estimates of the singular sets for parabolic systems. We exploited this idea in our paper.

Here we intend to relax known assumptions on the main data when we study regularity problem for systems (1) and assume in this paper only *boundedness* in t of the functions $a(x,t,p)$ and the matrix $\frac{\partial a(x,t,p)}{\partial p}$. We prove under such relaxation that

$$\dim_P \Sigma \le n + 2 - 2\beta, \qquad \beta \in (0,1). \tag{12}$$

We can estimate the singular set better provided that we assume that there exist derivatives $a'_x(x,t,p)$, i.e. $\beta = 1$ in our assumptions. In this case we prove that

$$\dim_P \Sigma \le n - \chi \tag{13}$$

where a number $\chi > 0$ is defined by the data.

The case $\beta \in (0,1)$ is considered in Theorem 2.1, and Theorem 2.2 is dedicated to the situation $\beta = 1$. The estimates (12) and (13) are justified in Theorem 2.3.

Under similar smoothness conditions in x and t for the principal matrix, the regularity question for a wide class of nonlinear *scalar* equations and $2m$-order parabolic *linear* systems ($m \ge 1$) was studied in a series of the works by N. V. Krylov, H. Dong, D. Kim (see [24], [25], [14], [15] and references therein). In these works the principal coefficients of the studied systems were also assumed bounded and measurable in t and VMO-smooth in the space variables. Our results concern nonlinear parabolic systems.

The paper is organized as follows: in Section 2 we list notation and main results; Section 3 contains auxiliary results, Section 4 is dedicated to properties of A(t)-caloric functions and we formulate the main A(t)-caloric lemma. In Section 5 we prove Theorem 2.1 and in Section 6 we prove Theorem 2.2. In the last Section 7 we prove Theorem 2.3.

2. Notation and main results

We assume that Ω is an open bounded domain in $\mathbb{R}^n$ and T is a positive number. We will use the following notation:

$$z=(x,t),\qquad z^0=(x^0,t^0)\in\Omega\times\Lambda=Q\subset\mathbb{R}^{n+1},\qquad \Lambda=(-T,0),$$
$$\Gamma=\partial\Omega\times\Lambda,\qquad \Lambda_r(t^0)=(t^0-r^2,t^0),\qquad B_r(x^0)=\{x\in\mathbb{R}^n\colon |x-x^0|<r\},$$
$$Q_r(z^0)=B_r(x^0)\times\Lambda_r(t^0),\qquad \Gamma_r(z^0)=\partial B_r(x^0)\times\Lambda_r(t^0),$$
$$\partial_p Q_r(z^0)=\Gamma_r(z^0)\cup(\overline{B_r(x^0)}\times\{t^0-r^2\}).$$

The Campanato space $\mathcal{L}^{2,\lambda}(Q;\delta)$ with $\lambda\in[0,n+4]$ is the set of all functions from $L^2(Q)$ with the finite seminorm

$$[u]_{\mathcal{L}^{2,\lambda}(Q;\delta)}=\left(\sup_{z^0\in Q,\,r\le d_0}\frac{1}{r^\lambda}\int_{Q_r(z^0)\cap Q}|u(z)-(u)_{r,z^0}|^2\,\mathrm{d}z\right)^{1/2}$$

where $d_0=\max_{z^1,z^2\in Q}\delta(z^1;z^2)$.

We recall that the space $\mathcal{L}^{2,n+2+2\alpha}(Q;\delta)$ is isomorphic to the Hölder space $C^{0,\alpha}(\overline{Q};\delta)$ for $\alpha\in(0,1]$, see [10].

Throughout the paper we use the standard notation for the Lebesgue and Sobolev spaces and we write $\|v\|_{p,\Omega}$ instead of $\|v\|_{L^p(\Omega)}$, $p\ge 1$.

Further we use the Hölder spaces $C^{0,\alpha}(Q)$ and Campanato spaces $\mathcal{L}^{2,\lambda}(Q)$ with respect to the *parabolic metric* δ.

Thus, for example, $C^{0,\alpha}(Q)=C^{\alpha,\alpha/2}_{x,t}(Q)$ in the euclidian metric in $\mathbb{R}^{n+1}$.

Next we denote the spaces

$$V(Q)=L^2(\Lambda;W^1_2(\Omega)),\qquad \overset{o}{W}{}^1_2(Q)=[\overline{C^\infty_0(Q)}]_{W^1_2(Q)},$$
$$V(Q_r(z^0))=L^2(\Lambda_r(t^0);W^1_2(B_r(x^0)))$$

for z^0,r such that $Q_r(z^0)\subset\subset Q$.

The space averages and the space-time averages of $u\in L^1(Q_r(z^0))$ are defined by

$$(u)_{r,x^0}(t)=\frac{1}{|B_r(x^0)|}\int_{B_r(x^0)}u(y,t)\,\mathrm{d}y;$$
$$(u)_{r,z^0}=\frac{1}{|Q_r(z^0)|}\int_{Q_r(z^0)}u(z)\,\mathrm{d}z=\fint_{Q_r(z^0)}u(z)\,\mathrm{d}z.$$

Space averages of functions $a(z,p)$ are defined by

$$(a)_{r,x^0}(t;p) = \frac{1}{|B_r(x^0)|}\int_{B_r(x^0)} a(y,t,p)\,\mathrm{d}y$$
$$= \fint_{B_r(x^0)} a(y,t,p)\,\mathrm{d}y, \qquad t\in\Lambda_r(t^0),\quad p\in\mathbb{R}^{nN}.$$

Here $|B_r|$ and $|Q_r|$ stand for the Lebesgue measures of B_r and Q_r in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively.

We often use the minimizing property of the averages, namely

$$(14)\qquad \int_{Q_r(z^0)} |u(z)-(u)_{r,z^0}|^2\,\mathrm{d}z \le \int_{Q_r(z^0)} |u(z)-c|^2\,\mathrm{d}z, \qquad \forall\, c\in\mathbb{R}^N,$$

which is a consequence of the fact that the function $\Phi(c)=\int_{Q_r(z^0)}|u(z)-c|^2\,\mathrm{d}z$ attains its minimum for $c=(u)_{r,z^0}$.

We write $v\in H^{1/2}(\Lambda;L^2(\Omega))$ provided that $v\in L^2(Q)$ and the following seminorm is finite:

$$[v]_{H^{1/2}(\Lambda;L^2(\Omega))} = \left(\int_\Lambda\int_\Lambda \frac{\|v(\cdot,t+h)-v(\cdot,t)\|^2_{L^2(\Omega)}}{|h|^2}\,\mathrm{d}t\,\mathrm{d}h\right)^{1/2} < \infty.$$

We also recall the definition of the parabolic fractional Sobolev spaces $W_2^{\alpha,\gamma}(Q)$, $Q=\Omega\times\Lambda$, $\alpha,\gamma\in(0,1)$. A function v from $L^2(Q)$ belongs to the space $W_2^{\alpha,\gamma}(Q)$ if

$$[v]^2_{W_2^{\alpha,\gamma}(Q)} := \int_\Lambda\int_\Omega\int_\Omega \frac{|v(x,t)-v(y,t)|^2}{|x-y|^{n+2\alpha}}\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_\Omega\int_\Lambda\int_\Lambda \frac{|v(x,t)-v(x,s)|^2}{|t-s|^{1+2\gamma}}\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}x < \infty.$$

We write $A(z,p)\in\{\nu,\mu\}$ if the matrix $A(z,p)$ satisfies the ellipticity condition [H3] with the parameters $0<\nu\le\mu$ for almost all $z\in Q$ and all $p\in\mathbb{R}^{nN}$.

To save the space, we write $v\in\mathbb{B}(\cdot)$ instead of $v\in\mathbb{B}(\cdot\,;\mathbb{R}^N)$ for a functional space $\mathbb{B}$ of N-vector functions.

In what follows we will use the notation Q_r, V_r, $(u)_r$ without denoting center of the ball or the cylinder if it does not cause misunderstandings.

In order to concentrate our attention on the properties of the principal functions $a(z,p)$, we omit additional nonlinear terms of the lower order. Certainly, we can also add in the right-hand side of (1) functions from appropriate Campanato spaces.

Next we formulate the main results of the paper.

Theorem 2.1. *Let the assumptions* [H1]–[H5] *hold with* $\beta\in(0,1)$ *and* $u\in V(Q)$ *be a weak solution to* (1). *Then for any number* $M\ge 1$ *there exist numbers*

$\tau, \theta \in (0,1)$ and $R_0 > 0$ such that if $Q_r(z^0) \subset Q$ with some $r < R_0$ and

$$(15) \qquad \fint_{Q_r(z^0)} |\nabla u(z)|^2 \,\mathrm{d}z < M, \qquad \fint_{Q_r(z^0)} |\nabla u(z) - (\nabla u)_{r,z^0}|^2 \,\mathrm{d}z < \theta,$$

then u belongs to $C^{0,1}(\overline{Q_{\tau r}(z^0)}; \delta)$, and $\nabla u \in C^{0,\beta}(\overline{Q_{\tau r}(z^0)}; \delta)$ where $\beta \in (0,1)$ is fixed in the assumption [H2]. *The correspondent norms of u and ∇u can be estimated by the data of the problem, $\|u\|_{V(Q)}$, M, and r^{-1}.*

We also consider the case when $\beta = 1$ in the assumption [H2]. More exactly, we assume that there exist derivatives $a'_x(x,t,p) = \frac{\partial a}{\partial x}$ and
[H2']

$$(16) \qquad |a'_x(x,t,p)| \le L(1 + |p|)$$

holds.

In this situation we can slightly change some steps of the proof of Theorem 2.1 and we formulate it as a special result.

Theorem 2.2. *Let the assumptions* [H1], [H3]–[H5], *and* [H2'] *hold and u be a weak solution to system* (1). *Then there exist numbers $\tau, \theta \in (0,1)$ and $R_0 > 0$ such that if*

$$(17) \qquad \fint_{Q_r(z^0)} |\nabla u(z) - (\nabla u)_{r,z^0}|^2 \,\mathrm{d}z < \theta, \qquad r^2 \fint_{Q_r(z^0)} |\nabla u(z)|^2 \,\mathrm{d}z < \theta$$

in some cylinder $Q_r(z^0) \subset Q$, $r \le R_0$, then $u \in C^{0,1}(\overline{Q_{\tau r}(z^0)}; \delta)$, and $\nabla u \in C^{\alpha}(\overline{Q_{\tau r}(z^0)}; \delta)$ with any $\alpha \in (0,1)$, and the corresponding norms are estimated by constants depending on $\nu, \mu, L, r^{-1}, \alpha$, and $\|u\|_{V(Q)}$.

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *hold and $u \in V(Q)$ be a weak solution to system* (1). *Then u is the $C^{0,1}$-function and its spatial derivatives $u_{x_1}, \ldots, u_{x_n}$ are Hölder continuous functions with the exponent $\beta \in (0,1)$ (in the parabolic metric) on an open set $Q_0 \subset Q$, $Q_0 = Q \setminus \Sigma$ where Σ is the closed singular set of u and*

$$(18) \qquad \dim_P \Sigma \le n + 2 - 2\beta.$$

If the conditions of Theorem 2.2 *hold then $u \in C^{0,1}(Q_0; \delta)$ and $\nabla u \in C^{\alpha}(Q_0; \delta)$ with any $\alpha \in (0,1)$ where Q_0 is an open set in Q and*

$$(19) \qquad \dim_P(\Sigma) \le n - \chi_0$$

with some $\chi_0 = \chi_0(\mu/\nu, n) > 0$ for the closed singular set $\Sigma = Q \setminus Q_0$.

3. Auxiliary results

In this section we recall several results needed further.

Lemma 3.1. *Let $w \in V(Q)$ be a weak solution to the system*

$$w_t(z) - \operatorname{div}(\boldsymbol{A}(z)\nabla w(z)) = F_1(z) - \operatorname{div} F_2(z), \tag{20}$$

where $[nN \times nN]$-matrix $\boldsymbol{A}(z) \in \{\nu, \mu\}$ for almost all $z \in Q$, F_1, $F_2 \in L^2(Q)$. Then $w \in H^{1/2}_{\mathrm{loc}}(\Lambda; L^2(\Omega))$ and the following estimates hold in any cylinder $Q_{2R}(z^0) \subset Q$:

$$\begin{aligned} \|w-k\|^2_{V(Q_R(z^0))} &:= \sup_{\Lambda_R(t^0)} \int_{B_R(x^0)} |w(x,t)-k|^2 \,\mathrm{d}x + \int_{Q_R(z^0)} |\nabla w(z)|^2 \,\mathrm{d}z \\ &\le \frac{c}{R^2} \int_{Q_{2R}(z^0)} |w(z)-k|^2 \,\mathrm{d}z \\ &\quad + c\int_{Q_{2R}(z^0)} (R^2|F_1(z)|^2 + |F_2(z)|^2)\,\mathrm{d}z, \qquad \forall\, k \in \mathbb{R}^N; \end{aligned} \tag{21}$$

$$\begin{aligned} \int_{Q_R(z^0)} |w(z) - (w)_{R,z^0}|^2 \,\mathrm{d}z \le cR^2 &\int_{Q_{2R}(z^0)} |\nabla w|^2 \,\mathrm{d}z \\ &+ c\int_{Q_{2R}(z^0)} (R^4|F_1|^2 + R^2|F_2|^2)\,\mathrm{d}z; \end{aligned} \tag{22}$$

$$\begin{aligned} [w]^2_{H^{1/2}(\Lambda_R(t^0);B_R(x^0))} \le c\{&\|\nabla w\|^2_{2,Q_{2R}(z^0)} + R^{-2}\|w\|^2_{2,Q_{2R}(z^0)} \\ &+ R^2\|F_1\|^2_{2,Q_{2R}(z^0)} + \|F_2\|^2_{2,Q_{2R}(z^0)}\}. \end{aligned} \tag{23}$$

Moreover, $w \in L^{2(n+2)/n}(Q)$ and

$$\|w\|^2_{2(n+2)/n,Q_R} \le c(n)\big\{\sup_{\Lambda_R} \|w(\cdot,t)\|^2_{2,B_R} + \|\nabla w\|^2_{2,Q_R}\big\}. \tag{24}$$

The constants in inequalities (21)–(23) *depend on μ/ν and n only.*

Inequalities (21) and (22) are the well known Caccioppoli and Poincaré inequalities. They can be derived from identity (7) applying in advance the Steklov average procedure (see, for example [11], [21], [22]). Proofs of (23) and (24) for systems and scalar parabolic equations are the same, one can find them in [26], Chapter 2 and 3.

Lemma 3.2. *Let the matrix $\mathbf{A}(z) \in \{\nu, \mu\}$ for almost all $z \in Q$, $F_1 \in L^{(2(n+2)/(n+4))+\varepsilon}(Q)$, $F_2 \in L^{2+\varepsilon}(Q)$ with an $\varepsilon > 0$ and $w \in V(Q)$ be a weak solution to system* (20) *then $\nabla w \in L^{q_o}_{\mathrm{loc}}(Q)$ for some $q_o \in (2, 2+\varepsilon)$, and the* *inequality holds*

$$\left(\fint_{Q_r} |\nabla w|^{q_0}\,\mathrm{d}z\right)^{2/q_0} \le c\fint_{Q_{2r}} |\nabla w|^2\,\mathrm{d}z + c\left(\fint_{Q_{2r}} |F_2|^{q_o}\,\mathrm{d}z\right)^{2/q_0} + c\left(\fint_{Q_{2r}} |F_1|^{s}\,\mathrm{d}z\right)^{2/s}, \tag{25}$$

for any $Q_{2r} \subset Q$ *and* $s = 2(n+2)/(n+4) + (q_o - 2)$. *The constant* c *in* (25) *depends on* μ/ν *and* n.

This lemma is a local variant of the Gehring lemma on the reverse Hölder inequalities [20] proved by M. Giaquinta and G. Modica. In the parabolic metric the lemma was applied first time in [22]. The result was extended for the case of additional functions from the anisotropic spaces and integrals over manifolds of the lower dimensions [1], [2]. The extension of the result for the case of p-parabolic metric, $p \neq 2$, see [23].

Lemmas 3.1 and 3.2 guarantee few useful results for nonlinear systems (1). Indeed, we can use the condition [H3] to obtain the equality

$$\begin{aligned} &a(z,p) = a(z,0) + \mathbb{A}(z,p)p,\\ &\mathbb{A}(z,p) = \int_0^1 \frac{\partial a}{\partial p}(z, s\,p)\,\mathrm{d}s \in \{\nu,\mu\} \qquad \text{a.a. } z \in Q, \quad \forall\, p \in \mathbb{R}^{nN}. \end{aligned} \tag{26}$$

We consider a weak solution $u \in V(Q)$ of system (1) as a weak solution to system

$$u_t - \operatorname{div}(\mathbb{A}(z,\nabla u)\nabla u) = \operatorname{div} a(z,0). \tag{27}$$

More exactly, a weak solution $u \in V(Q)$ to system (27) (or to system (1)) satisfies the identity

$$\int_Q [-u\cdot\varphi_t + \mathbb{A}(z,\nabla u)\nabla u \cdot \nabla\varphi]\,\mathrm{d}z = -\int_Q a(z,0)\cdot\nabla\varphi\,\mathrm{d}z, \qquad \varphi \in \overset{o}{W}{}_2^1(Q). \tag{28}$$

We put $u = w$, $\mathbb{A}(z,\nabla u(z)) = \mathbf{A}(z)$, $-a(z,0) = F_2(z)$, $|F_2(z)| \le L$, $F_1 = 0$, and consider (27) as the linear system (20). Applying Lemma 3.1 and Lemma 3.2, we obtain that the following assertion is valid.

Lemma 3.3. *Let the assumptions* [H1] *and* [H3] *hold,* $u \in V(Q)$ *be a weak solution to system* (1). *Then* $u \in H^{1/2}_{\mathrm{loc}}(\Lambda; L^2(\Omega))$ *and the following inequalities are valid in any* $Q_{2R}(z^0) \subset Q$:

$$\begin{aligned} \|u-k\|^2_{V(Q_R(z^0))} &= \sup_{\Lambda_R(t^0)} \|u-k\|^2_{2,B_R(x^0)} + \fint_{Q_R(z^0)} |\nabla u|^2\,\mathrm{d}z\\ &\le \frac{c}{R^2}\fint_{Q_{2R}(z^0)} |u(z)-k|^2\,\mathrm{d}z + cL^2, \qquad \forall\, k \in \mathbb{R}^N, \end{aligned} \tag{29}$$

$$\fint_{Q_R(z^0)} |u(z)-(u)_{R,z^0}|^2\,\mathrm{d}z \le cR^2 \fint_{Q_{2R}(z^0)} (|\nabla u|^2+L^2)\,\mathrm{d}z. \tag{30}$$

$$[u]^2_{H^{1/2}(\Lambda_R;\,L^2(B_R))} \le c\{\|\nabla\,u\|^2_{2,Q_{2R}}+L^2+R^{-2}\|u-k\|^2_{2,Q_{2R}}\}, \qquad \forall\,k\in\mathbb{R}^N. \tag{31}$$

Moreover, there exists number $q_o>2$ such that $\nabla u\in L^q_{\mathrm{loc}}(Q)$ for $q\in[2,q_o)$ and the inequality

$$\left(\fint_{Q_R(z^0)} |\nabla u|^q\,\mathrm{d}z\right)^{2/q} \le c\fint_{Q_{2R}(z^0)} (|\nabla u|^2+L^2)\,\mathrm{d}z, \qquad Q_{2R}(z^0)\subset Q, \tag{32}$$

holds. The constants c in (29)–(32) depends on μ/ν and n only.

Lemma 3.4. 1) *Let a function $v\in H^{1/2}(\Lambda;L^2(\Omega))$. Then for a fixed $h>0$*

$$\int_{-T+h}^{-h} \|v(\cdot\,,t+\tau)-v(\cdot\,,t)\|^2_{2,\Omega}\mathrm{d}t \le 8|\tau|\,[v]^2_{H^{1/2}(\Lambda;L^2(\Omega))}, \qquad |\tau|<\frac{h}{4}. \tag{33}$$

2) *If $v\in W_2^{\alpha,\alpha/2}(Q_R(z^0))$ then*

$$\|v-(v)_{R,z^0}\|^2_{2,Q_R} \le c\,R^{2\alpha}[v]^2_{W_2^{\alpha,\alpha/2}(Q_R(z^0))} \tag{34}$$

with the absolute constant $c>0$.

Proof: 1) Let a function $v\in H^{1/2}(\Lambda;\,L^2(\Omega))$ and $h>0$ be a fixed number. We put $\lambda_\tau=(-\tau,\tau)$ and estimate the expression $J_h(\tau)=\int_{-T+h}^{-h}\|v(\cdot\,,t+\tau)-v(\cdot\,,t)\|^2_{2,\Omega}\,\mathrm{d}t$ for $|\tau|<h/4$. We start with the case $\tau>0$. Then

$$\begin{aligned}
J_h(\tau) &= \fint_{\lambda_\tau} \mathrm{d}\xi\, J_h(\tau)\\
&= \fint_{\lambda_\tau}\int_{-T+h}^{-h} \|[v(\cdot\,,t+\tau)-v(\cdot\,,t+\xi)]+[v(\cdot,t+\xi)-v(\cdot\,,t)]\|^2_{2,\Omega}\,\mathrm{d}t\,\mathrm{d}\xi\\
&\le 4\tau\int_{\lambda_\tau}\int_{-T+h}^{-h} \frac{\|v(\cdot\,,t+\tau)-v(\cdot\,,t+\xi)\|^2_{2,\Omega}}{|\tau-\xi|^2}\,\mathrm{d}t\,\mathrm{d}\xi\\
&\quad + 4\tau\int_{\lambda_\tau}\int_{-T+h}^{-h} \frac{\|v(\cdot\,,t+\xi)-v(\cdot\,,t)\|^2_{2,\Omega}}{|\xi|^2}\,\mathrm{d}t\,\mathrm{d}\xi =: j_1+j_2.
\end{aligned}$$

For a fixed τ we put $s=t+\tau$ and $\xi'=\xi+t$ in j_1. Then $\mathrm{d}s=\mathrm{d}t$ and $\mathrm{d}\xi'=\mathrm{d}\xi$. Now it follows that

$$j_1 \le 4\tau\int_\Lambda\int_\Lambda \frac{\|v(\cdot\,,s)-v(\cdot\,,\xi')\|^2_{2,\Omega}}{|s-\xi'|^2}\mathrm{d}s\,\mathrm{d}\xi' = 4\tau[v]^2_{H^{1/2}(\Lambda;\,L^2(\Omega))}.$$

Further, we put $\xi + t = s$ in j_2 then

$$j_2 \le 4\tau \int_\Lambda \int_\Lambda \frac{\|v(\cdot\,,s) - v(\cdot\,,t)\|_{2,\Omega}^2}{|s-t|^2}\,\mathrm{d}t\,\mathrm{d}s \le 4\tau[v]^2_{H^{1/2}(\Lambda;L^2(\Omega))}.$$

Estimate (33) follows with $\tau > 0$. The case $\tau < 0$ can be proved by the same way.

To justify estimate (34) we put $\eta = (y,\tau)$, $z = (x,t)$ and write the inequalities

$$\begin{aligned}\|v-(v)_R\|^2_{2,Q_R} &= \int_{Q_R} \left| v(z) - \fint_{Q_R} v(\eta)\,\mathrm{d}\eta \right|^2 \mathrm{d}z \\ &\le 2\int_{Q_R} \fint_{Q_R} (|v(x,t)-v(y,t)|^2 + |v(y,t)-v(\eta)|^2)\,\mathrm{d}\eta\,\mathrm{d}z \\ &\le c(n)R^{2\alpha}[v]^2_{W_2^{\alpha,\alpha/2}(Q_R(z^0))}.\end{aligned}$$

To obtain the last inequality we have used the definition of $W_2^{\alpha,\alpha/2}(Q_R(z^0))$. $\square$

Now we recall some results on the *A(t)-caloric functions* from [8].

We fix two positive numbers $0 < \nu \le \mu$ and consider $[nN \times nN]$ matrices $A(t)$ with the entries from $L^\infty(\Lambda_R(t^0))$, $\Lambda_R(t^0) \subset \Lambda$, and $A(t) \in \{\nu,\mu\}$ for almost all $t \in \Lambda_R(t^0)$.

Definition 3.1. We say that a weak solution $h \in V(Q_R(z^0))$ to the system

$$h_t - \operatorname{div}(A(t)\nabla h) = 0, \qquad z = (x,t) \in Q_R(z^0), \tag{35}$$

is an $A(t)$-*caloric function in* $Q_R(z^0)$.

Obviously, any $A(t)$-caloric function satisfies Caccioppoli and Poincaré inequalities (21) and (22) with $F_1 = F_2 = 0$.

Moreover, weak solutions h of system (35) have an additional smoothness in any $Q_\varrho(z^0)$ for $\varrho < r$. The spatial derivatives $D^\alpha h$, $|\alpha| < \infty$, are continuous functions and $(D^\alpha h)_t$ are bounded in $Q_\varrho(z^0)$, $\varrho < r$, see [8].

For $A(t)$-caloric functions the following Campanato type integral estimates hold.

Lemma 3.5 (Lemma 5 in [8]). *Let $h \in V(Q_r)$ be an $A(t)$-caloric function in Q_r then*

$$\fint_{Q_\varrho(z^0)} |h(z) - h_{\varrho,z^0}|^2\,\mathrm{d}z \le c\Big(\frac{\varrho}{r}\Big)^2 \fint_{Q_r(z^0)} |h(z) - h_{r,z^0}|^2\,\mathrm{d}z, \qquad \varrho < r; \tag{36}$$

$$\fint_{Q_\varrho(z^0)} |\nabla h(z)|^2\,\mathrm{d}z \le c \fint_{Q_r(z^0)} |\nabla h(z)|^2\,\mathrm{d}z, \qquad \varrho < r; \tag{37}$$

$$\begin{aligned}(38)\qquad &\fint_{Q_\varrho(z^0)}|h(z)-(h)_{\varrho,z^0}-(\nabla h)_{\varrho,z^0}(x-x^0)|^2\,\mathrm{d}z\\ &\quad\le c\Big(\frac{\varrho}{r}\Big)^4\fint_{Q_r(z^0)}|h(z)-(h)_{r,z^0}-(\nabla h)_{r,z^0}(x-x^0)|^2\,\mathrm{d}z,\qquad \varrho<r.\end{aligned}$$

In inequalities (36), (37), (38) *the constants* $c>0$ *depend on* $\nu,\mu,n,$ *and* N *only.*

Lemma 3.6 ($A(t)$-caloric approximation lemma, Lemma 7 in [8])**.** *Let* μ,ν *be positive numbers,* $\nu\le\mu$. *Then for any* $\varepsilon>0$ *there exists a constant* $C_\varepsilon=C(\varepsilon,\nu,\mu,n,N)>0$ *such that whenever a matrix* $A(t)$ *with entries in* $L^\infty(\Lambda_r)$ *satisfies the condition* $A(t)\in\{\nu,\mu\}$ *for almost all* $t\in\Lambda_r$ *then for any* $u\in V(Q_r)$ *there exist an* $A(t)$*-caloric function* $h\in V(Q_{r/2})$*, and a function* $\varphi\in C_0^1(Q_r)$, $\sup_{Q_r}|\nabla\varphi|\le 1$, *such that*

$$(39)\qquad \fint_{Q_{r/2}}(|h(z)-(h)_{r/2}|^2+r^2|\nabla h(z)|^2)\,\mathrm{d}z \le 2^{n+4}\fint_{Q_r}(|u(z)-(u)_r|^2+r^2|\nabla u(z)|^2)\,\mathrm{d}z,$$

$$(40)\qquad \fint_{Q_{r/2}}|u(z)-h(z)|^2\,\mathrm{d}z\le\varepsilon\fint_{Q_r}(|u(z)-(u)_r|^2+r^2|\nabla u(z)|^2)\,\mathrm{d}z + C_\varepsilon r^2\mathcal{L}_r^2(u,\varphi)$$

where

$$(41)\qquad \mathcal{L}_r^2(u,\varphi)=\Big|\fint_{Q_r}[-u\cdot\varphi_t+A(t)\nabla u\cdot\nabla\varphi]\,\mathrm{d}z\Big|^2.$$

4. Proof of Theorem 2.1

Let $u\in V(Q)$ be a weak solution to system (1). We put

$$v(z)=u(z)-k-l\,(x-x^0)$$

for any $k\in\mathbb{R}^N$, $l\in\mathbb{R}^{nN}$. The function $v\in V(Q)$ is a weak solution to system

$$(42)\qquad v_t-\operatorname{div}a(x,t,\nabla v(z)+l)=0.$$

It satisfies the identity

$$(43)\qquad \int_Q[-v(z)\cdot\varphi_t(z)+a(z,\nabla v(z)+l)\cdot\nabla\varphi(z)]\,\mathrm{d}z=0,\qquad\forall\,\varphi\in\overset{o}{W}{}_2^1(Q).$$

Taking into account that

$$\begin{aligned}&a(z,\nabla v(z)+l)-a(z,l)+a(z,l)\\&\qquad=\int_0^1\frac{\partial a}{\partial p}(z,s\nabla v(z)+l)\,\mathrm{d}s\nabla v(z)+[a(x,t,l)-a(x^0,t,l)]+a(x^0,t,l),\end{aligned}$$

we can write identity (43) in the form

$$\int_Q[-v\cdot\varphi_t+\mathbb{A}(z;\nabla v,l)\nabla v\cdot\nabla\varphi]\,\mathrm{d}z=-\int_Q\Delta' a\cdot\nabla\varphi\,\mathrm{d}z, \tag{44}$$

where

$$\mathbb{A}(z;\nabla v,l)=\int_0^1\frac{\partial a}{\partial p}(z,s\nabla v(z)+l)\,\mathrm{d}s\in\{\nu,\mu\}\qquad\text{a.a. } z\in Q,$$

$$\int_Q a(x^0,t,l)\cdot\nabla\varphi(z)\,\mathrm{d}z=0,$$

and

$$\Delta' a(x,t,l)=a(x,t,l)-a(x^0,t,l),\qquad|\Delta' a(x,t,l)|\le L|x-x^0|^\beta(|l|+1).$$

By Lemma 3.1, $v\in H^{1/2}_{\mathrm{loc}}(\Lambda;L^2(\Omega))$ and the Caccioppoli and Poincaré inequalities are valid. It follows that

$$⨍_{Q_r}|v(z)-(v)_r|^2\,\mathrm{d}z\le cr^2⨍_{Q_{2r}}(|\nabla v(z)|^2+L^2r^{2\beta}(|l|^2+1))\,\mathrm{d}z, \tag{45}$$

$$\begin{aligned}⨍_{Q_r(z^0)}|\nabla v(z)|^2\,\mathrm{d}z\le\frac{c}{r^2}&⨍_{Q_{2r}(z^0)}|v(z)|^2\,\mathrm{d}z\\&+c\,L^2r^{2\beta}(|l|^2+1),\qquad c=c\Big(\frac{\mu}{\nu},n\Big),\end{aligned} \tag{46}$$

in any $Q_{2r}(z^0)\subset\subset Q$. As

$$⨍_{Q_r}|\nabla u-(\nabla u)_{r,z^0}|^2\,\mathrm{d}z\le⨍_{Q_r}|\nabla u-l|^2\,\mathrm{d}z,\qquad\forall\,l\in\mathbb{R}^{nN},$$

then using inequality (46) for $v(z)=u(z)-k-l(x-x^0))$, we obtain the following relation

$$\begin{aligned}⨍_{Q_r(z^0)}|\nabla u-(\nabla u)_{r,z^0}|^2\,\mathrm{d}z\le\frac{c}{r^2}&⨍_{Q_{2r}(z^0)}|u(z)-k-l(x-x^0)|^2\,\mathrm{d}z\\&+cL^2r^{2\beta}(|l|^2+1).\end{aligned} \tag{47}$$

For the fixed $Q_r(z^0)$ such that $Q_{2r}(z^0) \subset\subset Q$, we put

$$\Phi(\varrho, z^0) = \Phi(\varrho) = \fint_{Q_\varrho(z^0)} |\nabla u - (\nabla u)_{\varrho,z^0}|^2 \,\mathrm{d}z,$$

$$\Psi(\varrho, z^0) = \Psi(\varrho) = \fint_{Q_\varrho(z^0)} |\nabla u|^2 \,\mathrm{d}z,$$

and further we consider

$$v(z) = u(z) - (u)_{r,z^0} - (\nabla u)_{r,z^0}(x - x^0).$$

Thus, $v_t(z) = u_t(z)$, $\nabla v(z) = \nabla u(z) - (\nabla u)_{r,z^0}$.

Below we apply Lemma 3.6 to this function v. For a fixed $\varepsilon > 0$ and the matrix

$$A(t) = \fint_{B_r(x^0)} \frac{\partial a(x, t, (\nabla u)_{r,z^0})}{\partial p} \,\mathrm{d}x \in \{\nu, \mu\} \qquad \text{a.a. } t \in \Lambda_r(t^0) \tag{48}$$

there exist an $A(t)$-caloric function h from $V(Q_{r/2}(z^0))$, a constant C_ε, and a smooth function $\varphi \in C_0^1(Q_r(z^0))$, $\sup_{Q_r} |\nabla\varphi| \le 1$, such that inequalities (39) and (40) hold.

By (47) (with $\varrho/2$ instead of r), we have the inequality

$$\Phi\Big(\frac{\varrho}{2}\Big) \le \frac{c}{\varrho^2} \fint_{Q_\varrho} |u(z) - k - l(x - x^0)|^2 \,\mathrm{d}z + cL^2\varrho^{2\beta}(|l|^2 + 1), \qquad \varrho \le \frac{r}{2},$$

where we choose $k = (u)_{r,z^0} + (h)_{\varrho,z^0}$, $l = (\nabla u)_{r,z^0} + (\nabla h)_{\varrho,z^0}$. Then

$$\begin{aligned}\Phi\Big(\frac{\varrho}{2}\Big) &\le \frac{c}{\varrho^2} \fint_{Q_\varrho} |u - h + h - (u)_r - (h)_\varrho - (\nabla u)_r(x - x^0) - (\nabla h)_\varrho(x - x^0)|^2 \,\mathrm{d}z \\ &\quad + c\,L^2\varrho^{2\beta}(|(\nabla u)_r|^2 + |(\nabla h)_\varrho|^2 + 1) \\ &\le \frac{c}{\varrho^2}\Big\{ \fint_{Q_\varrho} |v - h|^2 \,\mathrm{d}z + \fint_{Q_\varrho} |h - (h)_\varrho - (\nabla h)_\varrho(x - x^0)|^2 \,\mathrm{d}z \Big\} \\ &\quad + cL^2\varrho^{2\beta}(|(\nabla u)_r|^2 + |(\nabla h)_\varrho|^2 + 1).\end{aligned}$$

Now we apply estimates (37), (38) to obtain the inequalities

$$\begin{aligned}\Phi\Big(\frac{\varrho}{2}\Big) &\le \frac{c}{\varrho^2}\Big(\frac{r}{\varrho}\Big)^{n+2} \fint_{Q_{r/2}} |v - h|^2 \,\mathrm{d}z \\ &\quad + \frac{c}{\varrho^2}\Big(\frac{\varrho}{r}\Big)^4 \fint_{Q_{r/2}} |h - (h)_{r/2} - (\nabla h)_{r/2}(x - x^0)|^2 \,\mathrm{d}z \\ &\quad + cL^2\varrho^{2\beta}(1 + |(\nabla u)_r|^2) + cL^2\varrho^{2\beta} \fint_{Q_{r/2}} |\nabla h|^2 \,\mathrm{d}z\end{aligned}$$

$$\leq \frac{c}{r^2}\Big(\frac{r}{\varrho}\Big)^{n+4} \fint_{Q_{r/2}} |v-h|^2 \,\mathrm{d}z$$
$$+ \frac{c}{\varrho^2}\Big(\frac{\varrho}{r}\Big)^4 \fint_{Q_{r/2}} (|h-(h)_{r/2}|^2 + r^2|\nabla h|^2)\,\mathrm{d}z$$
$$+ cL^2\varrho^{2\beta}(1+|(\nabla u)_r|^2) + cL^2\Big(\frac{\varrho}{r}\Big)^{2\beta} r^{2\beta} \fint_{Q_{r/2}} |\nabla h|^2\,\mathrm{d}z \leq (*).$$

We will estimate the inequality $(*)$ with the help of relations (39), (40) for v but not u in the way

$$(*) \leq cr^{-2}\Big(\frac{r}{\varrho}\Big)^{n+4}\{\varepsilon M_r + C_\varepsilon r^2\mathcal{L}_r^2(v,\varphi)\} + c\Big(\frac{\varrho}{r}\Big)^2 r^{-2}M_r$$
$$+ cL^2\varrho^{2\beta}(1+|(\nabla u)_r|^2) + c\Big(\frac{\varrho}{r}\Big)^{2\beta} r^{2\beta}r^{-2}M_r$$

where

$$M_r = \fint_{Q_r} (|v(z)-(v)_r|^2 + r^2|\nabla v|^2)\,\mathrm{d}z.$$

Using the Caccioppoli inequality (45) for the function v, $(v)_r = 0$, we obtain the estimate of M_r:

$$\begin{aligned} M_r &\leq cr^2\Big\{\fint_{Q_{2r}} |\nabla v|^2\,\mathrm{d}z + r^{2\beta}(|(\nabla u)_{2r}|^2+1)\Big\} \\ &\leq c\,r^2\{\Phi(2r) + r^{2\beta}(|(\nabla u)_{2r}|^2+1)\}. \end{aligned} \tag{49}$$

Thus, we have

$$\begin{aligned} \Phi\Big(\frac{\varrho}{2}\Big) \leq c\Big\{&\Big(\frac{r}{\varrho}\Big)^{n+4}\varepsilon + \Big(\frac{\varrho}{r}\Big)^2 + \Big(\frac{\varrho}{r}\Big)^{2\beta} r^{2\beta})\Big\}\Phi(2r) \\ &+ C_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+4}\mathcal{L}_r^2(v,\varphi) \\ &+ cL^2\Big[\Big(\frac{r}{\varrho}\Big)^{n+4}\varepsilon + \Big(\frac{\varrho}{r}\Big)^{2\beta}\Big] r^{2\beta}(|(\nabla u)_{2r}|^2+1). \end{aligned} \tag{50}$$

The next step is to estimate the expression $\mathcal{L}_r^2(v,\varphi)$ defined in (41).

Taking into account the definition (48) of the matrix $A(t)$, we apply identity (44) for $v(z)$ with $l = (\nabla u)_r$ and obtain the relation

$$\mathcal{L}_r^2(v,\varphi) = \Big| \fint_{Q_r} \{-v\cdot\varphi_t + A(t)\nabla v\cdot\nabla\varphi + [\mathbb{A}(z;\nabla v,(\nabla u)_r)$$
$$- \mathbb{A}(z;\nabla v,(\nabla u)_r)]\nabla v\cdot\nabla\varphi\}\,\mathrm{d}z\Big|^2$$

$$
\begin{aligned}
&= \Big| \fint_{Q_r} [(A(t) - \mathbb{A}(z;\nabla v, (\nabla u)_r))\nabla v \cdot \nabla\varphi - \Delta' a \cdot \nabla\varphi] \,\mathrm{d}z \Big|^2 \\
&\leq 2 \fint_{Q_r} |\Delta\,\mathbb{A}|^2 \,\mathrm{d}z \fint_{Q_r} |\nabla v|^2 \,\mathrm{d}z + 2L^2 r^{2\beta}(|(\nabla u)_r|^2 + 1)
\end{aligned}
$$

where

$$|\Delta\,\mathbb{A}| = |A(t) - \mathbb{A}(z, \nabla v, (\nabla u)_r)|.$$

We estimate $|\Delta\,\mathbb{A}|$ as follows

$$|\Delta\,\mathbb{A}| \leq |A(t) - \mathbb{A}(z; 0, (\nabla u)_r)| + |\mathbb{A}(z; 0, (\nabla u)_r) - \mathbb{A}(z; \nabla v, (\nabla u)_r))| =: j_1(z) + j_2(z)$$

where

$$\mathbb{A}(z; 0, (\nabla u)_r) = \frac{\partial a(z, (\nabla u)_r)}{\partial p}.$$

We remark that

$$j_1(z) = |\mathbb{A}(z; 0, (\nabla u)_r) - A(t)| = \Big| \frac{\partial a(z, (\nabla u)_r)}{\partial p} - \Big(\frac{\partial a(z, (\nabla u)_r)}{\partial p}\Big)_{r,x^0} \Big|. \tag{51}$$

Using the assumption [H4] we get the inequality

$$
\begin{aligned}
j_2(z) &= \Big| \int_0^1 \Big[\frac{\partial a(z, s\nabla u(z) + (1-s)(\nabla u)_r))}{\partial p} - \frac{\partial a(z, (\nabla u)_r)}{\partial p} \Big] \,\mathrm{d}s \Big| \\
&\leq \int_0^1 \omega(|s\nabla u(z) - s(\nabla u)_r)|^2) \,\mathrm{d}s \leq \omega(|\nabla v(z)|^2).
\end{aligned} \tag{52}
$$

With the help of relations (51), (52), and the assumptions [H4], [H5], we derive the inequality

(53)

$$
\begin{aligned}
\mathcal{L}_r^2(v,\varphi) &\leq 2L^2 r^{2\beta}(|(\nabla u)_r|^2 + 1) + 4 \fint_{Q_r} (j_1^2(z) + j_2^2(z)) \,\mathrm{d}z \fint_{Q_r} |\nabla v|^2 \,\mathrm{d}z \\
&\leq 4L^2 r^{2\beta}(|(\nabla u)_r|^2 + 1) + 4[q(r) + \omega_0\,\omega(\Phi(r))]\Phi(r), \qquad \omega_0 = \sup_{s\in[0,\infty)} \omega(s).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Phi\Big(\frac{\varrho}{2}\Big) \leq c\Big\{ &\Big(\frac{\varrho}{r}\Big)^2 + \varepsilon\Big(\frac{r}{\varrho}\Big)^{n+4} + \Big(\frac{\varrho}{r}\Big)^{2\beta} r^{2\beta} \\
&+ C_\varepsilon \Big(\frac{r}{\varrho}\Big)^{n+4} [\omega(\Phi(r)) + q(r)] \Big\} \Phi(2r) \\
&+ cL^2 \Big\{ \varepsilon\Big(\frac{r}{\varrho}\Big)^{n+4} + \Big(\frac{\varrho}{r}\Big)^{2\beta} + C_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+4} \Big\} r^{2\beta}(|l_{2r}|^2 + 1).
\end{aligned} \tag{54}
$$

Here and below $l_\varrho = (\nabla u)_\varrho$.

Now we fix

$$\gamma = \frac{1+\beta}{2}, \qquad R = 2r, \qquad \tau \leq \frac{1}{4}, \qquad \varrho = 2\tau R,$$

and note that $\beta < \gamma < 1$. We obtain from (54) that

$$\begin{aligned} \Phi(\tau R) \leq c_0\big\{&\tau^2 + \tau^{2\beta}R^{2\beta} + \varepsilon\tau^{-(n+4)} \\ &+ C_\varepsilon\tau^{-(n+4)}[\omega(2^{n+2}\Phi(R)) + q(R)]\big\}\Phi(R) \\ &+ c_1\big\{\tau^2 + \varepsilon\tau^{-(n+4)} + C_\varepsilon\tau^{-(n+4)}\big\}R^{2\beta}(|l_R|^2+1). \end{aligned} \tag{55}$$

We choose the parameter $\tau \leq 1/4$ such that

$$c_0\tau^2 \leq \frac{\tau^{2\gamma}}{8} \tag{56}$$

and then fix $\varepsilon > 0$ to obtain the relation

$$c_0\varepsilon\tau^{-(n+4)} \leq \frac{\tau^{2\gamma}}{8}. \tag{57}$$

The next step is to choose such small $\theta \in (0,1)$ that the inequality

$$c_0C_\varepsilon\tau^{-(n+4)}\omega(2^{n+2}\theta) < \frac{\tau^{2\gamma}}{8} \tag{58}$$

holds.

At last we fix $R_0 > 0$ to satisfy the condition

$$c_0(\tau^{2\beta}R_0^{2\beta} + C_\varepsilon\tau^{-(n+4)}q(R_0)) \leq \frac{\tau^{2\gamma}}{8}. \tag{59}$$

Let the assumptions (15) hold in the fixed point z^0 for some $M \geq 1$ with a radius $R < R_0$ and θ fixed by (58). It means that

$$\Phi(R) = \Phi(R, z^0) < \theta, \qquad \Psi(R) = \Psi(R, z^0) < M. \tag{60}$$

Evidently, $|l_R|^2 < M$ and $|l_R|^2 + 1 < 2M$ in this case.

We additionally assume that $\theta = \theta(M)$ and $R_0 = R_0(M)$ satisfy the restriction

$$\frac{\tau^{-(n+2)}\sqrt{\theta}}{1-\tau^\beta} < \sqrt{M}, \qquad \frac{K}{\tau^{2\beta}-\tau^{2\gamma}}R_0^{2\beta} < \frac{\theta}{4M} \tag{61}$$

where

$$K = 2c_1\{\tau^2 + \varepsilon\tau^{-(n+4)} + C_\varepsilon\tau^{-(n+4)}\}.$$

Now it follows from (55) that

$$\Phi(\tau\, R) \leq \frac{\tau^{2\gamma}}{2}\,\Phi(R) + 2\,K\,R^{2\beta}\,M. \tag{62}$$

In particular, by (60) and (61) it follows from (62) that

$$\Phi(\tau R) < \theta. \tag{63}$$

It means that the first assumption (60) is valid if we change R by τR. Then

$$\Phi(\tau^2 R) \le \frac{\tau^{2\gamma}}{2}\,\Phi(\tau R) + 2\,K(\tau R)^{2\beta} M. \tag{64}$$

Next we make the iterations in (64) for the sequence $\{\tau^j R\},\ j \in \mathbb{N}$, and obtain the inequalities

$$\begin{aligned} \Phi(\tau^j R) &\le \frac{\tau^{2\gamma}}{2}\Phi(\tau^{j-1}R) + 2KM(\tau^{j-1}R)^{2\beta} \\ &\le \Big(\frac{\tau^{2\gamma}}{2}\Big)^j \Phi(R) + \tau^{2\beta j}\,\frac{2KMR^{2\beta}}{\tau^{2\beta}(1-\tau^{2(\gamma-\beta)})}. \end{aligned} \tag{65}$$

Thanks to conditions (60) and (61), it follows from (65) that

$$\Phi(\tau^j R) \le \tau^{2\beta j}\Big[\frac{\Phi(R)}{2} + \frac{2KMR^{2\beta}}{\tau^{2\beta}-\tau^{2\gamma}}\Big] \le \tau^{2\beta j}\theta, \qquad j \in \mathbb{N}.$$

Now we assert that the sequence $\{l_{\tau^j r}\}$ has a finite limit when $j \to \infty$.
Let $j, m \in \mathbb{N}$, $j > m$, then

$$\begin{aligned} |l_{\tau^j R} - l_{\tau^m R}| &\le \sum_{k=m}^{j-1} |l_{\tau^{k+1}R} - l_{\tau^k R}| \le \tau^{-(n+2)}\Sigma_{k=m}^{j-1}\Phi^{1/2}(\tau^k R) \\ &\le \tau^{-(n+2)}\sum_{k=m}^{j-1}\tau^{\beta k}\sqrt{\theta} \le \frac{\tau^{-(n+2)}}{1-\tau^\beta}\sqrt{\theta}\,\tau^{\beta m} < \sqrt{M}\,\tau^{\beta m} \to 0 \end{aligned} \tag{66}$$

when $m \to \infty$.
Thus, there exists a finite limit $l_0 = \lim_{j\to\infty} l_{\tau^j R}$.
By (61),

$$|l_{\tau^j R} - l_R| \le \frac{\tau^{-(n+2)}}{1-\tau^\beta}\sqrt{\theta} < \sqrt{M}$$

and

$$|l_{\tau^j R}| \le |l_{\tau^j R} - l_R| + |l_R| < 2\sqrt{M}, \qquad \forall\, j \in \mathbb{N}.$$

For any $\varrho \in (0, R)$ there exists $\tau^j R$ such that $\tau^{j+1}R \le \varrho < \tau^j R$. It follows from (61) and (65) that

$$\Phi(\varrho, z^0) \le c\varrho^{2\beta}\{R^{-2\beta}\Phi(R, z^0) + M\}, \qquad \forall\, \varrho \le R. \tag{67}$$

Here the constant $c > 0$ depends on the parameters from the assumptions [H1]–[H5] but does not depend on z^0.

Moreover, there exists the limit of an arbitrary sequence l_{ϱ_m}, $\varrho_m \to 0$. Indeed, for any ϱ_m there exists $\tau^{j_m}R$ such that $\tau^{j_m+1}R \le \varrho_m < \tau^{j_m}R$ and

$$|l_{\varrho_m} - l_{\tau^{j_m}R}| \le \tau^{-(n+2)}\Phi^{1/2}(\tau^{j_m}R) \le \tau^{-(n+2)}\tau^{\beta j_m}\sqrt{\theta} \to 0, \qquad j_m \to \infty.$$

Then

$$|l_{\varrho_m} - l_0| \le |l_{\varrho_m} - l_{\tau^{j_m}R}| + |l_{\tau^{j_m}} - l_0| \ \to 0$$

provided that $\varrho_m \to 0$, $\tau^{j_m}R \to 0$.

We assumed that the assumptions (60) hold in a fixed point z^0. But it is easy to see that they are also valid in some neighborhood $Q_{\varrho_0}(z^0)$ (for the fixed earlier R), i.e.

$$\Phi(R,\xi) < \theta, \qquad \Psi(R,\xi) < M, \qquad \forall\, \xi \in Q_{\varrho_0}(z^0). \tag{68}$$

It means that inequality (67) holds for all $\xi \in Q_{\varrho_0}(z^0)$ but not only for z^0.

Thus

$$\sup_{\varrho \le R,\, \xi \in Q_{\varrho_0}(z^0)} \varrho^{-2\beta}\Phi(\varrho,\xi) \le c(R^{-1})\{\|\nabla u\|^2_{2,Q} + M\}. \tag{69}$$

This estimate guarantees us that the norm of the gradient of u in

$$\mathcal{L}^{2,n+2+2\beta}(Q_{\varrho_0}(z^0);\delta)$$

is bounded. Due to the isomorphism of this space to $C^\beta(\overline{Q_{\varrho_0}(z^0)};\delta)$, we can conclude Hölder continuity of the gradient of u in the vicinity of the point where conditions (60) hold.

Smoothness of u in $Q_{\varrho_0}(z^0)$ follows from the Poincaré inequality (30) and the second assumption (60). Thus, $u \in \mathcal{L}^{2,n+4}(Q_{\varrho_0}(z^0);\delta)$ and we can conclude that $u \in C^{0,1}(\overline{Q_{\varrho_0}(z^0)};\delta)$. □

5. Proof of Theorem 2.2

Lemma 5.1. *Let the conditions* [H1], [H2′], *and* [H3] *hold. Then*

1) *there exist the derivatives* $\nabla^2 u$, $u_t \in L^2_{\mathrm{loc}}(Q)$;
2) *there exists a number* $q' > 2$ *such that* $\nabla^2 u$, $u_t \in L^{q'}_{\mathrm{loc}}(Q)$.

Proof: Let $U(z) = \nabla u(z)$. Then formally U is a solution to the system

$$U_t - \operatorname{div}(\mathbb{A}(z,U)\nabla U) = -\operatorname{div} F(z), \qquad z \in Q' \subset\subset Q \tag{70}$$

where $\mathbb{A}(z,p) = \frac{\partial a(z,p)}{\partial p} \in \{\nu,\mu\}$, $p \in \mathbb{R}^{nN}$, $F(z) = -a'_x(z,\nabla u(z))$, and $|F(z)| \le L(1 + |\nabla u(z)|)$ by the assumption [H2'].

To justify existence of ∇U we should consider the difference $U_h(z) = (\nabla u(x + he_s,t) - \nabla u(x,t))/h$, $s \le n$, where $e_1,\dots,e_n$ is the canonical basis in $\mathbb{R}^n$, $|h| < \delta(Q';\partial Q)$, and prove uniform boundedness of $\|U_h\|_{2,Q'}$ in h. Then the existence of $\nabla U \in L^2(Q')$ follows and one can assert that $U \in V(Q')$ is a weak solution of (70) in $V(Q')$, $Q' \subset\subset Q$.

Moreover, $U \in \widetilde{V}_{\rm loc}(Q)$, where $\widetilde{V}(Q) = C(\Lambda; L^2(\Omega)) \cap L^2(\Lambda; W_2^1(\Omega))$. It satisfies the identity

$$\text{(71)} \qquad \int_{Q'} [-U \cdot \varphi_t + \mathbb{A}(z, U)\nabla U \cdot \nabla\varphi] \, \mathrm{d}z = \int_{Q'} F(z) \cdot \nabla\varphi \, \mathrm{d}z, \qquad \varphi \in \overset{o}{W}{}_2^1(Q'), \quad Q' \subset\subset Q.$$

By Lemma 3.3, $\nabla u \in L^q_{\rm loc}(Q)$ with some $q > 2$. It means that $F \in L^q_{\rm loc}(Q)$ and applying Lemmas 3.1 and 3.2 we obtain that the first and the second assertions of this lemma are valid, $q' \in (2, q]$. □

Moreover, we have the following local estimates for U:

$$\text{(72)} \qquad \begin{aligned} \mathbf{I} U - (U)_{2r} \mathbf{I}^2_{\widetilde{V}(Q_r)} &:= \sup_{\Lambda_R} \int_{B_r} |U - (U)_{2r}|^2 \, \mathrm{d}x + \int_{Q_r} |\nabla U|^2 \, \mathrm{d}z \\ &\le c r^{-2} \int_{Q_{2r}} |U - (U)_{2r}|^2 \, \mathrm{d}z + c \int_{Q_{2r}} |F|^2 \, \mathrm{d}z, \end{aligned}$$

$$\text{(73)} \qquad \int_{Q_r} |U - (U)_r|^2 \, \mathrm{d}z \le c r^2 \int_{Q_{2r}} |\nabla U|^2 \, \mathrm{d}z + c r^2 \int_{Q_{2r}} |F|^2 \, \mathrm{d}z, \qquad Q_{2r} \subset Q.$$

Inequality (72) follows from (21) for $w = U$, $k = U_{2r}$, $F_1 = 0$, $F_2 = F$. Inequality (73) also is a consequence of inequality (22) for $w = U$.

The constants c in the Caccioppoli inequality (72) and in the Poincaré inequality (73) depend on ν, μ, n, and N only. Moreover, we have for $q' \in (2, q]$ that

$$\text{(74)} \quad \left(\fint_{Q_r} |\nabla U|^{q'} \, \mathrm{d}z \right)^{2/q'} \le c_1 \fint_{Q_{2r}} |\nabla U|^2 \, \mathrm{d}z + c_2 \left(\fint_{Q_{2r}} |F|^{q'} \, \mathrm{d}z \right)^{2/q'}, \quad Q_{2r} \subset Q.$$

It follows from (24) that

$$\text{(75)} \qquad \|\widehat{U}\|^2_{2(n+2)/n, Q_r} \le c(n) \mathbf{I} \widehat{U} \mathbf{I}^2_{\widetilde{V}(Q_r)}, \qquad \widehat{U} = U - (U)_{2r}.$$

Now we claim that

$$\text{(76)} \quad r^2 \fint_{Q_r} |U|^{2(n+2)/n} \, \mathrm{d}z \le c\gamma(r) \left\{ \fint_{Q_{2r}} |U - (U)_{2r}|^2 \, \mathrm{d}z + r^2 \fint_{Q_{2r}} (1 + |U|^2) \, \mathrm{d}z \right\}$$

where

$$\gamma(r) = \|U\|^{4/n}_{2(n+2)/n, Q_r} \to 0 \qquad r \to 0.$$

Indeed, by (75) we have the inequalities

$$
\begin{aligned}
\int_{Q_r} |U|^{2(n+2)/n}\,\mathrm{d}z &= \|U\|^{2+4/n}_{2(n+2)/n,Q_r} = \gamma(r)\,\|U\|^2_{2(n+2)/n,Q_r} \\
&\le c(n)\gamma(r)\{\|\widehat{U}\|^2_{2(n+2)/n,Q_r} + |U_{2r}|^2|Q_r|\} \\
&\le c(n)\gamma(r)\{\mathbf{I}\widehat{U}\mathbf{I}^2_{\widetilde{V}(Q_r)} + \|U\|^2_{2,Q_{2r}}\}.
\end{aligned} \tag{77}
$$

Applying (72) to (77) we obtain that

$$
\begin{aligned}
\int_{Q_r} |U|^{2(n+2)/n}\,\mathrm{d}z \le c(\nu,\mu,n)\gamma(r)\biggl\{ r^{-2}\int_{Q_{2r}} |U-(U)_{2r}|^2\,\mathrm{d}z \\
+ \int_{Q_{2r}} (|F|^2+|U|^2)\,\mathrm{d}z\biggr\}.
\end{aligned} \tag{78}
$$

Using the definition of $F(z)$ we derive from (78) inequality (76).

Now we fix $Q_r(z^0)\subset Q$ such that $Q_{8r}(z^0)\subset Q$, and for a fixed $\varepsilon>0$ and $U\in V(Q_r(z^0))$ we apply $A(t)$-caloric lemma (Lemma 3.6) with the matrix

$$
A(t) = \fint_{B_r(x^0)} \frac{\partial a(x,t,(U)_{r,z^0})}{\partial p}\,\mathrm{d}x \in \{\nu,\mu\} \qquad \text{a.a. } t\in\Lambda_r(t^0). \tag{79}
$$

By Lemma 3.6, there exist an $A(t)$-caloric function $h\in V(Q_{r/2}(z^0))$, a constant $C_\varepsilon>0$, and a function $\varphi\in C_0^1(Q_r(z^0))$, $\sup_{Q_r(z^0)}|\nabla\varphi|\le 1$, such that inequalities (39) and (40) hold for the function U with

$$
\mathcal{L}_r^2(U,\varphi) = \left|\fint_{Q_r(z^0)} [-U\cdot\varphi_t + A(t)\nabla U\cdot\nabla\varphi]\,\mathrm{d}z\right|^2.
$$

Now we put

$$
\Phi(\varrho,z^0) = \fint_{Q_\varrho(z^0)} |U-(U)_{\varrho,z^0}|^2\,\mathrm{d}z, \qquad \Psi(\varrho,z^0) = \varrho^2\fint_{Q\varrho(z^0)} (1+|U|^2)\,\mathrm{d}z,
$$
$$
J(\varrho,z^0) = \Phi(\varrho,z^0) + \Psi(\varrho,z^0), \qquad \varrho\le 8r.
$$

We do not change the point z^0 up to the end of the proof and that is why we omit further dependence of the functions and the sets on this point: $\Phi(\varrho,z^0)=\Phi(\varrho)$, $Q_r(z^0)=Q_r$, and so on.

Note that from the Caccioppoli inequality (72) it follows that

$$
r^2\fint_{Q_r} |\nabla U|^2\,\mathrm{d}z \le c\Phi(2r) + cr^2\fint_{Q_{2r}} |F|^2\,\mathrm{d}z \le cJ(2r). \tag{80}
$$

Inequalities (39) and (40) guarantee now that

$$\fint_{Q_{r/2}} (|h(z)-(h)_{r/2}|^2 + r^2|\nabla h(z)|^2)\,\mathrm{d}z \le c\,J(2r), \tag{81}$$

$$\fint_{Q_{r/2}} |U(z)-h(z)|^2\,\mathrm{d}z \le c\varepsilon J(2r) + C_\varepsilon r^2 \mathcal{L}_r^2(U,\varphi). \tag{82}$$

The following inequality holds for $\varrho \le r/2$:

$$\begin{aligned}\Phi(\varrho) &\le 2\fint_{Q_\varrho} |(U-h)-(U-h)_\varrho|^2\,\mathrm{d}z + 2\fint_{Q_\varrho} |h-h_\varrho|^2\,\mathrm{d}z \\ &\le 2\fint_{Q_\varrho} |U-h|^2\,\mathrm{d}z + c\Big(\frac{\varrho}{r}\Big)^2 \fint_{Q_{r/2}} |h-(h)_{r/2}|^2\,\mathrm{d}z.\end{aligned}$$

Estimate (36) for h was applied in the last inequality.

We continue to estimate $\Phi(\varrho)$ with the help of inequalities (81) and (82). Then

$$\Phi(\varrho) \le c\Big(\frac{\varrho}{r}\Big)^2 J(2r) + c\Big(\frac{r}{\varrho}\Big)^{n+2}\{\varepsilon J(2r) + C_\varepsilon r^2\mathcal{L}_r^2(U,\varphi)\}. \tag{83}$$

To estimate $\mathcal{L}_r^2(U,\varphi)$, we address to identity (71) and obtain the relations

$$\begin{aligned}\mathcal{L}_r^2(U,\varphi) &= \Big|\fint_{Q_r} [-U\cdot\varphi_t + \mathbb{A}\nabla U\cdot\nabla\varphi + \Delta\mathbb{A}\nabla U\cdot\nabla\varphi]\,\mathrm{d}z\Big|^2 \\ &= \Big|\fint_{Q_r} (F\cdot\nabla\varphi + \Delta\mathbb{A}\,\nabla U\cdot\nabla\varphi)\,\mathrm{d}z\Big|^2.\end{aligned}$$

Here the difference

$$\Delta\mathbb{A} = A(t) - \mathbb{A}(z,U)$$

we estimate in the way:

$$|\Delta\mathbb{A}| \le \Big|\frac{\partial a(z,U)}{\partial p} - \frac{\partial a(z,(U)_r)}{\partial p}\Big| + \Big|\frac{\partial a(z,(U)_r)}{\partial p} - A(t)\Big|.$$

Observe that $\sup_{z\in Q_r}|\nabla\varphi(z)| \le 1$.

Further we use the assumptions [H4] and [H5] to derive the following inequalities:

$$\begin{aligned}r^2\mathcal{L}_r^2(U,\varphi) &\le 2r^2\fint_{Q_r}|F|^2\,\mathrm{d}z + 2r^2\fint_{Q_r}|\Delta\mathbb{A}|^2\,\mathrm{d}z\fint_{Q_r}|\nabla U|^2\,\mathrm{d}z \\ &\le c\Big\{\fint_{Q_r}\Big|\frac{\partial a(z,U)}{\partial p} - \frac{\partial a(z,(U)_r)}{\partial p}\Big|^2\,\mathrm{d}z \\ &\quad + \fint_{Q_r}\Big|\frac{\partial a(z,(U)_r)}{\partial p} - \Big(\frac{\partial a}{\partial p}\Big)_{x^0,r}(t;U_r)\Big|^2\,\mathrm{d}z\Big\}\end{aligned}$$

$$\times r^2 \fint_{Q_r} |\nabla U|^2 \, \mathrm{d}z + c\Psi(r)$$
$$\leq c\Psi(r) + c\Big[\omega\Big(\fint_{Q_r} |U - (U)_r|^2 \, \mathrm{d}z\Big) + q(r)\Big] r^2 \fint_{Q_r} |\nabla U|^2 \, \mathrm{d}z.$$

Applying relation (80) to the last inequality, we obtain that

$$r^2 \mathcal{L}_r^2(U, \varphi) \leq c\Psi(r) + c[\omega(\Phi(r)) + q(r)] J(2r). \tag{84}$$

It follows from (83) and (84) that

$$\begin{aligned} \Phi(\varrho) \leq c\Big\{\Big(\frac{\varrho}{r}\Big)^2 + \varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2} + C_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2} [\omega(\Phi(r)) + q(r)]\Big\} J(2r) \\ + C_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2} \Psi(r), \qquad \varrho \leq \frac{r}{2}. \end{aligned} \tag{85}$$

On the next step of the proof we will estimate the function $\Psi(\varrho)$. To do this, we put

$$I(\varrho, z^0) = I(\varrho) = \fint_{Q_\varrho(z^0)} |u(z) - (u)_{\varrho, z^0}|^2 \, \mathrm{d}z \tag{86}$$

and apply once more $A(t)$-caloric lemma (Lemma 3.6).

We put

$$A^0(t) = \mathbb{A}^0_{r,x^0}(t; (\nabla u)_r), \qquad \mathbb{A}^0(z, p) = \int_0^1 \frac{\partial a(z, sp)}{\partial p} \, \mathrm{d}s. \tag{87}$$

For the fixed earlier ε, the cylinder $Q_r(z^0)$, and the function $u \in V(Q_r(z^0))$ there exist an $A^0(t)$-caloric function $\eta \in V(Q_{r/2}(z^0))$, a constant $C^0_\varepsilon > 0$, and a function $\varphi \in C^1_0(Q_r(z^0))$, $\sup_{z \in Q_r} |\nabla \varphi(z)| \leq 1$, such that

$$\begin{aligned} \fint_{Q_{r/2}} (|\eta(z) - (\eta)_{r/2}|^2 + r^2 |\nabla \eta(z)|^2) \, \mathrm{d}z \\ \leq 2^{n+4} \fint_{Q_r} (|u(z) - (u)_r|^2 + r^2 |\nabla u(z)|^2) \, \mathrm{d}z, \end{aligned} \tag{88}$$

$$\begin{aligned} \fint_{Q_{r/2}} |u(z) - \eta(z)|^2 \, \mathrm{d}z \\ \leq \varepsilon \fint_{Q_r} (|u(z) - (u)_r|^2 + r^2 |\nabla u(z)|^2) \, \mathrm{d}z + C^0_\varepsilon r^2 \mathcal{L}_r^2(u, \varphi) \end{aligned} \tag{89}$$

where

$$\mathcal{L}_r^2(u, \varphi) = \Big| \fint_{Q_r} [-u \cdot \varphi_t + A^0(t) \nabla u \cdot \nabla \varphi] \, \mathrm{d}z \Big|^2.$$

Using the Caccioppoli inequality (29) we obtain the relation

$$
\fint_{Q_r} (|u(z)-(u)_r|^2 + r^2|\nabla u(z)|^2)\,\mathrm{d}z \le c(I(2r)+L^2r^2), \tag{90}
$$

here L is the constant from the assumption [H3].

Now we have the following relations for $I(\varrho)$, $\varrho \le r/2$:

$$
\begin{aligned}
I(\varrho) &\le 2\fint_{Q_\varrho} |(u-\eta)-(u-\eta)_\varrho|^2\,\mathrm{d}z + 2\fint_{Q_\varrho} |\eta-(\eta)_\varrho|^2\,\mathrm{d}z \\
&\le c\Big(\frac{r}{\varrho}\Big)^{n+2}\fint_{Q_{r/2}} |u-\eta|^2\,\mathrm{d}z + c\Big(\frac{\varrho}{r}\Big)^2\fint_{Q_{r/2}} |\eta-(\eta)_{r/2}|^2\,\mathrm{d}z \\
&\le c\Big\{\Big(\frac{\varrho}{r}\Big)^2 + \Big(\frac{r}{\varrho}\Big)^{n+2}(C^0_\varepsilon r^2\mathcal{L}^2_r(u,\varphi)+\varepsilon)\Big\}I(2r) + cC^0_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2}r^2.
\end{aligned}
$$

We applied relation (36) to justify the second inequality. The last relation holds due to inequalities (88), (89) and (90).

Now we use the identity (28) to estimate the expression $\mathcal{L}^2_r(u,\varphi)$:

$$
\mathcal{L}^2_r(u,\varphi) \le 2\fint_{Q_r} |\Delta\mathbb{A}|^2\,\mathrm{d}z \fint_{Q_r} |\nabla u|^2\,\mathrm{d}z + 2L^2
$$

where

$$
|\Delta\mathbb{A}| = |\mathbb{A}^0(z,\nabla u) - A^0(t)| \le |\mathbb{A}^0(z,\nabla u) - \mathbb{A}^0(z,(\nabla u)_r)| + |\mathbb{A}^0(z,(\nabla u)_r) - A^0(t)|.
$$

With the help of the assumptions [H4], [H5], and inequality (90), we obtain the estimate

$$
\begin{aligned}
r^2\mathcal{L}^2_r(u,\varphi) &\le \Big[\fint_{Q_r} \omega^2(|\nabla u-(\nabla u)_r|^2)\,\mathrm{d}z + q(r)\Big] r^2\fint_{Q_r} |\nabla u(z)|^2\,\mathrm{d}z + 2\,L^2r^2 \\
&\le c[\omega(\Phi(r))+q(r)]I(2r) + cL^2r^2.
\end{aligned}
$$

It follows from the estimate for $I(\varrho)$ and the last inequality that

$$
I(\varrho) \le c\Big\{\Big(\frac{\varrho}{r}\Big)^2 + \varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2} + C^0_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2}[\omega(\Phi(r))+q(r)]\Big\}I(2r) + cC^0_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2}L^2r^2.
$$

Applying the Caccioppoli and the Poincaré inequalities, we derive from the last inequality that

$$
\begin{aligned}
\Psi\Big(\frac{\varrho}{2}\Big) \le c\Big\{\Big(\frac{\varrho}{r}\Big)^2 + \varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2} + C^0_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2}[\omega(\Phi(r))+q(r)]\Big\}\Psi(4r) \\
+ c\,C^0_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2}L^2r^2.
\end{aligned} \tag{91}
$$

Now we add inequality (91) to (85) and obtain for $\varrho/2 \le r$ the inequality

$$
\begin{aligned}
J\Big(\frac{\varrho}{2}\Big) \le c\Big\{\Big(\frac{\varrho}{r}\Big)^2 + \varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2} + \widehat{C}_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2}[\omega(\Phi(r)) + q(r)]\Big\}J(4r) \\
+ cC_\varepsilon^0\Big(\frac{r}{\varrho}\Big)^{n+2}L^2r^2 + c\widehat{C}_\varepsilon\Big(\frac{r}{\varrho}\Big)^{n+2}\Psi(4r), \qquad \widehat{C}_\varepsilon = C_\varepsilon + C_\varepsilon^0.
\end{aligned}
\tag{92}
$$

Further we apply the Cauchy inequality and relation (76) to estimate $\Psi(4r) = \Psi(R/2)$, $R = 8r$, as follows

$$
\begin{aligned}
\Psi\Big(\frac{R}{2}\Big) &= \Big(\frac{R}{2}\Big)^2 \fint_{Q_{R/2}} (|U|^2 + 1)\,\mathrm{d}z \le R^2 \fint_{Q_{R/2}} (|U|^{2(n+2)/n} + c(n))\,\mathrm{d}z \\
&= R^2 \fint_{Q_{R/2}} |U|^{2(n+2)/n}\,\mathrm{d}z + c(n)R^2 \\
&\le cR^2 + c\gamma(R)(\Phi(R) + \Psi(R)) \\
&= cR^2 + \gamma(R)J(R), \qquad \gamma(R) \to 0, \quad R \to 0.
\end{aligned}
\tag{93}
$$

Now we put in (92) $r = R/8$, $\varrho = \tau R$ where $\tau \le 1/16$ we will choose below. Thus,

$$
\begin{aligned}
J(\tau R) \le c_0\{\tau^2 + \varepsilon\tau^{-(n+2)} + \widehat{C}_\varepsilon\tau^{-(n+2)}[\omega(8^{n+2}\Phi(R)) + q(R) \\
+ \gamma(R)]J(R)\} + c_1\widehat{C}_\varepsilon\tau^{-(n+2)}R^2.
\end{aligned}
\tag{94}
$$

In (94) the constants c_0 and c_1 depend on the parameters ν, μ, L, n, N, but do not depend on the fixed point z^0.

Now we make the choice of the parameters τ, ε and the maximal radius R_0. For any number $\alpha \in (0,1)$ we fix a number $\alpha' \in (\alpha, 1)$ and choose $\tau \le 1/16$ to satisfy the relation

$$
c_0\tau^2 \le \frac{\tau^{2\alpha'}}{8}. \tag{95}
$$

Then we fix $\varepsilon < 1$ such that

$$
\varepsilon c_0\tau^{-(n+2)} \le \frac{\tau^{2\alpha'}}{8}. \tag{96}
$$

As the function $\omega(s) \to 0$ when $s \to 0$, we can fix a number $\theta \in (0,1)$ satisfying the condition

$$
c_0\widehat{C}_\varepsilon\tau^{-(n+2)}\omega(8^{n+2}\theta) \le \frac{\tau^{2\alpha'}}{8}. \tag{97}
$$

Using the condition [H5] and relation (76), we find R_0 such that

$$
c_0\,\widehat{C}_\varepsilon\tau^{-(n+2)}(q(R_0) + \gamma(R_0)) \le \frac{\tau^{2\alpha'}}{8}, \qquad c_1\widehat{C}_\varepsilon\tau^{-(n+2)}R_0^2 \le \frac{\theta}{2}. \tag{98}
$$

Let us assume that for some $R \le R_0$ the following inequality is valid in the fixed point z^0:

$$\begin{aligned} J(R) &= J(R, z^0) \\ &= \fint_{Q_R(z^0)} |U(z) - (U)_{R,z^0)}|^2 \, \mathrm{d}z + R^2 \fint_{Q_R(z^0)} (|U|^2 + 1) \, \mathrm{d}z < \theta. \end{aligned} \tag{99}$$

As $\Phi(R) < J(R)$, the assumption (99) supplies the condition $\Phi(R) < \theta$, and the inequality

$$c_0 \widehat{C}_\varepsilon \tau^{-(n+2)} \omega(8^{n+2} \Phi(R)) \le \frac{\tau^{2\alpha'}}{8} \tag{100}$$

holds for such R due to (97). In this case we obtain from (94) the relation

$$J(\tau R) \le \frac{\tau^{2\alpha'}}{2} J(R) + KR^2, \qquad K = c_1 \widehat{C}_\varepsilon \tau^{-(n+2)}, \tag{101}$$

and $KR_0^2 \le \theta/2$ by (98).

In particular, it follows that

$$\Phi(\tau R) < J(\tau R) < \theta,$$

and inequality (100) holds with τR instead of R.

It allows us to repeat all considerations with τR instead of R. Thus,

$$J(\tau^2 R) \le \frac{\tau^{2\alpha'}}{2} J(\tau R) + K(\tau R)^2.$$

Now we can assert that the following inequalities hold for the sequence $R_j = \tau^j R$:

$$J(R_j) \le \frac{\tau^{2\alpha'}}{2} J(R_{j-1}) + K(R_{j-1})^2, \qquad j \in \mathbb{N}. \tag{102}$$

The iteration process guarantees us that

$$J(R_j) \le \tau^{2\alpha j} [J(R) + cK(R)^{2\alpha}], \qquad j \in \mathbb{N}, \quad \alpha < \alpha'. \tag{103}$$

It follows from (103) that the inequality

$$J(\varrho) = J(\varrho, z^0) \le c \Big(\frac{\varrho}{R}\Big)^{2\alpha} J(R, z^0) + cK\varrho^{2\alpha} \tag{104}$$

holds for all $\varrho \le R$. In particular, we obtain from (104) that

$$\frac{1}{\varrho^{n+2+2\alpha}} \int_{Q_\varrho(z^0)} |\nabla u(z) - (\nabla u)_{\varrho,z^0}|^2 \, \mathrm{d}z \le c(R^{-1}, \nu, \mu, L, \alpha, \|\nabla u\|_{2,Q}). \tag{105}$$

As inequality (99) holds (for the fixed $R \leq R_0$) in some neighborhood of the point z^0, we can assert that estimate (105) also is valid in some cylinder $Q_{\varrho_0}(z^0)$. More exactly,

$$\begin{aligned}(106)\qquad \frac{1}{\varrho^{n+2+2\alpha}} \int_{Q_\varrho(\xi)} |\nabla u(z) - (\nabla u)_{\varrho,\xi}|^2 \,\mathrm{d}z \\ \leq c(R^{-1}, \nu, \mu, L, \alpha, \|\nabla u\|_{2,Q}), \qquad \forall\, \xi \in Q_{\varrho_0}(z^0).\end{aligned}$$

This inequality guarantees estimate of the seminorm of the gradient of u in $\mathcal{L}^{2,n+2+2\alpha}(Q_{\varrho_0}(z^0);\delta)$. It follows that the norm in this space is also estimated. Due to the isomorphism of the Campanato space $\mathcal{L}^{2,n+2+2\alpha}(Q_{\varrho_o}(z^0);\delta)$ and $C^\alpha(\overline{Q_{\varrho_0}(z^0)};\delta)$, we have the estimate of the gradient of u in the Hölder norm in $Q_{\varrho_0}(z^0)$.

Moreover, $|\nabla u|$ is bounded near the point z^0 and we have also the following estimate for the function u:

$$I(\varrho,\xi) \leq c(\Psi(2\varrho,\xi) + \varrho^2) = c\varrho^2\big(1 + \sup_{Q_{\varrho_0}(z^0)} |\nabla u|^2\big) \leq c'(R^{-1}, \nu, \mu, L, n, N)\varrho^2, \qquad \forall\, \xi \in Q_{\varrho_0}(z^0).$$

It means that $u \in C^{0,1}(\overline{Q_{\varrho_0}(z^0)};\delta)$. Theorem 2.2 is proved. □

Remark 5.1. To prove local smoothness of u near the point z^0, we can change smallness condition (99) by the assumption that

$$(107)\qquad \liminf_{\varrho\to 0} \varrho^2 ⨍_{Q_\varrho(z^0)} (|\nabla^2 u|^2 + |\nabla u|^2)\,\mathrm{d}z = 0.$$

Indeed, let (107) hold. Taking into account estimate (73) for $U(z) = \nabla u(z)$ we can choose a radius R small enough to obtain validity of the assumption (99) for the function $J(R, z^0)$. It means that condition (107) guarantees smoothness of u in some neighborhood of z^0.

6. Fractional derivatives of the gradient

Let the assumptions [H1]–[H5] hold and $\beta \in (0,1)$ be the parameter from the condition [H2].

For cylinders $\widetilde{Q} \subset\subset Q'' \subset\subset Q' \subset\subset Q$ we denote $d_0 = \min\{\delta(Q';\partial Q), \delta(Q'';\partial Q'), \delta(\widetilde{Q};\partial Q'')\}$ and define the difference

$$\Delta_h^s u(z) = u(x + he_s, t) - u(x,t), \qquad s = 1, \dots, n,$$

where $z \in Q'$, $h \in \mathbb{R}^1$, $|h| < d_0$, and $e_1, \dots, e_n$ is the canonical basis in $\mathbb{R}^n$.

To estimate fractional derivatives of the gradient of u, we apply the difference quotient method. We start with defining the functions

$$U(z)=\frac{\Delta_h^s u(z)}{|h|^\beta},\qquad z=(x,t)\in Q',\quad s=1,\dots,n. \tag{108}$$

It follows from identity (7) that the functions $U(z)$ satisfy the equality

$$\begin{aligned}\int_Q\Big[-U\cdot\varphi_t+\frac{1}{|h|^\beta}\{a(x+he_s,t,\nabla u(x+he_s,t))\\ -a(x,t,\nabla u(x,t))\}\cdot\nabla\varphi\Big]\,\mathrm{d}z=0\end{aligned} \tag{109}$$

for all $\varphi\in\overset{o}{W}{}_2^1(Q')$.

The expression in the braces of relation (109) we rewrite in the form

$$\begin{aligned}\{\dots\}&=\int_0^1\frac{\partial a(x+he_s,t,\nabla u(z)+q\Delta_h^s\nabla u(z))}{\partial p}\,\mathrm{d}q\Delta_h^s\nabla u(z)+\Delta_h'a(z)\\ &=:\mathbb{A}(z)\nabla(\Delta_h^s u(z))+\Delta_h'a(z)\end{aligned}$$

where the bounded matrix $\mathbb{A}(z)\in\{\nu,\mu\}$ for almost all $z\in Q'$. By the condition [H2],

$$\begin{aligned}&|\Delta_h'a(z)|\le L|h|^\beta(1+|\nabla u(z)|),\\ &\Delta_h'a(z)=a(x+he_s,t,\nabla u(x,t))-a(x,t,\nabla u(x,t)).\end{aligned} \tag{110}$$

Now the equality (109) can be written in the form

$$\int_Q[-U\cdot\varphi_t+\mathbb{A}(z)\nabla U\cdot\nabla\varphi]\,\mathrm{d}z=\int_Q F(z)\cdot\nabla\varphi\,\mathrm{d}z,\qquad \varphi\in\overset{o}{W}{}_2^1(Q'), \tag{111}$$

where

$$F(z)=-\frac{\Delta_h'a(z)}{|h|^\beta},\qquad |F(z)|\le L(1+|\nabla u(z)|).$$

Thus, for any fixed $s=1,\dots,n$ and $|h|<d_0$, the function $U\in V(Q')$ is a weak solution to the linear system

$$U_t-\operatorname{div}(\mathbb{A}(z)\nabla U)=-\operatorname{div}F(z),\qquad z\in Q'. \tag{112}$$

We apply Lemma 3.1 to assert that the following Caccioppoli and Poincaré inequalities are valid for the function U:

$$\begin{aligned}\int_{Q_R}|\nabla U|^2\,\mathrm{d}z&=\|\nabla U\|^2_{2,Q_R}\\ &\le c\{R^{-2}\|U\|^2_{2,Q_{2R}}+L^2(1+\|\nabla u\|^2_{2,Q_{2R}})\},\qquad Q_{2R}\subset Q';\end{aligned} \tag{113}$$

$$\int_{Q_R} |U(z) - (U)_R|^2 \,\mathrm{d}z \le c\Big\{R^2 \int_{Q_{2R}} |\nabla U|^2 \,\mathrm{d}z \tag{114}$$
$$+ L^2R^2 \int_{Q_{2R}} (1 + |\nabla u|^2) \,\mathrm{d}z\Big\}, \qquad Q_{2R} \subset Q'.$$

From the definition of the function U and inequality (113) it follows that

$$\int_{Q''} |U(z)|^2 \,\mathrm{d}z \le c \int_{Q'} |\nabla u(z)|^2 \,\mathrm{d}z, \tag{115}$$
$$\int_{Q''} |\nabla U(z)|^2 \mathrm{d}z \le c(1 + \|\nabla u\|^2_{2,Q}) =: cM_0.$$

The constant c in (115) depends on the data and d_0^{-1}.

By Lemma 3.3, $F \in L^q(Q')$ with some $q > 2$ in (112).

Using Lemma 3.2, we can assert that there exists $p \in (2, q]$ such that $\nabla U \in L^p_{\mathrm{loc}}(Q')$ and the estimate

$$\|\nabla U\|^2_{p,\widetilde{Q}} \le c\{\|\nabla U\|^2_{2,Q''} + 1 + \|\nabla u\|^2_{p,Q''}\} \le cM_0, \qquad \forall\, \widetilde{Q} \subset\subset Q'', \tag{116}$$

is valid. In the last inequality (116) we have used estimates (32) and (115).

Now it follows from (116) that

$$\int_{Q_R} |\nabla U|^2 \,\mathrm{d}z \le \Big(\int_{Q_R} |\nabla U|^p \,\mathrm{d}z\Big)^{2/p} |Q_R|^{1-2/p} \tag{117}$$
$$\le c(n)\|\nabla U\|^2_{p,\widetilde{Q}} R^{(n+2)(1-2/p)} \le cM_0R^m,$$

$\forall\, Q_R \subset \widetilde{Q}$. Here and below

$$m = (n + 2)\Big(1 - \frac{2}{p}\Big) > 0.$$

In the definition of the function $U(z)$ the direction e_s was fixed arbitrary. If we take the number h sufficiently small we obtain from estimate (117) that

$$\int_{Q_R(z^0)} |\nabla u(x + he_s, t) - \nabla u(x, t)|^2 \,\mathrm{d}z \le c\,|h|^{2\beta} R^m, \tag{118}$$
$$s = 1, \dots, n, \quad \forall\, Q_R \subset \widetilde{Q}.$$

Here the constant c depends on $\|\nabla u\|_{2,Q}, \nu, \mu, L, \beta, n, d_0^{-1}$.

Note that all subdomains in Q were fixed arbitrarily. Thus, estimate (118) is valid for any $Q_r(z^0) \subset\subset Q$ and the constant c in (118) depends on d^{-1} where $d = \delta(Q_r(z^0); \partial Q)$.

By Lemma 3.1 the function U belongs to $H^{1/2}_{\mathrm{loc}}(\Lambda; L^2(\Omega))$. Let $Q_{4R} \subset Q_o \subset\subset Q$. Then the following estimate of the seminorm follows from (23):

$$\begin{aligned}[U]^2_{H^{1/2}(\Lambda_R;L^2(B_R))} \le c\{&\|\nabla U\|^2_{2,Q_{2R}} + R^{-2}\|U-(U)_{2R}\|^2_{2,Q_{2R}} \\ &+ L^2(R^{n+2} + \|\nabla u\|^2_{2,Q_{2R}})\}.\end{aligned} \tag{119}$$

Using Poincaré inequality (114), we can derive from (119) the following estimate:

$$[U]^2_{H^{1/2}(\Lambda_R;L^2(B_R))} \le c\{\|\nabla U\|^2_{2,Q_{4R}} + R^{n+2} + \|\nabla u\|^2_{2,Q_{4R}}\}. \tag{120}$$

As ∇U and ∇u are functions from $L^p_{\mathrm{loc}}(Q)$, $p>2$, we apply the Hölder inequality and inequality (116) to the right-hand side of inequality (120) and obtain that

$$\begin{aligned}[U]^2_{H^{1/2}(\Lambda_R;L^2(B_R))} &\le cR^{(n+2)(1-2/p)}\{\|\nabla U\|^2_{p,Q_{4R}} + 1 + \|\nabla u\|^2_{p,Q_{4R}}\} \\ &\le cR^m M_0.\end{aligned} \tag{121}$$

It follows from (121) and Lemma 3.4 that

$$\int_{Q_R} |U(x,t+\tau) - U(x,t)|^2\,\mathrm{d}z \le c|\tau|R^m, \qquad Q_{4R}\subset Q_o \subset\subset Q,\ \ |\tau|<R^2. \tag{122}$$

The constant c depends on $\|\nabla u\|^2_{2,Q}$, ν, μ, L, β, d^{-1}, n; here $m=(n+2)(1-2/p)$, $d=\delta(Q_o,\partial Q)$.

Now we prove integral continuity of the gradient of u in the time variable.

Lemma 6.1. *Let $Q_{4R}\subset \widetilde{Q} \subset\subset Q$ be fixed, $d=\delta(\widetilde{Q};\partial Q)$ and $\tau\in\mathbb{R}^1$, $|\tau|<R^2$. Then the following estimate is valid*

$$\int_{Q_R} |\nabla u(x,t+\tau) - \nabla u(x,t)|^2\,\mathrm{d}z \le c|\tau|^\beta R^m \tag{123}$$

where β is the exponent from the assumption [H2], *the exponent $m>0$ is the same as in estimate* (122), *and $c=c(\|u\|_{V(Q)},\nu,\mu,L,\beta,\,d^{-1},n)$.*

Proof: To estimate the left-hand side of inequality (123), we can fix any direction $s=1,\dots,n$ and explain how to derive the inequality

$$\begin{aligned}I^s := \int_{Q_R} &|u_{x_s}(x,t+\tau) - u_{x_s}(x,t)|^2\,\mathrm{d}z \\ &\le c|\tau|^\beta\, R^m, \qquad c=c(\|u\|_{V(Q)}, d^{-1},\nu,\mu,L,\beta,n)\end{aligned} \tag{124}$$

for any $s\le n$.

We illustrate the procedure for $s=n$ and denote below $u_{x_n}=\nabla_n u$. The other directions can be considered in the same way.

We fix a number $r\in(0,R)$ and put $y=(x',y_n)$, $y_n\in(x_n,x_n+r)$. We will define the number r below.

The following relations are valid:

$$
\begin{aligned}
I^n = &\int_{Q_R} \left| \fint_{x_n}^{x_n+r} \{[\nabla_n u(x,t+\tau) - \nabla_n u(y,t+\tau)] + [\nabla_n u(y,t+\tau) - \nabla_n u(y,t)] \right. \\
&\qquad\qquad \left. + [\nabla_n u(y,t) - \nabla_n u(x,t)]\}\,\mathrm{d}y_n \right|^2 \mathrm{d}z \\
&\le 4\int_{Q_R} \left(\fint_{x_n}^{x_n+r} |\nabla_n u(x,t+\tau) - \nabla_n u(y,t+\tau)|^2\,\mathrm{d}y_n \right) \mathrm{d}z \\
&\quad + 4\int_{Q_R} \left| \fint_{x_n}^{x_n+r} \nabla_n [u(y,t+\tau) - u(y,t)]\,\mathrm{d}y_n \right|^2 \mathrm{d}z \\
&\quad + 4\int_{Q_R} \left(\fint_{x_n}^{x_n+r} |\nabla_n u(x,t) - \nabla_n u(y,t)|^2 \mathrm{d}y_n \right) \mathrm{d}z \\
&=: 4(j_1 + j_2 + j_3).
\end{aligned}
$$

Note that the integrals j_1 and j_3 are estimated in the same way.
For example,

$$
\begin{aligned}
j_1 &= \int_{Q_R} \left(\fint_0^r |\nabla_n u(x', x_n, t+\tau) - \nabla_n u(x', x_n+\xi, t+\tau)|^2\,\mathrm{d}\xi \right) \mathrm{d}z \\
&\le \sup_{\xi\in[0,r]} \int_{Q_R} |\nabla_n u(x', x_n, t+\tau) - \nabla_n u(x', x_n+\xi, t+\tau)|^2\,\mathrm{d}z \\
&\le c_1 r^{2\beta} R^m
\end{aligned}
$$

where the last inequality follows from estimate (118), the constant c_1 depends on $\|u\|_{V(Q)}$, β, ν, μ, L and d^{-1}.

The same estimate we have for j_3.

Further we transform and estimate the integral j_2. After calculating the internal integral in j_2 we have the expression

$$
\begin{aligned}
j_2 = &\frac{1}{r^2} \int_{Q_R} |[u(x', x_n+r, t+\tau) - u(x', x_n+r, t)] \\
&\qquad - [u(x', x_n, t+\tau) - u(x', x_n, t)]|^2 \mathrm{d}z \\
= &\frac{1}{r^2} \int_{Q_R} |[u(x', x_n+r, t+\tau) - u(x', x_n, t+\tau)] \\
&\qquad - [u(x', x_n+r, t) - u(x', x_n, t)]|^2 \mathrm{d}z.
\end{aligned}
$$

If we put in the definition (108) of the function U the direction $s = n$ and $|h| = r$ we will have the function

$$
U(x,t) = \frac{u(x', x_n+r, t) - u(x,t)}{r^\beta} = \frac{u(x+re_n, t) - u(x,t)}{r^\beta}.
$$

Then it follows from estimate (122) that

$$j_2 = \frac{r^{2\beta}}{r^2} \int_{Q_R} |U(x,t+\tau) - U(x,t)|^2 \,\mathrm{d}z \le c_2 \frac{|\tau| R^m r^{2\beta}}{r^2}. \tag{125}$$

Here the constant c_2 depends on the same data as c_1 does.

Taking into account the estimates of j_1, j_2, j_3, we arrive at the following inequality for I^n:

$$I^n \le c\Big\{ r^{2\beta} + \frac{r^{2\beta}|\tau|}{r^2} \Big\} R^m, \qquad m = (n+2)\Big(1 - \frac{2}{p}\Big). \tag{126}$$

We choose now the number $r \le R$ to satisfy the equality

$$r^2 = |\tau|.$$

Then we obtain from (126) that

$$I^n \le c|\tau|^{\beta} R^m. \tag{127}$$

Repeating the proof for any direction $s = 1, \ldots, n-1$, we obtain estimate (124). Inequality (123) follows. $\square$

Lemma 6.2. *For any $\alpha \in (0,\beta)$ the gradient of u belongs to the space $W_2^{\alpha,\alpha/2}(Q')$, $\forall\, Q' \subset\subset Q$, and*

$$[\nabla u]^2_{W_2^{\alpha,\alpha/2}(Q')} \le c \tag{128}$$

where the constant c depends on $\|\nabla u\|_{2,Q}$, α, β, μ, ν, L, n, and $1/\delta(Q';\partial Q)$.

Validity of the assertion of Lemma 6.2 follows from estimates (118) and (123) if we apply Proposition 3.4 of [18].

Remark 6.1. Certainly, estimate (116) supplies better information on the behavior of the gradient of u in the space variables but we can not improve estimate (122) (and as a consequence (123)) in the situation when no smoothness in t of the functions $a(x,t,p)$ is assumed.

7. Estimates of the singular sets. Proof of Theorem 2.3

First, we consider the case $\beta = 1$.

Using Remark 5.2, we can assert that all singular points of the solution under consideration are described by the set

$$\Sigma = \Big\{ z^0 \in Q\colon \liminf_{\varrho \to 0} \varrho^2 \fint_{Q_\varrho(z^0)} (|\nabla^2 u|^2 + |\nabla u|^2)\,\mathrm{d}z > 0 \Big\}. \tag{129}$$

We recall that the functions ∇u and $\nabla^2 u$ are integrable with some degree $p > 2$ (we fix the degree p the same in estimates (32) and (74)). If $z^0 \in \Sigma$ then

$$\begin{aligned} 0 &< \frac{1}{\varrho^n} \int_{Q_\varrho(z^0)} (|\nabla^2 u|^2 + |\nabla u|^2)\,\mathrm{d}z \le \frac{|Q_\varrho|^{1-2/p}}{\varrho^n} \bigg(\int_{Q_\varrho(z^0)} (|\nabla^2 u| + |\nabla u|)^p\,\mathrm{d}z \bigg)^{2/p} \\ &= c(n) \bigg(\frac{1}{\varrho^{n-(p-2)}} \int_{Q_\varrho(z^0)} (|\nabla^2 u| + |\nabla u|)^p\,\mathrm{d}z \bigg)^{2/p}. \end{aligned}$$

It follows that $\Sigma \subset \Sigma_p$ where the set

$$\Sigma_p = \bigg\{ z^0 \in Q\colon \liminf_{\varrho \to 0} \frac{1}{\varrho^{n-(p-2)}} \int_{Q_\varrho(z^0)} (|\nabla^2 u| + |\nabla u|)^p\,\mathrm{d}z > 0 \bigg\}.$$

Then $\mathcal{H}_{n-(p-2)}(\Sigma_p; \delta) = 0$ (see, for example, [21], [16]). Thus, $\mathcal{H}_{n-(p-2)}(\Sigma; \delta) = 0$ and

$$\dim_P \Sigma \le n - \chi_0, \qquad \chi_0 = p - 2 > 0. \tag{130}$$

We have proved Theorem 2.3 for the case $\beta = 1$.

Now we assume that $\beta \in (0, 1)$. By Lemma 6.2, $\nabla u \in W^{\alpha,\alpha/2}_{\mathrm{loc}}(Q)$, and the Poincaré inequality (34) is valid for ∇u:

$$\int_{Q_\varrho(z^0)} |\nabla u(z) - (\nabla u)_{\varrho,z^0}|^2\,\mathrm{d}z \le c\varrho^{2\alpha} [\nabla u]^2_{W^{\alpha,\alpha/2}(Q_\varrho(z^0))}, \qquad \forall\, \alpha < \beta, \quad Q_\varrho(z^0) \subset\subset Q. \tag{131}$$

It follows that

$$⨍_{Q_\varrho(z^0)} |\nabla u(z) - (\nabla u)_{\varrho,z^0}|^2\,\mathrm{d}z \le c\varrho^{-(n+2)+2\alpha} [\nabla u]^2_{W^{\alpha,\alpha/2}(Q_\varrho(z^0))} \qquad \forall\, \alpha < \beta, \quad Q_\varrho(z^0) \subset\subset Q. \tag{132}$$

If we take into account the description of the set Q_0 of the regular points of u and denote as Σ the admissible closed singular set of this solution, $\Sigma = Q \setminus Q_0$, then we can assert that $\Sigma \subseteq \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_1 = \bigg\{ z^0 \in Q\colon \liminf_{\varrho \to 0} ⨍_{Q_\varrho(z^0)} |\nabla u(z) - (\nabla u)_{\varrho,z^0}|^2\,\mathrm{d}z > 0 \bigg\}$$

and

$$\Sigma_2 = \Big\{ z^0 \in Q\colon \limsup_{\varrho \to 0} |(\nabla u)_{\varrho,z^0}| = \infty \Big\}.$$

We will prove that $\dim_P \Sigma_1 \le n + 2 - 2\beta$ and $\dim_P \Sigma_2 \le n + 2 - 2\beta$.

Now we fix a number $\varepsilon \in (0,\beta)$ and put

$$S_1 = \Big\{ z^0 \in Q\colon\ \limsup_{\varrho\to 0} \varrho^{-(n+2)+2\alpha} [\nabla u]^2_{W_2^{\alpha,\alpha/2}(Q_\varrho(z^0))} > 0 \Big\}, \qquad \alpha = \beta - \frac{\varepsilon}{2}.$$

It follows from (132) with $\alpha = \beta - \varepsilon/2$ that $\Sigma_1 \subset S_1$. As ε was fixed arbitrary, we use Lemma 3.2 from [18] or Lemma 4.2 from [27] and can assert that

$$\dim_P \Sigma_1 \le \dim_P S_1 \le n + 2 - 2\beta. \tag{133}$$

To estimate the set Σ_2, we consider the set

$$S_2 = \{ z^0 \in Q\colon\ \limsup \varrho^{-(n+2)+2\alpha-\varepsilon} [\nabla u]^2_{W^{\alpha,\alpha/2}(Q_\varrho(z^0))} > 0 \}.$$

We have the estimate

$$\mathcal{H}_{n+2-2\beta+2\varepsilon}(S_2;\delta) = 0. \tag{134}$$

Further we will prove that $\Sigma_2 \subset S_2$. To prove the implication $\Sigma_2 \subset S_2$, we fix a point $z^0 \in Q \setminus S_2$ and will prove that $z^0 \in Q \setminus \Sigma_2$.

We fix a number $\varrho > 0$ and consider the sequence $\varrho_i = \tau^i \varrho$ with any fixed $\tau \in (0,1)$, $i \in \mathbb{N}$, $\varrho_i \to 0$, $i \to \infty$, and will prove that the following limit exists and is finite

$$\lim_{\varrho_k \to 0} |(\nabla u)_{\varrho_k, z^0}| < \infty. \tag{135}$$

To this end, we estimate the difference

$$\begin{aligned} J_k := |(\nabla u)_{\varrho_{k+1}, z^0} - (\nabla u)_{\varrho_k, z^0}|^2 &\le \fint_{Q_{\varrho_k+1}(z^0)} |\nabla u(z) - (\nabla u)_{\varrho_{k}, z^0}|^2 \,\mathrm{d}z \\ &\le \tau^{-(n+2)} \fint_{Q_{\varrho_k}(z^0)} |\nabla u(z) - (\nabla u)_{\varrho_k, z^0}|^2 \,\mathrm{d}z. \end{aligned}$$

We continue to estimate the right-hand side of the last inequality with the help of estimate (131) then

$$\begin{aligned} J_k &\le c\tau^{-(n+2)} \varrho_k^{-(n+2)+2\alpha} [\nabla u]^2_{W_2^{\alpha,\alpha/2}(Q_{\varrho_k}(z^0))} \\ &\le c \frac{\varrho_k^{\varepsilon}}{\varrho_k^{n+2-2\alpha+\varepsilon}} [\nabla u]^2_{W_2^{\alpha,\alpha/2}(Q_{\varrho_k}(z^0))} \le_{(*)} c_1 \varrho_k^{\varepsilon} \to 0, \qquad k \to \infty. \end{aligned}$$

At the last step in inequality $(*)$ we have used the definition of the set S_2.

For arbitrary $m > k$ we have now inequalities

$$\begin{aligned}|(\nabla u)_{\varrho_m,z^0} - (\nabla u)_{\varrho_k,z^0}| &\le \sum_{j=k}^{j=m-1} |(\nabla u)_{\varrho_{j+1},z^0} - (\nabla u)_{\varrho_j,z^0}| \\ &\le c \sum_{j=k}^{j=m-1} \varrho_j^{\varepsilon} \le c(\tau)\tau^{\varepsilon k}\varrho^{\varepsilon} \to 0, \qquad k \to \infty.\end{aligned}$$

It means that there exists $\lim_{k\to\infty} |(\nabla u)_{\varrho_k}| < \infty$. It is not difficult to justify that the finite limit of the sequence $|(\nabla u)_{r_j,z^0}|$ exists for arbitrary sequence of $r_j \to 0$. Thus, $z^0 \in Q \setminus \Sigma_2$. In a result we have that $Q \setminus S_2 \subset Q \setminus \Sigma_2$ and $\Sigma_2 \subset S_2$.

By (134), $\mathcal{H}_{n+2-2\beta+2\varepsilon}(\Sigma_2;\delta) = 0$. As this equality is valid for any $\varepsilon > 0$, we obtain the estimate

$$\dim_P(\Sigma_2) \le n + 2 - 2\beta. \tag{136}$$

The estimate

$$\dim_P(\Sigma) \le n + 2 - 2\beta$$

follows now from (133) and (136) in the case when $\beta \in (0,1)$. □

Acknowledgement. The authors are indebted to the referee for carefully reading the paper and for his remarks.

References

[1] Arkhipova A. A., *On the regularity of the solutions of the Neumann problem for quasilinear parabolic systems*, Izv. Ross. Akad. Nauk Ser. Mat. **58** (1994), no. 5, 3–25 (Russian); translation in Russian Acad. Sci. Izv. Math. **45** (1995), no. 2, 231–253.

[2] Arkhipova A. A., *Reverse Hölder inequalities with boundary integrals and L_p-estimates for solutions of nonlinear elliptic and parabolic boundary value problems*, Nonlinear Evolution Equations, Amer. Math. Soc. Transl. Ser. 2, 164, Adv. Math. Sci., 22, Amer. Math. Soc. (1995), 15–42.

[3] Arkhipova A. A., *Regularity of weak solutions to the model Venttsel problem for linear parabolic systems with nonsmooth in time principal matrix: $A(t)$-caloric approximation method*, Manuscripta Math. **151** (2016), no. 3–4, 519–548.

[4] Arkhipova A. A., *Regularity of solutions of the model Venttsel' problem for quasilinear parabolic systems with nonsmooth in time principal matrices*, Comput. Math. Math. Phys. **57** (2017), no. 3, 476–496.

[5] Arkhipova A. A., Stará J., *Boundary partial regularity for solutions of quasilinear parabolic systems with non smooth in time principal matrix*, Nonlinear Anal. **120** (2015), 236–261.

[6] Arkhipova A. A., Stará J., *Regularity of weak solutions to linear and quasilinear parabolic systems of non-divergence type with non-smooth in time principal matrix: $A(t)$-caloric method*, Forum Math. **29** (2017), no. 5, 1039–1064.

[7] Arkhipova A. A., Stará J., *Regularity problem for $2m$-order quasilinear parabolic systems with non smooth in time principal matrix. $(A(t), m)$-caloric approximation method*, Topol. Methods Nonlinear Anal. **52** (2018), no. 1, 111–146.

[8] Arkhipova A. A., Stará J., John O., *Partial regularity for solutions of quasilinear parabolic systems with nonsmooth in time principal matrix*, Nonlinear Anal. **95** (2014), 421–435.

[9] Bögelein V., Duzaar F., Mingione G., *The boundary regularity of non-linear parabolic systems. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), no. 1, 201–255.

[10] Campanato S., *Equazioni paraboliche del secondo ordine e spazi $L^{2,\theta}(\Omega,\delta)$*, Ann. Mat. Pura Appl. (4) **73** (1966), 55–102 (Italian).

[11] Campanato S., *On the nonlinear parabolic systems in divergence form. Hölder continuity and partial Hölder continuity of the solutions*, Ann. Mat. Pura Appl. (4) **137** (1984), 83–122.

[12] Diening L., Lengeler D., Stroffolini B., Verde A., *Partial regularity for minimizers of quasiconvex functionals with general growth*, SIAM J. Math. Anal. **44** (2012), no. 5, 3594–3616.

[13] Diening L., Schwarzacher S., Stroffolini B., Verde A., *Parabolic Lipschitz truncation and caloric approximation*, Calc. Var. Partial Differential Equations **56** (2017), no. 4, Art. 120, 27 pages.

[14] Dong H., Kim D., *Parabolic and elliptic systems with VMO coefficients*, Methods Appl. Anal. **16** (2009), no. 3, 365–388.

[15] Dong H., Kim D., *Global regularity of weak solutions to quasilinear elliptic and parabolic equations with controlled growth*, Communic. Partial Differential Equations **36** (2011), no. 10, 1750–1777.

[16] Dong H., Kim D., *On the L_p-solvability of higher order parabolic and elliptic systems with BMO coefficients*, Arch. Ration. Mech. Anal. **199** (2011), no. 3, 889–941.

[17] Duzaar F., Grotowski J. F., *Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation*, Manuscripta Math. **103** (2000), no. 3, 267–298.

[18] Duzaar F., Mingione G., *Second order parabolic systems, optimal regularity, and singular sets of solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), no. 6, 705–751.

[19] Duzaar F., Steffen K., *Optimal interior and boundary regularity for almost minimizers to elliptic variational integrals*, J. Reine Angew. Math. **546** (2002), 73–138.

[20] Gehring F. W., *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. **130** (1973), 265–277.

[21] Giaquinta M., Giusti E., *Partial regularity for solutions to nonlinear parabolic systems*, Ann. Mat. Pura Appl. (4) **97** (1973), 253–266.

[22] Giaquinta M., Struwe M., *On the partial regularity of weak solutions of nonlinear parabolic systems*, Math. Z. **179** (1982), no. 4, 437–451.

[23] Kinnunen J., Lewis J. L., *Higher integrability for parabolic systems of p-Laplacian type*, Duke Math. J. **102** (2000), no. 2, 253–271.

[24] Krylov N. V., *Parabolic and elliptic equations with VMO coefficients*, Comm. Partial Differential Equations **32** (2007), no. 1–3, 453–475.

[25] Krylov N. V., *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, Graduate Studies in Mathematics, 96, American Mathematical Society, Providence, 2008.

[26] Ladyzhenskaya O. A., Solonnikov V. A., Uraltseva N. N., *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, 23, American Mathematical Society, Providence, 1968.

[27] Mingione G., *The singular set of solutions to non differentiable elliptic systems*, Arch. Ration. Mech. Anal. **166** (2003), no. 4, 287–301.

A. A. Arkhipova:
Mathematical and Mechanical Faculty, St. Petersburg State University, 7/9 Universitetskaya nab., 199034, St. Petersburg, Russia

E-mail: arinaark@gmail.com

J. Stará:
Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8, Czech Republic

E-mail: stara@karlin.mff.cuni.cz

(*Received* May 17, 2018, *revised* March 13, 2019)