Results of nonexistence of solutions for some nonlinear evolution problems

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Abstract. In the present paper, we prove nonexistence results for the following nonlinear evolution equation, see works of T. Cazenave and A. Haraux (1990) and S. Zheng (2004),

$$u_{tt} + f(x)u_t + (-\Delta)^{\alpha/2}(u^m) = h(t,x)|u|^p,$$

posed in $(0,T)\times\mathbb{R}^N$, where $(-\Delta)^{\alpha/2}$, $0<\alpha\le 2$ is $\alpha/2$-fractional power of $-\Delta$. Our method of proof is based on suitable choices of the test functions in the weak formulation of the sought solutions. Then, we extend this result to the case of a 2×2 system of the same type.

Keywords: nonexistence; test functions; global weak solution; fractional Laplacian; critical exponent

Classification: 47J35, 35A01, 35D30

1. Introduction

In this article, we are concerned with the following problem:

$$\text{(1.1)}\qquad \begin{cases} u_{tt} + f(x)u_t + (-\Delta)^{\alpha/2}(|u|^m) = h(t,x)|u|^p, & (t,x)\in(0,T)\times\mathbb{R}^N,\\ u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^N \end{cases}$$

for some $0<T\le\infty$, where $(-\Delta)^{\alpha/2}$ with $0<\alpha\le 2$ is the fractional power of the $(-\Delta)$, $p>1$ and $1\le m<p$.

The integral representation of the fractional Laplacian in the N-dimensional space is

$$\text{(1.2)}\qquad (-\Delta)^{\beta/2}\psi(x) = -c_N(\beta)\int_{\mathbb{R}^N}\frac{\psi(x+z)-\psi(x)}{|z|^{N+\beta}}\,\mathrm{d}z \qquad \text{for all } x\in\mathbb{R}^N,$$

where $c_N(\beta)=\Gamma((N+\beta)/2)/(2\pi^{N/2+\beta}\Gamma(1-\beta/2))$, and Γ denotes the gamma function, see [10].

Note that the fractional Laplacian $((-\Delta)^{\alpha/2})$, see [8], [10], with $\alpha\in(0;2]$ is a pseudo-differential operator defined by:

$$(-\Delta)^{\alpha/2}u(x) = \mathcal{F}^{-1}\{|\zeta|^{\alpha}\mathcal{F}(u)(\zeta)\}(x) \qquad \text{for all } x\in\mathbb{R}^N,$$

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where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are Fourier transform and its inverse, respectively. Set $\Sigma_T = (0,T)\times(\mathbb{R}^N)$.

Before beginning this work, let us point out that many authors were interested in studying the following Cauchy problem for a nonlinear wave equation with damping term:

$$\begin{cases} u_{tt}+u_t-\Delta u = |u|^p, & (t,x)\in(0,\infty)\times\mathbb{R}^N,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^N. \end{cases} \tag{1.3}$$

G. Todorova and B. Yordanov in [13] showed that, if $p_c < p \le N/(N-2)$ for $n \ge 3$ and $p_c < p < \infty$ for $N = 1,2$, where $p_c = 1+2/N$, then (1.3) subjected to initial data $u(0,x) = \varepsilon u_0(x)$, $u_t(0,x) = \varepsilon u_1(x)$, $\varepsilon > 0$, $x \in \mathbb{R}^N$, admits a unique global solution, and they proved that if $1 < p < 1+2/N$, then the solution u blows up in a finite time.

Q. Zhang in [14] studied the case $1 < p < 1+2/N$, when u_i, $i = 0,1$, is compactly supported and $\int u_i(x)\,\mathrm{d}x > 0$, he proved that global solution of (1.3) does not exist. Therefore, he showed that $p = 1+2/N$ belongs to the blow-up case.

Let us point that T. Ogawa and H. Takeda in [9] showed that when $1 < p < 1+2/N$ and the support of data is not far away from the obstacle, then the weak solution of (1.3) does not exist globally, but if the supports of the initial data are sufficiently away from the boundary, they treated the problem as in the Cauchy problem.

A. Z. Fino, H. Ibrahim and A. Wehbe in [2] generalized the results of T. Ogawa and H. Takeda in [9] by proving the blow-up of solutions of (1.3) under weaker assumptions on the initial data and they extended this results to the critical case $p = 1+2/N$. Observe that A. Hakem in [6] treated the problem:

$$\begin{cases} u_{tt}+g(t)u_t-\Delta u = |u|^p, & (t,x)\in(0,\infty)\times\mathbb{R}^N,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.4}$$

as a generalized problem of (1.3), where $g(t)$ is a function behaving like t^β, $0 \le \beta < 1$. He obtained the non-existence of weak solution for the problem (1.4), when $1 < p \le 1+(N+2)(N+2\beta)$.

It should be noted that F. Sun and M. Wang in [11] worked on the system:

$$\begin{cases} u_{tt}-\Delta(u)+u_t = |v|^p, & (t,x)\in(0,\infty)\times\mathbb{R}^N,\\ v_{tt}-\Delta(v)+v_t = |u|^q, & (t,x)\in(0,\infty)\times\mathbb{R}^N,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \\ v(0,x)=v_0(x), \quad v_t(0,x)=v_1(x), \end{cases} \tag{1.5}$$

where $p,q \ge 1$ and satisfy $pq > 1$. They showed that if $\max\{(1+p)/(pq-1), (1+q)/(pq-1)\} \ge N/2$ for $N \ge 1$, then every solution with initial data having positive average value does not exist globally.

Our purpose of this work is to generalize some of the above results, so in the first part of our research and with the suitable choice of the test function, we prove the non-existence of nontrivial global weak solution of (1.1), and in the second part we extend the results of A. Hakem's work [6] to the fractional Laplacian, see [4], [7], [8], [10]. The same technique is used to prove the non-existence of solutions to the system:

$$\begin{cases} u_{tt} + (-\Delta)^{\alpha/2}(u) + f(x)u_t = h(t,x)|v|^{p+1}, & (t,x) \in (0,\infty) \times \mathbb{R}^N, \\ v_{tt} + (-\Delta)^{\beta/2}(v) + g(x)v_t = h(t,x)|u|^{q+1}, & (t,x) \in (0,\infty) \times \mathbb{R}^N, \end{cases} \tag{1.6}$$

subjected to the conditions

$$\begin{aligned} u(0,x) &= u_0(x), \qquad & u_t(0,x) &= u_1(x), \\ v(0,x) &= v_0(x), \qquad & v_t(0,x) &= v_1(x). \end{aligned}$$

The results of our research are based on the following definitions:

Definition 1.1. We say that u is a local weak solution to (1.1), defined in Σ_T, $0 < T < \infty$, if u is a locally integrable function such that $u^p h \in L^1_{\mathrm{loc}}(\Sigma_T)$ and

$$\begin{aligned} &\int_{\Sigma_T} h|u|^p \Psi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} f(x)u_0(x)\Psi(0,x)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^N} u_0(x)\Psi_t(0,x)\,\mathrm{d}x \\ &= \int_{\Sigma_T} u\Psi_{tt}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Sigma_T} f(x)u\Psi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{\Sigma_T} |u|^m(-\Delta)^{\alpha/2}\Psi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

is satisfied for any $\Psi \in C_0^\infty(\overline{\Sigma_T})$ which vanishes for large $|x|$ and at $t = T$.

Definition 1.2. We say that u is a global weak solution to (1.1) if it is a local solution to (1.1) defined in Σ_T for any $T > 0$, see [5].

The integrals in the above definition are supposed to be convergent.

2. Nonexistence results for nonlinear evolution equation

We consider the following Cauchy problem:

$$\begin{cases} u_{tt} + f(x)u_t + (-\Delta)^{\alpha/2}|u|^m = h(t,x)|u|^p, & (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N \end{cases} \tag{2.1}$$

for some $0 < T \le \infty$, where $(-\Delta)^{\alpha/2}$ with $0 < \alpha \le 2$ is the fractional power of the $-\Delta$, $p > 1$, $m \ge 1$, and the functions f and h are non-negative and satisfy the conditions:

- $f \in L^\infty(\mathbb{R}^N)$.

- For every compact $\Omega \subset \mathbb{R}_+ \times \mathbb{R}^N$, there exists a real $l \geq 0$ such that with $(t,x) \in \Omega$ we have:

$$\begin{aligned} h(t,x) = O(R^l) \quad \text{and} \quad h(t,x) \ &\text{behave like } Ct^\mu |x|^\nu, \\ &\text{where } \ R > 0 \ \text{ large }, \ C > 0. \end{aligned} \tag{2.2}$$

The assumption on the positive real numbers R, l, μ and ν will be determined later in the proof, see bellow (2.8).

Theorem 2.1. *Assume that $1 \leq m < p$ and the conditions (2.2) are fulfilled and the initial data satisfies*

$$\int_{\mathbb{R}^N} f(x) u_0(x)\,\mathrm{d}x > 0 \qquad \textit{and} \qquad \int_{\mathbb{R}^N} u_1(x)\,\mathrm{d}x > 0. \tag{2.3}$$

If

$$\frac{(p-1)N}{\alpha} - 1 \ \leq \ l, \tag{2.4}$$

then every weak solution of the problem (2.1) does not exist globally in time.

Proof: The proof proceeds by contradiction. Suppose u is a solution which exists globally in time. Let Φ be the test function such that

$$\Phi(r) = \begin{cases} 0 & \text{if } \ r \geq 2, \\ 1 & \text{if } \ r \leq 1, \end{cases}$$

and

$$0 \leq \Phi \leq 1, \quad |\Phi'| \leq \frac{C}{r} \qquad \text{for all } \ r > 0.$$

Now multiplying the equation (2.1) by Ψ and integrating by parts on $\Sigma_T = (0,T) \times \mathbb{R}^N$, we get

$$\begin{aligned} &\int_{\Sigma_T} h|u|^p \Psi\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} f(x) u_0(x) \Psi(0,x)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^N} u_0(x)\Psi_t(0,x)\,\mathrm{d}x \\ = &\int_{\Sigma_T} u\Psi_{tt}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Sigma_T} f(x) u \Psi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{\Sigma_T} |u|^m (-\Delta)^{\alpha/2}\Psi\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{2.5}$$

where

$$\Psi(t,x) = \Phi\Big(\frac{t^2 + |x|^{2\alpha}}{R^2}\Big), \qquad R > 0.$$

With the fact that

$$\Psi_t(t,x) = 2tR^{-2}\Phi'\Big(\frac{t^2 + |x|^{2\alpha}}{R^2}\Big),$$

we have

$$\Psi_t(0,x) = 0.$$

Thus the formula (2.5) will be on the shape

$$
\begin{aligned}
&\int_{\Sigma_T} h|u|^p\Psi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} f(x)u_0(x)\Psi(0,x)\,\mathrm{d}x + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,\mathrm{d}x \\
&= \int_{\Sigma_T} u\Psi_{tt}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Sigma_T} f(x)u\Psi_t\,\mathrm{d}x\,\mathrm{d}t + \int_{\Sigma_T} |u|^m((-\Delta)^{\alpha/2}\Psi)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.6}
$$

To estimate

$$\int_{\Sigma_T} |u|\Psi_{tt}\,\mathrm{d}x\,\mathrm{d}t,$$

we observe that

$$\int_{\Sigma_T} |u|\Psi_{tt}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Sigma_T} |u|(h\Psi)^{1/p}\Psi_{tt}(h\Psi)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t,$$

we have also

$$\int_{\Sigma_T} f(x)|u|\Psi_t\,\mathrm{d}x\,\mathrm{d}t = \int_{\Sigma_T} f(x)|u|(h\Psi)^{1/p}\Psi_t(h\Psi)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\int_{\Sigma_T} |u|^m((-\Delta)^{\alpha/2}\Psi)\,\mathrm{d}x\,\mathrm{d}t = \int_{\Sigma_T} |u|^m(h\Psi)^{m/p}((-\Delta)^{\alpha/2}\Psi)(h\Psi)^{-m/p}\,\mathrm{d}x\,\mathrm{d}t.$$

An application of the following ε-Young's inequality

$ab \le \varepsilon a^p + C(\varepsilon)b^q$ where $a>0$, $b>0$, $\varepsilon>0$, $pq=p+q$ and $C(\varepsilon)=(\varepsilon p)^{-q/p}q^{-1}$,

in the first integral of the right hand side of (2.6), we obtain

$$\int_{\Sigma_T} |u|\Psi_{tt}\,\mathrm{d}x\,\mathrm{d}t \le \varepsilon\int_{\Sigma_T} |u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t + C(\varepsilon)\int_{\Sigma_T} |\Psi_{tt}|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t,$$

in the second integral of the right hand side of (2.6), we get

$$
\begin{aligned}
&\int_{\Sigma_T} f(x)|u|\Psi_t\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \varepsilon\int_{\Sigma_T} |u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t + C(\varepsilon)\int_{\Sigma_T} (f(x)|\Psi_t|)^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \varepsilon\int_{\Sigma_T} |u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t + \widetilde{C}(\varepsilon)\int_{\Sigma_T} |\Psi_t|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

where $\widetilde{C}(\varepsilon) = C(\varepsilon)\|f(x)\|_\infty^{p/(p-1)}$. And in the third integral of the right hand side of (2.6), we have

$$\begin{aligned}\Big|\int_{\Sigma_T} |u|^m (-\Delta)^{\alpha/2}\Psi \,\mathrm{d}x\,\mathrm{d}t\Big| \\ \leq \varepsilon \int_{\Sigma_T} |u|^p h\Psi \,\mathrm{d}x\,\mathrm{d}t + C(\varepsilon)\int_{\Sigma_T} |(-\Delta)^{\alpha/2}(\Psi)|^{p/(p-m)}(h\Psi)^{-m/(p-m)}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Finally, we get

$$\begin{aligned}\int_{\Sigma_T} |u|^p h\Psi \,\mathrm{d}x\,\mathrm{d}t \leq C_1 \int_{\Sigma_T} |\Psi_{tt}|^{p-(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t \\ + C_2 \int_{\Sigma_T} |\Psi_t|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t \\ + C_3 \int_{\Sigma_T} |(-\Delta)^{\alpha/2}(\Psi)|^{p/(p-m)}(h\Psi)^{-m/(p-m)}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.7}$$

At this stage, we introduce the scaled variables $\tau = tR^{-1}$, $\zeta = xR^{-1/\alpha}$ and use the fact that $\Psi_t = R^{-1}\Psi_\tau$, $\Psi_{tt} = R^{-2}\Psi_{\tau\tau}$, $(-\Delta)_x^{\alpha/2}\Psi = R^{-1}(-\Delta)_\zeta^{\alpha/2}\Psi$, and also

$$h(t,x) = h(\tau R, \zeta R^{1/\alpha}) = C\tau^\mu|\zeta|^{\nu/\alpha}R^{\mu+\nu/\alpha} = O(R^l), \quad \text{where } \ l = \mu+\nu/\alpha. \tag{2.8}$$

By setting

$$\Omega = \{(\tau,\zeta)\in\mathbb{R}^+\times\mathbb{R}^N : 1 \leq \tau^2 + |\zeta|^{2\alpha} \leq 2\}, \qquad \varphi(\tau,\zeta) = \tau^2+|\zeta|^{2\alpha},$$

we arrive at

$$\begin{aligned}\int_{\Sigma_T} |u|^p h\Psi \,\mathrm{d}x\,\mathrm{d}t \leq C_1 R^{\theta_1}\int_\Omega |(\Psi_{\tau\tau})(\varphi)|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ + C_2 R^{\theta_2}\int_\Omega |(\Psi_\tau)(\varphi)|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ + C_3 R^{\theta_3}\int_\Omega |(-\Delta)^{\alpha/2}\Psi(\varphi)|^{p/(p-m)}(h\Psi)^{-m/(p-m)}\,\mathrm{d}\zeta\,\mathrm{d}\tau\end{aligned} \tag{2.9}$$

where

$$\begin{cases}\theta_1 = \dfrac{N}{\alpha} - 1 - \dfrac{2}{p-1} - \dfrac{l}{p-1}, \\ \theta_2 = \dfrac{N}{\alpha} - \dfrac{1}{p-1} - \dfrac{l}{p-1}, \\ \theta_3 = \frac{N}{\alpha} - \dfrac{m}{p-m} - \dfrac{lm}{p-m}.\end{cases}$$

One can easily observe that $\theta_1 < \theta_2$ and $\theta_3 < \theta_2$, we infer that

$$\begin{aligned}\int_{\Sigma_T} |u|^p h\Psi \,\mathrm{d}x\,\mathrm{d}t \le CR^{\theta}\Big[\int_{\Omega} &|(\Psi_{\tau\tau})(\varphi)|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_{\Omega} |(\Psi_{\tau})(\varphi)|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}\zeta\,\mathrm{d}\tau \\ &+ \int_{\Omega} |(-\Delta)^{\alpha/2}\Psi(\varphi)|^{p/(p-m)}(h\Psi)^{-m/(p-m)}\,\mathrm{d}\zeta\,\mathrm{d}\tau\Big],\end{aligned} \tag{2.10}$$

where $R > 0$ is large and $\theta := \theta_2 = (1/(p-1))[(N(p-1)/\alpha) - 1 - l]$. We have two cases:

- If
$$\frac{N(p-1)}{\alpha} - 1 - l < 0,$$
then the right-hand side of (2.10) goes to 0 when R tends to ∞. We pass to the limit in the left hand side, as R goes to ∞; we get
$$\lim_{R\to\infty}\int_{\Sigma_T} h|u|^p\Psi\,\mathrm{d}x\,\mathrm{d}t = 0.$$
Using the Lebesgue dominated convergence theorem, the continuity in time and space of u and the fact that $\Psi(t,x) \to 1$ as $R \to \infty$, we infer that
$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h|u|^p\,\mathrm{d}x\,\mathrm{d}t = 0.$$
Therefore, if u exists then necessarily $u \equiv 0$ a.e. on $\mathbb{R}^+ \times \mathbb{R}^N$. This is a contradiction to the assumptions (2.3).
- If
$$\frac{N(p-1)}{\alpha} - 1 - l = 0,$$
then we have
$$\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p h\,\mathrm{d}x\,\mathrm{d}t < \infty. \tag{2.11}$$
By using (2.6) we obtain
$$\begin{aligned}\int_{\Sigma_T} h|u|^p\Psi\,\mathrm{d}x\,\mathrm{d}t &+ \int_{\mathbb{R}^N} f(x)u_0(x)\Psi(0,x)\,\mathrm{d}x + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,\mathrm{d}x \\ &\le \int_{\Sigma_T} |u|(h\Psi)^{1/p}|\Psi_{tt}|(h\Psi)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\Sigma_T} |u|(h\Psi)^{1/p} f(x)|\Psi_t|(h\Psi)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\Sigma_T} |u|^m(h\Psi)^{m/p}|(-\Delta)^{\alpha/2}\Psi|(h\Psi)^{-m/p}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{2.12}$$

Accordingly, using Hölder's inequality in the right hand side of (2.12), yields

$$\begin{aligned}
&\int_{\Sigma_T} h|u|^p\Psi\,\mathrm{d}x\,\mathrm{d}t\\
&\leq \Big(\int_{\Sigma_T}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/p}\Big(\int_{\Sigma_T}|\Psi_{tt}|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-1)/p}\\
&\quad+\Big(\int_{\Sigma_T}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/p}\Big(\int_{\Sigma_T}(f(x)|\Psi_t|)^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-1)/p}\\
&\quad+\Big(\int_{\Sigma_T}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{m/p}\Big(\int_{\Sigma_T}(|(-\Delta)^{\alpha/2}\Psi|)^{p/(p-m)}(h\Psi)^{-m/(p-m)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-m)/p}.
\end{aligned}$$

Let

$$\Big(\int_{\Sigma_T}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{\kappa/p}=\max\Big\{\Big(\int_{\Sigma_T}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/p},\Big(\int_{\Sigma_T}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{m/p}\Big\},\qquad \kappa=1\quad\text{or}\quad\kappa=m,$$

we obtain

$$\begin{aligned}
&\int_{\Sigma_T} h|u|^p\Psi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq \Big(\int_{\Sigma_T}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{\kappa/p}\Big[\Big(\int_{\Sigma_T}|\Psi_{tt}|^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-1)/p}\\
&\qquad\quad+\Big(\int_{\Sigma_T}(f(x)|\Psi_t|)^{p/(p-1)}(h\Psi)^{-1/(p-1)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-1)/p}\\
&\qquad\quad+\Big(\int_{\Sigma_T}(|(-\Delta)^{\alpha/2}\Psi|)^{p/(p-m)}(h\Psi)^{-m/(p-m)}\,\mathrm{d}x\,\mathrm{d}t\Big)^{(p-m)/p}\Big].
\end{aligned}$$

Because $N(p-1)/\alpha-1-l=0$, we get from (2.11) that

$$\begin{aligned}
&\int_{\Sigma_T} h|u|^p\Psi\,\mathrm{d}x\,\mathrm{d}t\\
&\leq \Big(\int_{\Omega_2}|u|^p h\Psi\,\mathrm{d}x\,\mathrm{d}t\Big)^{\kappa/p}\Big[\Big(\int_{\Omega_1}|\Psi_{\tau\tau}(\varphi)|^{p/(p-1)}(h\Psi(\varphi))^{-1/(p-1)}\,\mathrm{d}\zeta\,\mathrm{d}\tau\Big)^{(p-1)/p}\\
&\quad+\Big(\int_{\Omega_1}(f(\zeta)|\Psi_\tau(\varphi)|)^{p/(p-1)}(h\Psi(\varphi))^{-1/(p-1)}\,d\zeta\,\mathrm{d}\tau\Big)^{(p-1)/p}\\
&\quad+\Big(\int_{\Omega_1}(|(-\Delta)^{\alpha/2}\Psi(\varphi)|)^{p/(p-m)}(h\Psi(\varphi))^{-m/(p-m)}\,\mathrm{d}\zeta\,\mathrm{d}\tau\Big)^{(p-m)/p}\Big],
\end{aligned}$$

where

$$\Omega_1=\{(\tau,\zeta)\in\mathbb{R}^+\times\mathbb{R}^N\colon 1\leq\tau^2+|\zeta|^{2\alpha}\leq 2\},$$

and
$$\Omega_2 = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \colon R^2 \leq t^2 + |x|^{2\alpha} \leq 2R^2\}.$$
Taking into account the fact that
$$\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p h \,\mathrm{d}x\,\mathrm{d}t < \infty,$$
we obtain
$$\lim_{R\to\infty} \int_{\Omega_2} |u|^p h\Psi \,\mathrm{d}x\,\mathrm{d}t = 0,$$
hence, we conclude that
$$\int_{\mathbb{R}^+\times\mathbb{R}^N} |u|^p h \,\mathrm{d}x\,\mathrm{d}t = 0.$$
Whereupon, $u \equiv 0$. This is also a contradiction. We deduce that no global solution to (2.1) is possible. This finishes the proof. □

Remark 2.2. We can observe that in the case $l = 0$ and $\alpha = 2$, we retrieve the critical exponent $p_c = 1 + 2/N$, see [3].

We conclude this section by the study of the inhomogeneous equation:
(2.13)
$$\begin{cases} u_{tt} + f(x)u_t + (-\Delta)^{\alpha/2}(|u|^m) = h(t,x)|u|^p + \varrho w(x), & (t,x) \in (0,T)\times\mathbb{R}^N, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^N, \end{cases}$$
where $\varrho > 0$, $u_0 \in L^1(\mathbb{R}^N)$, $u_1 \in L^1(\mathbb{R}^N)$ and the function w is positive and $w \not\equiv 0$.

Theorem 2.3. *Assume that the hypotheses of Theorem* 2.1 *are satisfied, then the problem* (2.13) *does not admit global solutions for* ϱ *large.*

Proof: All as in the proof of Theorem 2.1, we obtain
$$\begin{aligned} \int_{\Sigma_T} h|u|^p\Psi \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} f(x)u_0(x)\Psi(0,x)\,\mathrm{d}x + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,\mathrm{d}x \\ + \varrho\int_{\Sigma_T} w(t,x)\Psi\,\mathrm{d}x\,\mathrm{d}t \leq CR^\theta. \end{aligned} \tag{2.14}$$
For R large, where
$$\theta = \frac{1}{p-1}\Big(\frac{N(p-1)}{\alpha} - 1 - l\Big).$$
First of all we suppose that $w \in L^1(\mathbb{R}^+\times\mathbb{R}^N)$. Therefore, we obtain the inequality
$$\int_{\mathbb{R}^N} f(x)u_0(x)\Psi(0,x)\,\mathrm{d}x + \int_{\mathbb{R}^N} u_1(x)\Psi(0,x)\,\mathrm{d}x + \varrho\int_{\mathbb{R}^+\times\mathbb{R}^N} w(x)\Psi\,\mathrm{d}x\,\mathrm{d}t \leq C,$$

which is impossible if

$$\varrho > \frac{\|f\|_{L^\infty(\mathbb{R}^N)}\|u_0\|_{L^1(\mathbb{R}^N)} + \|u_1\|_{L^1(\mathbb{R}^N)} + C}{\|w\|_{L^1(\mathbb{R}^+\times\mathbb{R}^N)}}.$$

But if $\|w\|_1 = \infty$, then we arrive again at a contradiction with (2.14) for all $\varrho > 0$. $\square$

3. Case of system of equations

In this section we consider the problem

$$\text{(3.1)} \quad \begin{cases} u_{tt} + (-\Delta)^{\alpha/2}(u) + f(x)u_t = h(t,x)|v|^{p+1}, & (t,x) \in (0,\infty)\times\mathbb{R}^N, \\ v_{tt} + (-\Delta)^{\beta/2}(v) + g(x)v_t = h(t,x)|u|^{q+1}, & (t,x) \in (0,\infty)\times\mathbb{R}^N, \end{cases}$$

subjected to the conditions

$$\begin{aligned} u(0,x) &= u_0(x), \quad u_t(0,x) = u_1(x), \\ v(0,x) &= v_0(x), \quad v_t(0,x) = v_1(x), \end{aligned}$$

where $p > 0$, $q > 0$, $0 < \alpha \le 2$, $0 < \beta \le 2$, $h > 0$, f, g are positive functions and $f \in L^\infty(\mathbb{R}^N)$, $g \in L^\infty(\mathbb{R}^N)$.

Suppose that the function h satisfied (2.2) and the assumption on the positive real numbers l, μ and ν will be determined later in the proof, see bellow (3.11).

Using the same reasoning as above, one gets the following assertion.

Theorem 3.1. *Let*

$$p^\star := \frac{(p+1)[\alpha(q+1)+\beta] + \sigma[l(p+2) - (p+1)(q+1) + 1]}{(p+1)(q+1)-1},$$

and

$$q^\star := \frac{(q+1)[\beta(p+1)+\alpha] + \sigma[l(q+2) - (p+1)(q+1) + 1]}{(p+1)(q+1)-1},$$

where $p > 0$, $q > 0$ and $\sigma = \max\{\alpha, \beta\}$. Assume that the condition (2.2) is fulfilled and the initial data satisfies

$$\text{(3.2)} \quad \begin{aligned} &\int_{\mathbb{R}^N} f(x)u_0(x)\,\mathrm{d}x > 0, \qquad \int_{\mathbb{R}^N} u_1(x)\,\mathrm{d}x > 0 \qquad \textit{and} \\ &\int_{\mathbb{R}^N} g(x)v_0(x)\,\mathrm{d}x > 0, \qquad \int_{\mathbb{R}^N} v_1(x)\,\mathrm{d}x > 0. \end{aligned}$$

If

$$N \le \max\{p^\star; q^\star\},$$

then the solution $(u(t,x), v(t,x))$ of problem (3.1) does not exist globally.

Proof: We notice that, in all steps of proof, $C > 0$ is a real positive number which may change from line to line.

Set $\zeta(t,x) = \Phi(t^2 + |x|^{2\sigma}/R^2)$, where $\Phi \in C_c^\infty(\mathbb{R}^+)$ satisfies $0 \le \Phi \le 1$ and

$$\Phi(r) = \begin{cases} 1 & \text{if } \ r \le 1, \\ 0 & \text{if } \ r \ge 2. \end{cases}$$

Multiplying the first equation of (3.1) by ζ and integrating by parts on $Q_T = (0,T) \times \mathbb{R}^N$, we get

$$\begin{aligned} &\int_{Q_T} h|v|^{p+1}\zeta \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} f(x)u_0(x)\zeta(0,x)\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} u_1(x)\zeta(0,x)\,\mathrm{d}x - \int_{\mathbb{R}^N} u_0(x)\zeta_t(0,x)\,\mathrm{d}x \\ = &\int_{Q_T} u\zeta_{tt}\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} f(x)u\zeta_t\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u(-\Delta)^{\alpha/2}\zeta\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.3}$$

With the fact that $\zeta_t(0,x) = 0$, we obtain

$$\begin{aligned} &\int_{Q_T} h|v|^{p+1}\zeta \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^N} f(x)u_0(x)\zeta(0,x)\,\mathrm{d}x + \int_{\mathbb{R}^N} u_1(x)\zeta(0,x)\,\mathrm{d}x \\ &\quad = \int_{Q_T} u\zeta_{tt}\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} f(x)u\zeta_t\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} u(-\Delta)^{\alpha/2}\zeta\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} &\int_{Q_T} h|v|^{p+1}\zeta \,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{Q_T} |u||\zeta_{tt}|\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} f(x)|u||\zeta_t|\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} |u||(-\Delta)^{\alpha/2}\zeta|\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

We have also

$$\begin{aligned} &\int_{Q_T} h|u|^{q+1}\zeta \,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{Q_T} |v||\zeta_{tt}|\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} g(x)|v||\zeta_t|\,\mathrm{d}x\,\mathrm{d}t + \int_{Q_T} |v||(-\Delta)^{\beta/2}\zeta|\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.6}$$

To estimate

$$\int_{Q_T} |u||\zeta_{tt}|\,\mathrm{d}x\,\mathrm{d}t,$$

we observe that it can be rewritten as

$$\int_{Q_T} |u||\zeta_{tt}|\,\mathrm{d}x\,\mathrm{d}t = \int_{Q_T} |u|(h\zeta)^{1/(q+1)}|\zeta_{tt}|(h\zeta)^{-1/(q+1)}\,\mathrm{d}x\,\mathrm{d}t.$$

Using Hölder's inequality, we obtain

$$\begin{aligned}\int_{Q_T} |u||\zeta_{tt}| \,\mathrm{d}x\,\mathrm{d}t \\ \leq \Big(\int_{Q_T} |u|^{q+1}(h\zeta)\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/(q+1)}\Big(\int_{Q_T} |\zeta_{tt}|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t\Big)^{q/(q+1)}.\end{aligned}$$

Proceeding as above, we have

$$\begin{aligned}&\int_{Q_T} f|u||\zeta_t| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \Big(\int_{Q_T} |u|^{q+1}(h\zeta)\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/(q+1)}\Big(\int_{Q_T} f^{(q+1)/q}|\zeta_t|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t\Big)^{q/(q+1)},\end{aligned}$$

and

$$\begin{aligned}&\int_{Q_T} |u||(-\Delta)^{\alpha/2}\zeta| \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \Big(\int_{Q_T} |u|^{q+1}(h\zeta)\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/(q+1)}\Big(\int_{Q_T} |(-\Delta)^{\alpha/2}\zeta|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t\Big)^{q/(q+1)}.\end{aligned}$$

Finally, we infer

$$\int_{Q_T} h|v|^{p+1}\zeta\,\mathrm{d}x\,\mathrm{d}t \leq \Big(\int_{Q_T} |u|^{q+1}(h\zeta)\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/(q+1)}\mathcal{K}_q, \tag{3.7}$$

where

$$\begin{aligned}\mathcal{K}_q = &\Big(\int_{Q_T} |\zeta_{tt}|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t\Big)^{q/(q+1)} \\ &+ \Big(\int_{Q_T} f^{(q+1)/q}|\zeta_t|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t\Big)^{q/(q+1)} \\ &+ \Big(\int_{Q_T} |(-\Delta)^{\alpha/2}\zeta|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t\Big)^{q/(q+1)}.\end{aligned}$$

Arguing as above we have likewise

$$\int_{Q_T} h|u|^{q+1}\zeta\,\mathrm{d}x\,\mathrm{d}t \leq \Big(\int_{Q_T} |v|^{p+1}(h\zeta)\,\mathrm{d}x\,\mathrm{d}t\Big)^{1/(p+1)}\mathcal{L}_p, \tag{3.8}$$

where

$$\mathcal{L}_p = \Big(\int_{Q_T} |\zeta_{tt}|^{(p+1)/p}(h\zeta)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t\Big)^{p/(p+1)} + \Big(\int_{Q_T} g^{(p+1)/p}|\zeta_t|^{(p+1)/p}(h\zeta)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t\Big)^{p/(p+1)} + \Big(\int_{Q_T} |(-\Delta)^{\beta/2}\zeta|^{(p+1)/p}(h\zeta)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t\Big)^{p/(p+1)}.$$

By substituting (3.8) in (3.7), it yields

$$\Big(\int_{Q_T} h|v|^{p+1}\zeta\,\mathrm{d}x\,\mathrm{d}t\Big)^{((p+1)(q+1)-1)/((p+1)(q+1))} \le \mathcal{K}_q\mathcal{L}_p^{1/(q+1)}. \tag{3.9}$$

Similarly, we get

$$\Big(\int_{Q_T} h|u|^{q+1}\zeta\,\mathrm{d}x\,\mathrm{d}t\Big)^{((p+1)(q+1)-1)/((p+1)(q+1))} \le \mathcal{L}_p\mathcal{K}_q^{1/(p+1)}. \tag{3.10}$$

Now we consider the scale of variables

$$t = \tau R, \quad x = yR^{1/\sigma}, \qquad \text{where } \sigma = \max\{\alpha, \beta\},$$

and taking into account the fact that $f \in L^\infty(\mathbb{R}^N)$, $g \in L^\infty(\mathbb{R}^N)$ and

$$h(t,x) = h(\tau R, yR^{1/\sigma}) = C\tau^\mu|y|^{\nu/\sigma}R^{\mu+\nu/\sigma} = O(R^l), \qquad \text{where } l = \mu + \frac{\nu}{\sigma}, \tag{3.11}$$

namely we have

$$\int_{Q_T} f^{(q+1)/q}|\zeta_t|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t \le C\int_{Q_T} |\zeta_t|^{(q+1)/q}(h\zeta)^{-1/q}\,\mathrm{d}x\,\mathrm{d}t, \qquad C > 0,$$

and

$$\int_{Q_T} g^{(p+1)/p}|\zeta_t|^{(p+1)/p}(h\zeta)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t \le C\int_{Q_T} |\zeta_t|^{(p+1)/p}(h\zeta)^{-1/p}\,\mathrm{d}x\,\mathrm{d}t, \qquad C > 0.$$

We easily deduce that

$$\Big(\int_{Q_T} h|v|^{p+1}\zeta\,\mathrm{d}x\,\mathrm{d}t\Big)^{((p+1)(q+1)-1)/((p+1)(q+1))} \le C\big[R^{\gamma_1} + R^{\gamma_2} + R^{\gamma_3}\big] \times \big[R^{\lambda_1} + R^{\lambda_2} + R^{\lambda_3}\big]^{1/(q+1)}. \tag{3.12}$$

Similarly, we have

$$
\left(\int_{Q_T} h|u|^{q+1}\zeta \,\mathrm{d}x\,\mathrm{d}t\right)^{((p+1)(q+1)-1)/((p+1)(q+1))} \leq C\big[R^{\lambda_1}+R^{\lambda_2}+R^{\lambda_3}\big]\times\big[R^{\gamma_1}+R^{\gamma_2}+R^{\gamma_3}\big]^{1/(p+1)}, \tag{3.13}
$$

where

$$
\gamma_1 = \Big(\frac{N}{\sigma}+1\Big)\frac{q}{q+1} - 2 - \frac{l}{q+1},
$$
$$
\gamma_2 = \Big(\frac{N}{\sigma}+1\Big)\frac{q}{q+1} - 1 - \frac{l}{q+1},
$$
$$
\gamma_3 = \Big(\frac{N}{\sigma}+1\Big)\frac{q}{q+1} - \frac{\alpha}{\sigma} - \frac{l}{q+1},
$$

and

$$
\lambda_1 = \Big(\frac{N}{\sigma}+1\Big)\frac{p}{p+1} - 2 - \frac{l}{p+1},
$$
$$
\lambda_2 = \Big(\frac{N}{\sigma}+1\Big)\frac{p}{p+1} - 1 - \frac{l}{p+1},
$$
$$
\lambda_3 = \Big(\frac{N}{\sigma}+1\Big)\frac{p}{p+1} - \frac{\beta}{\sigma} - \frac{l}{p+1}.
$$

We remark that $\gamma_1 < \gamma_2 \leq \gamma_3$ and $\lambda_1 < \lambda_2 \leq \lambda_3$, hence

$$
\left(\int_{Q_T} h|v|^{p+1}\zeta \,\mathrm{d}x\,\mathrm{d}t\right)^{((p+1)(q+1)-1)/((p+1)(q+1))} \leq CR^{\gamma_3+\lambda_3/(q+1)}, \tag{3.14}
$$

and

$$
\left(\int_{Q_T} h|u|^{q+1}\zeta \,\mathrm{d}x\,\mathrm{d}t\right)^{((p+1)(q+1)-1)/((p+1)(q+1))} \leq CR^{\lambda_3+\gamma_3/(p+1)}. \tag{3.15}
$$

We conclude that

- If $\gamma_3 + \lambda_3/(q+1) < 0$, the right hand side of (3.14) goes to 0, when R tends to ∞, while the left hand side converges to

$$
\left(\int_{\mathbb{R}^+\times\mathbb{R}^N} h|v|^{p+1} \,\mathrm{d}x\,\mathrm{d}t\right)^{((p+1)(q+1)-1)/((p+1)(q+1))}.
$$

  This implies that $v \equiv 0$ and hence $u \equiv 0$. We arrive at a contradiction with (3.2).
- If $\gamma_3 + \lambda_3/(q+1) = 0$, we get

$$
\left(\int_{\mathbb{R}^+\times\mathbb{R}^N} h|v|^{p+1} \,\mathrm{d}x\,\mathrm{d}t\right) < \infty.
$$

Using again Hölder's inequality we obtain

$$\int_{Q_T} h|u|^{q+1}\zeta \,\mathrm{d}x\,\mathrm{d}t \le \left(\int_{B_R} |v|^{p+1}(h\zeta)\,\mathrm{d}x\,\mathrm{d}t\right)^{1/(p+1)} \mathcal{L}_p,$$

where

$$B_R = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N :\ R^2 \le t^2 + |x|^{2\theta} \le 2R^2\}.$$

Since

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h|v|^{p+1}\,\mathrm{d}x\,\mathrm{d}t < \infty,$$

we get

$$\lim_{R\to\infty} \int_{B_R} |v|^{p+1} h\zeta \,\mathrm{d}x\,\mathrm{d}t = 0,$$

hence, we infer that

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h|u|^{q+1}\,\mathrm{d}x\,\mathrm{d}t = 0,$$

this leads to $u \equiv 0$ a.e. on $\mathbb{R}^+ \times \mathbb{R}^N$, which contradicts our assumption (3.2). This completes the proof.

□

Remark 3.2. We notice that, in the case where $l = 0$, $\alpha = \beta = 2$, we obtain the same result of A. Hakem, when $g(t), f(t)$ behave like t^β, t^α and $\beta = \alpha = 0$, see [6] in Section 5 and [12]. Also we recover the case studied by F. Sun and M. Wang, see [11].

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