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On some properties of the upper central series in Leibniz algebras

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Abstract. This article discusses the Leibniz algebras whose upper hypercenter has finite codimension. It is proved that such an algebra L includes a finite dimensional ideal K such that the factor-algebra L/K is hypercentral. This result is an extension to the Leibniz algebra of the corresponding result obtained earlier for Lie algebras. It is also analogous to the corresponding results obtained for groups and modules.

Keywords: Leibniz algebra; Lie algebra; center; central serie; hypercenter; nilpotent residual

Classification: 17A32, 17A60, 17A99

Some algebraic structures, which at first glance look very far from each other, in fact have common deep connections. These relationships are expressed in the presence of objects that can be called analogs, and in the presence of properties of these analogs, which in one sense or another, can also be called similar. As a well-known example, one can point out the connection between different types of generalized nilpotent groups and Lie algebras. Nilpotency is a concept that arises (in one form or another) in different algebraic structures. In turn, nilpotency entails the existence of specific series of substructure—the central series, and hence the existence of such an object as a center. The concept of center is also connected with commutativity: it is an indicator of the noncommutativity of the structure (the difference between the algebraic structure and its center shows how close or far from a commutative structure it is). Therefore, the concept of a center arises in different ways in distinct algebraic structures. In groups, associative rings, and associative algebras, the center is the set of all those elements that commute with any other element. In nonassociative rings and algebras this concept is transformed. Thus, in a Lie algebra L, the center is the set of all those elements z for which $[z, x] = 0$ for every element $x \in L$. In a module M over a ring R, the center is the set of all elements $a \in M$ such that $\alpha a = a$ for all elements $\alpha \in R$, and so on. The following question naturally arises: what properties associated with the center in one of such algebraic structures have their analogs in other algebraic structures? In other words, what properties are common, i.e. which have some deep nature?

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One of the important results of the theory of infinite groups which became the basis for a whole direction, is the following theorem:

If the center $\zeta(G)$ of a group G has finite index, then the derived subgroup $[G, G]$ is finite.

This result was proved by B. H. Neumann in the paper [13], but was named Schur's theorem by P. Hall. This result was discussed in many works and led to natural questions, one of which was the following:

Which properties of the factor group over the center are carried over to its commutator subgroup?

A series of works, among which we can mention [7], [11], [2], were dedicated to the development of this topic. Another direction in the development of this result is connected with the replacement of the center by hypercenters. The first result here, derived from the work of B. H. Neumann, was due to R. Baer, see [1], where he proved the following:

If the factor-group $G/\zeta_k(G)$ is finite, where $\zeta_k(G)$ is the kth term of the upper central series, then the $(k+1)$st term of the lower central series is also finite.

Baer's result was extended in various ways. One of the final is the following statement, see [6], [9]:

If the upper hypercenter of G has finite index, then G includes a finite normal subgroup K such that G/K is hypercentral.

This result also had further extensions, see for example [4], [5]. The above mentioned fundamental results and other results of this topic have analogs in other algebraic structures, see for example [12]. In particular, they have analogs also in Lie algebras. These analogs were obtained in [10]. A natural and essential generalization of Lie algebras are Leibniz algebras.

Let L be an algebra over a field F with the binary operations $+$ and $[\,,]$. Then L is called a Leibniz algebra (more precisely a left Leibniz algebra) if it satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

for all $a, b, c \in A$.

If L is a Lie algebra, then L is a Leibniz algebra. Conversely, if L is a Leibniz algebra such that $[a, a] = 0$ for each element $a \in L$, then L is a Lie algebra. Therefore, Lie algebras can be characterized as the Leibniz algebras in which $[a, a] = 0$ for every element a. In other words, Lie algebras can be described as anticommutative Leibniz algebras.

This shows that the differences between Lie algebras and Leibniz algebras are very significant. An analogy with groups is relevant here, differences of approximately the same nature exist between abelian and nonabelian groups. The very first example that shows this is cyclic algebras. Cyclic Lie algebras are abelian and have dimension 1. The situation with cyclic Leibniz algebras is significantly complicated, as can be seen from the results of [3] where their structure was described.

A Leibniz algebra L has one important ideal. Denote by $\mathrm{Leib}(L)$ the subspace, generated by the elements $[a, a]$, $a \in L$. It is possible to prove that $\mathrm{Leib}(L)$ is an ideal of L. Moreover, $L/\operatorname{Leib}(L)$ is a Lie algebra. Conversely, if H is an ideal of L such that L/H is a Lie algebra, then $\operatorname{Leib}(L) \leq H$.

The ideal $\operatorname{Leib}(L)$ *is called the Leibniz kernel of the algebra* L.

With regard to the left center it can be mentioned in passing that it is an ideal because it is the kernel of the Leibniz algebra homomorphism $\Phi\colon L \to \mathbf{ad}(L)$; $x \to \mathbf{ad}(x)$ where $\mathbf{ad}(x)(y) = [x, y]$. The difference between Lie algebras and Leibniz algebras is in that fact that the latter have a nonzero Leibniz kernel. Again an analogy with group theory arises: the difference between abelian groups and nonabelian groups consists in the presence of a nontrivial derived subgroup in the latter.

In the paper [8], it has been considered a Leibniz algebra L whose hypercenter of finite number k has finite codimension. The constraints for the dimension of finite $\gamma_{k+1}(L)$ were obtained in this case.

The fact that the operation in Leibniz algebras is not anticommutative, leads us to three centers.

The left (or right) center $\zeta^{\text{left}}(L)$ ($\zeta^{\text{right}}(L)$, respectively) of L is defined by the rule

$$\zeta^{\text{left}}(L) = \{x \in L\colon [x, y] = 0 \text{ for each element } y \in L\}$$

($\zeta^{\text{right}}(L) = \{x \in L\colon [y, x] = 0$ for each element $y \in L\}$, respectively).

In general, $\zeta^{\text{left}}(L) \neq \zeta^{\text{right}}(L)$; it is possible to prove that the left center of L is an ideal, but the right center of L is only a subalgebra, see corresponding examples in [8].

Put

$$\zeta(L) = \{x \in L\colon [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The subset $\zeta(L)$ *is called the center of* L. It is not hard to see that the center is an ideal of L, so that we can speak about the factor–algebra $L/\zeta(L)$.

As usual, a Leibniz algebra L is called abelian, if $[x, y] = 0$ for all elements $x, y \in L$. We note that $\zeta^{\text{left}}(L)$ and $\zeta^{\text{right}}(L)$ (and hence the center of L) are abelian.

We note the following important property of the Leibniz kernel:

$$[[a, a], x] = [a, [a, x]] - [a, [a, x]] = 0.$$

This property shows that $\operatorname{Leib}(L) \leq \zeta^{\text{left}}(L)$, in particular, $\operatorname{Leib}(L)$ is an abelian subalgebra of L.

We have the usual correspondence between subalgebras of a factor algebra by a given ideal and subalgebras of the original algebra containing the ideal. Starting from the center, we can define the upper central series

$$\langle 0\rangle = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \cdots \leq \zeta_\alpha(L) \leq \zeta_{a+1}(L) \leq \cdots \leq \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively $\zeta_{a+1}(L)/\zeta_a(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) =$

$\bigcup_{\mu<\lambda} \zeta_\mu(L)$ for limit ordinals λ. By definition, each term of this series is a two-sided ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter of L*.

A Leibniz algebra L is said to be hypercentral if it coincides with the upper hypercenter.

Denote by $\mathrm{zl}(L)$ the length of upper central series of L.

Dual to the concept of the upper central series is the concept of the lower central series of L. This is the series

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \cdots \geq \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \cdots \geq \gamma_\sigma(L)$$

defined by the following rule: $L = \gamma_1(L)$, $\gamma_2(L) = [L, L]$, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α, and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\sigma(L)$ is called the lower hypocenter of L. We have $\gamma_\sigma(L) = [L, \gamma_\sigma(L)]$.

If $\sigma = k$ is a positive integer, then $\gamma_k(L) = \underbrace{[L, [L, [L, \dots, L] \dots]]}_{k}$.

The Leibniz algebra concepts of upper and lower central series, introduced here for the first time, are analogs of similar concepts defined in other algebraic structures such as Lie algebras and groups. These concepts play a key role there.

An important notion of nilpotency of a Leibniz algebra L is associated with these two series. In the majority of works it is introduced using the lower central series.

We say that a Leibniz algebra L is called nilpotent, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be nilpotent of nilpotency class c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote the nilpotency class of L by $\mathrm{ncl}(L)$.

It is a well-known fact that the lower and the upper central series in nilpotent Lie algebras and nilpotent groups have the same length. For Leibniz algebras this statement also holds, it was proved in [8]. In other words, a Leibniz algebra L is nilpotent of nilpotency class c if and only if $\zeta_c(L) = L$.

The paper [10] considered Lie algebras whose upper hypercenters have finite codimension. In addition, some restrictions were obtained for the dimension of this finite dimensional ideal. A similar result for groups was obtained in [9]. These results relate to a rather extensive topic linked to the study of the relationships between the upper and lower central series in various algebraic structures, see the survey [12]. The paper [8] considered a Leibniz algebra L with hypercenter of finite index k having finite codimension. It was proved that in this case, $\gamma_{k+1}(L)$ has finite dimension and some limitations for the dimension of a finite $\gamma_{k+1}(L)$ were obtained.

In the current paper, we expanded to Leibniz algebras the main result of the paper [10]. More precisely we prove the following

Theorem B. *Let L be a Leibniz algebra over a field F. Suppose that $\zeta_\infty(L)$ has finite codimension, say d. Then L includes a finite dimensional ideal E such that the factor-algebra L/E is hypercentral. Moreover,* $\dim_F(E) \leq d(d+1)$.

Another main result relates to the refinement of Theorem A of the paper [8]. The fact that $\gamma_{k+1}(L)$ has finite dimension, entails the finite dimensionality of the nilpotent residual of L. As it turned out, the constraints on the dimension of the nilpotent residual are much simpler than the restrictions for the dimension of finite $\gamma_{k+1}(L)$. This statement is justified by another main result of our work.

Theorem A. *Let L be a Leibniz algebra over a field F, and R be the nilpotent residual of L. Suppose that there exists a positive integer n such that $\mathrm{codim}_F(\zeta_n(L)) = d$. Then L/R is nilpotent, and R has dimension at most $d(d+1)$.*

We note that the last theorem is one of the steps of the proof of Theorem B.

1. On some direct decomposition in abelian ideals of Leibniz algebras

The first section of the current work is of a preparatory nature. It is devoted to obtaining important direct decompositions in abelian ideals of Leibniz algebras. The presence of such direct decompositions plays a significant role in the proving the main results of this work.

Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A, B]$ denotes a subspace generated by all elements $[a, b]$ where $a \in A$, $b \in B$.

Let L be a Leibniz algebra over a field F, M be a nonempty subset of L. Then $\langle M\rangle$ denotes the subalgebra of L generated by M.

Let L be a Leibniz algebra over a field F, M be a nonempty subset of L and H be a subalgebra of L. Put

$$\mathrm{Ann}_H^{\mathrm{left}}(M) = \{a \in H \colon [a, M] = 0\}, \qquad \mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \colon [M, a] = 0\}.$$

The subset $\mathrm{Ann}_H^{\mathrm{left}}(M)$ is called the left annihilator or left centralizer of M in a subalgebra H; the subset $\mathrm{Ann}_H^{\mathrm{right}}(M)$ is called the right annihilator or right centralizer of M in a subalgebra H. The intersection

$$\mathrm{Ann}_H(M) = \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \colon [a, M] = \langle 0\rangle = [M, a]\}$$

is called the annihilator or centralizer of M in a subalgebra H.

It is not hard to see that all these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then $\mathrm{Ann}_H^{\mathrm{left}}(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathrm{Ann}_H^{\mathrm{left}}(M)$, $b \in M$. Then

$$[[a, x], b] = [a, [x, b]] - [x, [a, b]] = 0 - [x, 0] = 0,$$

and

$$[[x, a], b] = [x, [a, b]] - [a, [x, b]] = [x, 0] - 0 = 0.$$

If M is an ideal of L, then $\mathrm{Ann}_L(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathrm{Ann}_H(M)$, $b \in M$. Using the above arguments, we obtain that $[[a, x], b] = [[x, a], b] = 0$. Further,

$$[b, [a, x]] = [[b, a], x]] + [a, [b, x]] = [0, x] + 0 = 0,$$

and

$$[b,[x,a]] = [[b,x],a] + [x,[b,a]] = 0 + [x,0] = 0.$$

Let A, B be ideals of L such that $B \le A$. Then we define $\operatorname{Ann}_L^{\text{left}}(A/B)$, $\operatorname{Ann}_L^{\text{right}}(A/B)$, and $\operatorname{Ann}_L(A/B)$ by the following

$$\begin{aligned}
\operatorname{Ann}_L^{\text{left}}(A/B) &= \{x \in L \colon [x,A] \le B\},\\
\operatorname{Ann}_L^{\text{right}}(A/B) &= \{y \in L \colon [A,y] \le B\},\\
\operatorname{Ann}_L(A/B) &= \operatorname{Ann}_L^{\text{left}}(A/B) \cap \operatorname{Ann}_L^{\text{right}}(A/B)\\
&= \{z \in L \colon [z,A] \le B \text{ and } [A,z] \le B\}.
\end{aligned}$$

By the statements proved above the left center of L is an ideal, moreover $\operatorname{Leib}(L) \le \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra.

The center $\zeta(L)$ of L is the intersection of annihilators of all elements of L. In other words, the center is the annihilator of the whole algebra L, and by the statements proved above it is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

Let L be a Leibniz algebra, B, C be ideals of L such that $B \le C$. The factor C/B is called L-central, if $C/B \le \zeta z(L/B)$. In other words, $[C,L],[L,C] \le B$ or $\operatorname{Ann}_{L/B}(C/B) = L/B$.

The factor C/B is called *L-eccentric if* $\operatorname{Ann}_{L/B}(C/B) \ne L/B$.

Lemma 1.1. *Let L be a Leibniz algebra over a field F, and A be an ideal of L. Suppose that A satisfies the following conditions:*

(i) *A is abelian;*
(ii) *$L/\operatorname{Ann}_L(A)$ is hypercentral;*
(iii) *A includes an ideal C of L such that $\operatorname{Ann}_L(C) \ne L$ and C is L-chief;*
(iv) *$A/C \le \zeta(L/C)$.*

Then A includes an ideal D of L such that $A = C \oplus D$, in particular, D is L-central.

Proof: Let $Y = \operatorname{Ann}_L(A)$ and $Y \in z + Y \ne \zeta(L/Y)$. Consider the mapping $\xi_z \colon A \longrightarrow A$ defined by the rule $\xi_z(a) = [z,a]$ for each $a \in A$. Clearly, this mapping is F-linear, $\operatorname{Ker}(\xi_z) = \operatorname{Ann}_A^{\text{right}}(z)$, $\operatorname{Im}(\xi_z) = [z,A]$, and we have the following F-isomorphism $A/\operatorname{Ker}(\xi_z) \cong_F \operatorname{Im}(\xi_z)$.

Let $x \in L$ and $c \in \operatorname{Ann}_A^{\text{right}}(z)$. We have

$$[z,[c,x]] = [[z,c],x] + [c,[z,x]].$$

From the choice of z we obtain that $[z,x] \in \operatorname{Ann}_L(A)$, so that $[c,[z,x]] = 0$. The choice of c implies that $[z,c] = 0$, thus $[[z,c],x] = 0$, which shows that $[c,x] \in \operatorname{Ann}_A^{\text{right}}(z)$. Further,

$$[z,[x,c]] = [[z,x],c] + [x,[z,c]].$$

Again, the inclusion $[z,x] \in \mathrm{Ann}_L(A)$ implies $[[z,x],c] = 0$, and $[x,[z,c]] = [x,0] = 0$, so that $[x,c] \in \mathrm{Ann}_A^{\mathrm{right}}(z)$. This proves that $\mathrm{Ann}_A^{\mathrm{right}}(z)$ is an ideal of L.

Since A is abelian, $D = [z,A]$ is a subalgebra. Let $x \in L$, $a \in A$. We have

$$[[z,a],x] = [z,[a,x]] - [a,[z,x]].$$

Since A is an ideal, $[a,x] \in A$, so that $[z,[a,x]] \in [z,A]$. The choice of z implies that $[z,x] \in \mathrm{Ann}_L(A)$, and hence $0 = [a,[z,x]]$. Furthermore,

$$[x,[z,a]] = [[x,z],a] + [z,[x,a]].$$

Since A is an ideal, $[x,a] \in A$, so that $[z,[x,a]] \in [z,A]$. The choice of z implies that $[x,z] \in \mathrm{Ann}_L(A)$, and hence $[[x,z],a] = 0$. It follows that D is an ideal of L.

By our condition, $[z,A] \leq C$. Since C is L-chief, then either $[z,A] = C$ or $[z,A] = \langle 0\rangle$.

Consider the first case. Then $\mathrm{Im}(\zeta_z) = C$. Suppose that $[z,C] = \langle 0\rangle$. Then we obtain the inclusion $C \leq \mathrm{Ker}(\zeta_z)$. Since A/C is L-central, the factor $A/\mathrm{Ker}(\zeta_z)$ is also L-central. Put $K = \mathrm{Ker}(\zeta_z)$. Let c be an arbitrary element of C. Then $c = [z,a]$ for some element $a \in A$. Since A/K is L -central, $K = [a+K, x+K] = [x+K, a+K]$ for every element $x \in L$.

The equations $[a+K, x+K] = [a,x]+K$ and $[x+K, a+K] = [x,a]+K$ imply that $[a,x], [x,a] \in K = \mathrm{Ker}(\zeta_z)$. It implies that $[z,[a,x]] = 0 = [z,[x,a]]$. Then

$$[c,x] = [[z,a],x] = [z,[a,x]] - [a,[z,x]].$$

From the choice of z we obtain that $[z,x] \in \mathrm{Ann}_L(A)$, so that $[a,[z,x]] = 0$. Then $[c,x] = [z,[a,x]] = 0$. Similarly,

$$[x,c] = [x,[z,a]] = [[x,z],a]] + [z,[x,a]].$$

We have again $[x,z] \in \mathrm{Ann}_L(A)$, so that $[[x,z],a] = 0$. Then $[x,c] = [z,[x,a]] = 0$.

But this means that C is L-central, and we obtain a contradiction with the choice of C. This contradiction shows that $[z,C] \neq \langle 0\rangle$.

Let b be an arbitrary element of $[z,C]$, and x be an arbitrary element of L, then $b = [z,u]$ for some element $u \in C$. Since $[u,[z,x]] = 0$, we have

$$[b,x] = [[z,u],x] = [z,[u,x]] - [u,[z,x]] = [z,[u,x]].$$

The fact that C is an ideal of L implies that $[u,x] \in C$, and therefore, $[z,[u,x]] \in [z,C]$. Furthermore,

$$[x,b] = [x,[z,u]] = [[x,z],u] + [z,[x,u]] = [z,[x,u]].$$

The fact that C is an ideal of L implies that $[x,u] \in C$, and therefore, $[z,[x,u]] \in [z,C]$, which proves that $[z,C]$ is an ideal of L.

Since $[z, C] \neq \langle 0 \rangle$, condition (iii) implies that $[z, C] = C = [z, A]$. Then for every element $a \in A \cap C$ there exists an element $v \in C$ such that $[z, a] = [z, v]$. It follows that $[z, a - v] = 0$. The choice of a yields that $a - v \neq \langle 0 \rangle$. The equation $a = v + (a - v)$ shows that $A = C + \mathrm{Ann}_A^{\mathrm{right}}(z)$. We noted above that $\mathrm{Ann}_A^{\mathrm{right}}(z)$ is an ideal of L. Finally, the intersection $C \cap \mathrm{Ann}_A^{\mathrm{right}}(z)$ is an ideal of L, so that either $\langle 0 \rangle = C \cap \mathrm{Ann}_A^{\mathrm{right}}(z)$, or $C \cap \mathrm{Ann}_A^{\mathrm{right}}(z) = C$. As we saw above, the last equation is impossible. Hence $A = C \oplus \mathrm{Ann}_A^{\mathrm{right}}(z)$.

Suppose now that $[z, A] = \langle 0 \rangle$. The choice of z implies that $z \in \mathrm{Ann}_L(A) = \mathrm{Ann}_A^{\mathrm{right}}(z) \cap \mathrm{Ann}_A^{\mathrm{left}}(z)$. Then our assumption implies that $z \notin \mathrm{Ann}_L^{\mathrm{right}}(A)$. Consider the mapping $\eta_z \colon A \longrightarrow A$ defined by the rule $\eta_z(a) = [a, z]$ for each $a \in A$. Clearly, this mapping is F-linear, $\mathrm{Ker}(\eta_z) = \mathrm{Ann}_A^{\mathrm{left}}(z)$, $\mathrm{Im}(\eta_z) = [A, z]$ and we have the following F-isomorphism $A/\mathrm{Ker}(\eta_z) \cong_F \mathrm{Im}(\eta_z)$.

Let $x \in L$ and $c \in \mathrm{Ann}_A^{\mathrm{left}}(z)$. We have

$$[[c, x], z] = [c, [x, z]] - [x, [c, z]].$$

From the choice of z we obtain that $[x, z] \in \mathrm{Ann}_L(A)$, so that $[c, [x, z]] = 0$. The choice of c yields that $[c, z] = 0$, thus $[x, [c, z]] = 0$, which shows that $[c, x] \in \mathrm{Ann}_A^{\mathrm{left}}(z)$. Further,

$$[[x, c], z] = [x, [c, z]] - [c, [x, z]].$$

As we have seen above, $[x, [c, z]] = [c, [x, z]] = 0$, so that $[x, c] \in C$. This proves that $\mathrm{Ann}_A^{\mathrm{left}}(z)$ is an ideal of L.

Since A is abelian, $V = [A, z]$ is a subalgebra. Let again $x \in L$, $a \in A$. We have

$$[[a, z], x] = [a, [z, x]] - [z, [a, x]].$$

Since A is an ideal, $[a, x] \in A$, so that $[z, [a, x]] = 0$, because $[z, A] = \langle 0 \rangle$. As above $[a, [z, x]] = 0$, and thus $[[a, z], x] = 0$. Furthermore,

$$[x, [a, z]] = [[x, a], z] + [a, [x, z]] = [[x, a], z] \in [A, z].$$

It follows that V is an ideal of L.

By our condition, $[A, z] \leq C$. Since C is L-chief, then either $[A, z] = C$, or $[A, z] = \langle 0 \rangle$. But in the last case, $z \in \mathrm{Ann}_L^{\mathrm{right}}(A)$, and we obtain a contradiction. Thus $[A, z] = C$. Suppose that $[C, z] = \langle 0 \rangle$. Then we obtain the inclusion $C \leq \mathrm{Ker}(\eta_z)$. Since A/C is L-central, the factor $A/\mathrm{Ker}(\eta_z)$ is also L-central. Put $T = \mathrm{Ker}(\eta_z)$. Let c be an arbitrary element of C, then $c = [a, z]$ for some element $a \in A$. Since A/K is L-central, then as above $[a, x], [x, a] \in T$. It implies that $[[a, x], z] = 0 = [[x, a], z]$. Since $[z, x] \in \mathrm{Ann}_L(A)$ and $[z, A] = \langle 0 \rangle$, we obtain

$$[c, x] = [[a, z], x] = [a, [z, x]] - [z, [a, x]] = 0,$$

so that $[c, x] = 0$.

Similarly,

$$[x, c] = [x, [a, z]] = [[x, a], z]] + [a, [x, z]] = [[x, a], z] = 0.$$

Thus $[x, c] = 0$. As we seen above, this means that C is L-central, and we obtain a contradiction with a choice of C. This contradiction shows that $[C, z] \neq \langle 0 \rangle$. Let b be an arbitrary element of $[C, z]$ and x be an arbitrary element of L. Then $b = [c, z]$ for some element $c \in C$. We have

$$[b, x] = [[c, z], x] = [c, [z, x]] - [z, [c, x]] = 0,$$

because $[c, [z, x]] = 0$ and $[z, A] = \langle 0 \rangle$. In particular, $[b, x] \in [C, z]$. Furthermore,

$$[x, b] = [x, [c, z]] = [[x, c], z] + [c, [x, z]] = [[x, c], z].$$

The fact that C is an ideal of L implies that $[x, c] \in C$, and, therefore, $[[x, c], z] \in [C, z]$, which proves that $[C, z]$ is an ideal of L. Since $[C, z] \neq \langle 0 \rangle$, condition (iii) implies that $[C, z] = C = [A, z]$. Then for every element $a \in A \cap C$ there exists an element $w \in C$ such that $[a, z] = [w, z]$. It follows that $[a - w, z] = 0$. The choice of a yields that $a - w \neq 0$. The equation $a = w + (a - w)$ shows that $A = C + \mathrm{Ann}_A^{\mathrm{left}}(z)$. We noted above that $\mathrm{Ann}_A^{\mathrm{left}}(z)$ is an ideal of L. Finally, the intersection $C \cap \mathrm{Ann}_A^{\mathrm{left}}(z)$ is an ideal of L, so that either $C \cap \mathrm{Ann}_A^{\mathrm{left}}(z) = \langle 0 \rangle$, or $C \cap \mathrm{Ann}_A^{\mathrm{left}}(z) = C$. As we saw above, the last equation is impossible. Hence $A = C \oplus \mathrm{Ann}_A^{\mathrm{left}}(z)$. □

Corollary 1.2. *Let L be a Leibniz algebra over a field F and A be an ideal of L. Suppose that A satisfies the following conditions:*

(i) *A is abelian;*
(ii) *$L/\mathrm{Ann}_L(A)$ is hypercentral;*
(iii) *A has a series*

$$\langle 0 \rangle = C_0 \leq C_1 \leq \cdots \leq C_n = C \leq A$$

of ideals of L such that the factors C_j/C_{j-1} are L-eccentric and L-chief, $1 \leq j \leq n$ and A/C is a L-central factor.

Then A includes an ideal D of L such that $A = C \oplus D$.

If A is an ideal of L, then we define the upper L-central series

$$\langle 0 \rangle = \zeta_{0L}(A) \leq \zeta_{1L}(A) \leq \zeta_{2L}(A) \leq \cdots \leq \zeta_{aL}(A) \leq \zeta_{\gamma L}(A) = \zeta_{\infty L}(A)$$

of an ideal A by the following rule $\zeta_{1L}(A) = A \cap \zeta(L)$, $\zeta_{\alpha+1_L}(A)/\zeta_{\alpha_L}(A) = \zeta(L/\zeta_{\alpha_L}(A)) \cap A/\zeta_{\alpha_L}(A)$ for all ordinals α, and $\zeta_{\lambda L}(A) = \bigcup_{\mu<\lambda} \zeta_{\mu L}(A)$ for the limit ordinals λ. The last term $\zeta_{\infty L}(A)$ of this series is called the *upper L-hypercenter of A*. By this definition, every term $\zeta_{aL}(A)$ of the upper central series of A is an ideal of L.

An ideal C of L is said to be *L-hypereccentric*, if it has an ascending series

$$\langle 0\rangle \leq\ C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots \leq C_\gamma = C$$

of ideals of L such that each factor $C_{\alpha+1}/C_\alpha$ is an *L-eccentric* and *L-chief* for every $\alpha < \gamma$.

We say that the ideal A of L has the *Z-decomposition* if

$$A = \zeta_{\infty L}(A) \oplus \eta_{\infty L}(A)$$

where $\eta_{\infty L}(A)$ is the maximal L-hypereccentric ideal of A.

Note that in this case, $\eta_{\infty L}(A)$ includes every L-hypereccentric ideal of L, in particular, it is unique.

In fact, let B be an L-hypercentric ideal of L such that $B \leq A$ and put $E = \eta_{\infty L}(A)$. If $(B+E)/E$ is nonzero, it includes a nonzero L-chief ideal U/E of L/E. Then $U = (B\cap U) + E$. Put $V = B\cap U$. Since $V \leq B$ and the factor $V/(V\cap E)$ is L-chief, $\mathrm{Ann}_L(V/(V\cap E)) \neq L$. On the other hand, U/E is a nonzero L-chief ideal of the hypercentral algebra L/E, which implies that $L = \mathrm{Ann}_L(U/E)$. By Lemma 2.1 of [3], $\mathrm{Ann}_L(U/E) = \mathrm{Ann}_L((V+E)/E) = \mathrm{Ann}_L(V/(V\cap E))$, and we obtain a contradiction. This contradiction proves the inclusion $B \leq\ E$. Hence $\eta_{\infty L}(A)$ includes every L-hypereccentric ideal of L and, as we claimed, it is unique.

Corollary 1.2 and Corollary 2.3 of [3] together imply

Proposition 1.3. *Let L be a Leibniz algebra over a field F, and A be an abelian ideal of L. Suppose that A has a finite series of ideals of L, whose factors are either L-central or L-eccentric and L-chief. If the factor-module $L/\mathrm{Ann}_L(A)$ is nilpotent, then A has a Z-decomposition.*

Corollary 1.4. *Let L be a Leibniz algebra over a field F and A be an abelian ideal of L. Suppose that $\dim_F(A)$ is finite. If the factor-module $L/\mathrm{Ann}_L(A)$ is nilpotent, then A has a Z-decomposition.*

In fact, since A has finite dimension over F, then A has a finite L-chief series of ideals of L, and we can apply Proposition 1.3.

Proposition 1.5. *Let L be an Leibniz algebra over a field F, and K be an ideal of L. Suppose that K has a finite series*

$$\langle 0\rangle = K_0 \leq K_1 \leq \cdots \leq K_n = K$$

of ideals of L such that every factor K_j/K_{j-1} is L-central, $1 \leq j \leq n$. Then the factor-algebra $L/\mathrm{Ann}_L(K)$ is nilpotent of class at most $n-1$.

Proof: We use induction on n. Let $n = 2$ and x, y be an arbitrary elements of L. If $a \in K$, then $[[x,y],a] = [x,[y,a]] - [y,[x,a]]$. Since the factor K_2/K_1 is L-central, $[x,a],[y,a] \in K_1$. Then the inclusion $K_1 \leq \zeta(L)$ implies that $[x,[y,a]] = [y,[x,a]] = 0$. Furthermore, $[a,[x,y]] = [[a,x],y] + [x,[a,y]]$. We have $[a,x]$, $[a,y] \in K_1$, which implies that $[[a,x],y] = [x,[a,y]] = 0$. Thus $[x,y] \in \mathrm{Ann}_L(K)$. This means that $L/\mathrm{Ann}_L(K)$ is abelian.

Suppose now that $n > 2$ and our assertion is proved for ideals having finite central series of length less than n. Let $B_1 = \mathrm{Ann}_L(K_{n-1}), B_2 = \mathrm{Ann}_L(K/K_1)$. By the induction hypothesis, L/B_1 and L/B_2 are nilpotent of class at most $n-2$. Put $B_3 = B_1 \cap B_2$. Then by Remak theorem L/B_3 is embedded in the direct product $L/B_1 \times L/B_2$, which implies that L/B_3 is nilpotent of class at most $n-2$. Let $b \in B_3$, $g \in L$, and a be an arbitrary element of K. We have $[[g,b],a] = [g,[b,a]] - [b,[g,a]$. Since $[b,a] \in K_1$ and $K_1 \le \zeta(L)$, $[g,[b,a]] = 0$. Since $[g,a] \in K_{n-1}$ and $b \in \mathrm{Ann}_L(K_{n-1})$, $[b,[g,a]] = 0$. Thus $[[g,b],a] = 0$. Further, $[a,[g,b]] = [[a,g],b] + [g,[a,b]]$. We have $[a,g] \in K_{n-1}$ and, therefore, $[[a,g],b] = 0$. Since $[a,b] \in K_1$ and $K_1 \le \zeta(L)$, $[g,[a,b]] = 0$. Thus $[a,[g,b]] = 0$. It follows that $[g,b] \in \mathrm{Ann}_L(K)$. Similarly,

$$[[b,g],a] = [b,[g,a]] - [g,[b,a]] = 0 \quad \text{and} \quad [a,[b,g]] = [[a,b],g] + [b,[a,g]] = 0,$$

which implies that $[b,g] \in \mathrm{Ann}_L(K)$. In turn out, this shows that $B_3/\mathrm{Ann}_L(K) \le \zeta(L/\mathrm{Ann}_L(K))$. It follows that the factor-algebra $L/\mathrm{Ann}_L(K))$ is nilpotent of class at most $n-1$. □

2. Proofs of the main results

Let L be a Leibniz algebra and X be a class of Leibniz algebras, then put

$$\mathrm{Res}_X(L) = \{H \colon H \text{ is an ideal of } L \text{ such that } L/H \in X\}.$$

Then the intersection L^X of all ideals from a family $\mathrm{Res}_X(L)$ is called the X-residual of L. If $\mathrm{Res}_X(L)$ has the least element R, then $R = L^X$ and $L/L^X \in X$. But, in general, $L/L^X \notin X$.

If $X = A$ is the class of all abelian Leibniz algebras, then the A-residual L^A is exactly the derived ideal $[L,L]$ of L. In particular, $L/L^A \in A$.

If $X = N_c$ is the class of all nilpotent Leibniz algebras having nilpotency class at most c, then the N_c-residual L^{N_c} is exactly the subalgebra $\gamma_{c+1}(L)$. In particular, $L/L^{N_c} \in N_c$.

However, such situations do not always occur. In particular, if $X = N$ is the class of all nilpotent Leibniz algebras, then, in general, the factor-algebra L/L^N is not nilpotent.

Theorem A. *Let L be a Leibniz algebra over a field F, and R be the nilpotent residual of L. Suppose that there exists a positive integer n such that $\mathrm{codim}_F(\zeta_n(L)) = d$. Then L/R is nilpotent, and R has dimension at most $d(d+1)$.*

Proof: Put $Z = \zeta_n(L)$ and $C = \mathrm{Ann}_L(Z)$. Since L/Z has finite dimension, L/Z has a finite series of ideals, whose factors are L-chief. Proposition 1.5 shows that the factor-algebra L/C is nilpotent of class at most $n-1$. The intersection $B = Z \cap C$ is an ideal of L (in particular, of C) and $B \le \gamma(C)$. Using this inclusion and the isomorphism $C(Z \cap C) \cong (C+Z)/Z$, we obtain that $\dim F(C/\zeta(C)) \le d$. An application of Corollary B1 of the paper [8] implies that the derived subalgebra

$[C, C] = D$ of C has dimension at most d^2. The factor-algebra C/D is abelian. By Proposition 2.2 of [8], D is an ideal of L.

Since C/D is abelian, $C/D \leq \operatorname{Ann}_{L/D}(C/D)$. The fact that L/C is nilpotent implies that $(L/D)/\operatorname{Ann}_{L/D}(C/D)$ is a nilpotent Leibniz algebra. Using Proposition 1.3 we obtain that C/D has the Z-decomposition:

$$C/D = \zeta_{\infty L/D}(C/D) \oplus \eta_{\infty L/D}(C/D).$$

The choice of Z secures the inclusion $(Z + D)/D \leq \zeta_{\infty L/D}(C/D)$, from which, in its turn, it follows that $\dim_F((C/D)/\zeta_{\infty L/D}(C/D)) \leq d$. The latter inclusion shows that ideal $\zeta_{\infty L/D}(C/D))$ has dimension at most d. Put $E/D = \eta_{\infty L/D}(C/D)$. The isomorphism $C/E \cong (C/D)/(E/D) \cong \zeta_{\infty L/D}(C/D)$ shows that $C/E \leq \zeta_{\infty L/E}(C/E)$. Since L/C is nilpotent, L/E is also nilpotent. It follows that $R \leq E$. Finally, $\dim_F(E) = \dim_F(D) + \dim_F(E/D) \leq d^2 + d = d(d + 1)$. □

Corollary 2.1. *Let L be a finite dimensional Leibniz algebra over a field F and R be the nilpotent residual of L. If $\operatorname{codim}_F(\zeta_\infty(L)) = d$, then R has dimension at most $d(d + 1)$. Moreover, L/R is nilpotent.*

To prove the second main result of this paper, we need some additional ones, which, however, have an independent interest.

Proposition 2.2. *Let L be a finitely generated Leibniz algebra over a field F. Suppose that H is an ideal of L having finite codimension. Then H is finitely generated as an ideal.*

Proof: Let $M = \{a_1, \dots, a_n\}$ be a finite subset which generates L, and let B be a subspace of L such that $L = B \oplus H$. Let $\operatorname{codim}_F(H) = d$, then $\dim_F(B) = d$. Choose in B some basis $\{b_1, \dots, b_d\}$. Denote by pr_B (or pr_H) the canonical projection of L on B (H, respectively). Let E be the ideal, generated by the elements

$$\{\operatorname{pr}_H(a_j), \operatorname{pr}_H([a_j, b_m]), \operatorname{pr}_H([b_m, a_j]) \colon 1 \leq j \leq n,\ 1 \leq m \leq d\}.$$

By such a choice H includes E and E is finitely generated as an ideal of L. If x is an arbitrary element of $E + B$, then $x = u + b$ where $u \in E$, $b \in B$. Furthermore, $b = \alpha_1 b_1 + \dots + \alpha_d b_d$ for suitable elements $\alpha_1, \dots, \alpha_d \in F$. We have

$$\begin{aligned}
[b, a_j] &= [\alpha_1 b_1 + \dots + \alpha_d b_d, a_j] = \alpha_1[b_1, a_j] + \dots + \alpha_d[b_d, a_j] \\
&= \alpha_1(\operatorname{pr}_H([b_1, a_j]) + \operatorname{pr}_B([b_1, a_j]) + \dots + \alpha_d(\operatorname{pr}_H([b_d, a_j]) + \operatorname{pr}_B([b_d, a_j]) \\
&= \alpha_1 \operatorname{pr}_H([b_1, a_j]) + \dots + \alpha_d \operatorname{pr}_H([b_d, a_j]) \\
&\quad + \alpha_1 \operatorname{pr}_B([b_1, a_j]) + \dots + \alpha_d \operatorname{pr}_B([b_d, a_j]); \\
[a_j, b] &= [a_j, \alpha_1 b_1 + \dots + \alpha_d b_d] = \alpha_1[a_j, b_1] + \dots + \alpha_d[a_j, b_d] \\
&= \alpha_1(\operatorname{pr}_H([a_j, b_1]) + \operatorname{pr}_B([a_j, b_1]) + \dots + \alpha_d(\operatorname{pr}_H([a_j, b_d]) + \operatorname{pr}_B([a_j, b_d])
\end{aligned}$$

$$= \alpha_1 \operatorname{pr}_H([a_j, b_1]) + \cdots + \alpha_d \operatorname{pr}_H([a_j, b_d]) + \alpha_1 \operatorname{pr}_B([a_j, b_1]) + \cdots + \alpha_d \operatorname{pr}_B([a_j, b_d]).$$

The elements $\sum_{1\leq m\leq d}(\alpha_m \operatorname{pr}_H([b_m, a_j]) + \alpha_m \operatorname{pr}_B([b_m, a_j]))$ and $\sum_{1\leq m\leq d}(\alpha_m \times \operatorname{pr}_H([a_j, b_m]) + \alpha_m \operatorname{pr}_B([a_j, b_m]))$ clearly belong to $E + B$. It follows that $E + B$ is an ideal of A. Since $a_j = \operatorname{pr}_H(a_j) + \operatorname{pr}_B(a_j) \in E + B$, $1 \leq j \leq n$, then $E+B = A = H+B$. The inclusion $E \leq H$ and the equation $H \cap B = \langle 0\rangle$ implies that $H = E$. In particular, H is a finitely generated as an ideal. □

Proposition 2.3. *Let L be a finitely generated Leibniz algebra over a field F. If $\zeta_\infty(L)$ has finite codimension, then $\mathrm{zl}(L)$ is finite, so that $\zeta_\infty(L)$ is nilpotent.*

Proof: Let

$$\langle 0\rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_\alpha \leq Z_{\alpha+1} \leq \cdots \leq Z_\gamma = \zeta_\infty(L) \leq L$$

be the upper central series of L. Since $\dim_F(L/\zeta_\infty(L))$ is finite, Proposition 2.2 shows that $\zeta_\infty(L)$ is finitely generated as an ideal. This fact implies that γ is not a limit ordinal. Suppose that γ is infinite, then $\gamma = \tau + n$ for some limit ordinal τ. Let $M = \{v_1, \dots, v_n\}$ be a finite subset of L such that $Z\gamma$ is generated by M as an ideal. The fact that $Z_\gamma/Z_{\gamma-1}$ is the center of $L/Z_{\gamma-1}$ implies the equation $Z_\gamma/Z_{\gamma-1} = (v_1F + \cdots + v_nF)Z_{\gamma-1}/Z_{\gamma-1}$. In particular, $Z_\gamma/Z_{\gamma-1}$ has finite dimension at most n. Proposition 2.2 implies that $Z\gamma - 1$ is finitely generated as an ideal. Using the above arguments and the equation $Z_{\gamma-1}/Z_{\gamma-2} = \zeta(L/Z_{\gamma-2})$, we obtain that the factor $Z_{\gamma-1}/Z_{\gamma-2}$ has finite dimension, and an application of Proposition 2.2 shows that $Z_{\gamma-2}$ is finitely generated as an ideal. Using the same arguments, after finitely many steps, we obtain that Z_τ is finitely generated as an ideal. We noted that A/Z_τ is nilpotent and has finite dimension over F. Let $W = \{w_1, \dots, w_m\}$ be a finite subset such that Z_τ is generated by W as an ideal. From the equation $Z_\tau = \bigcup_{\beta<\tau} Z_\beta(A)$ we obtain that $w_j \in Z_{\beta(j)}$ for some $\beta(j) < \tau$, $1 \leq j \leq m$. Let σ be the greatest ordinal from the set $\{\beta(1), \dots, \beta(m)\}$. Then $w_j \in Z_\sigma$ for all j, $1 \leq j \leq m$. Since Z_σ is an ideal, it follows that it includes the ideal of L generated by elements $w_1, \dots, w_m$. But the last coincides with Z_τ, so that $Z_\tau \leq Z_\sigma$, and we obtain a contradiction. This contradiction shows that γ must be finite. □

Corollary 2.4. *Let L be a finitely generated Lie algebra over a field F. If L is hypercentral, then L is nilpotent. Moreover, L has finite dimension.*

In fact, L is nilpotent by Proposition 2.3. Using the arguments from the proof of Proposition 2.3, we obtain that $\dim_F(L)$ is finite.

A Leibniz algebra L is said to be *locally nilpotent*, if every finite subset of L generates a nilpotent subalgebra.

Corollary 2.5. *Let L be a hypercentral Leibniz algebra over a field F. Then L is locally nilpotent.*

Corollary 2.6. *Let L be a finitely generated Leibniz algebra over a field F. If $\zeta_\infty(L)$ has finite codimension, then L has finite dimension.*

Proof: In fact, $\mathrm{zl}(L)$ is finite by Proposition 2.3. Now we can repeat the arguments from the first part of the proof of Proposition 2.3, and obtain that L is finite dimensional. □

Theorem B. *Let L be a Leibniz algebra over a field F. Suppose that $\zeta_\infty(L)$ has finite codimension, say d. Then L contains a finite dimensional ideal E such that the factor-algebra L/E is hypercentral. Moreover, $\dim_F(E) \le d(d+1)$.*

Proof: Put $Z = \zeta_\infty(L)$ and choose the basis $\{a_1 + Z, \dots, a_d + Z\}$ in L/Z. Denote by H the subalgebra of L generated by the elements $a_1, \dots, a_d$, and let Φ be a family of all finitely generated subalgebras including H. By Corollary 2.6, every subalgebra $S \in \Phi$ is finite dimensional. Denote the nilpotent residual of S by $R(S)$. If $U, V \in \Phi$ and $U \le V$, then, taking into account the isomorphism $U/(U \cap R(V)) \cong (U + R(V))/R(V)$ and the obvious inclusion $(U + R(V))/R(V) \le V/R(V)$, we obtain that the factor-algebra $U/(U \cap R(V))$ is nilpotent. It follows that $R(U) \le U \cap R(V) \le R(V)$. Hence the family $\{R(S)\colon S \in \Phi\}$ is local, which implies that $\bigcup_{S\in\Phi} R(S) = E$ is a subalgebra. Moreover, since $R(S)$ is an ideal of S, then E is an ideal of L. By Corollary 2.1, $\dim_F(R(S)) \le d(d+1)$ for each $S \in \Phi$. Then we can choose the subalgebra $T \in \Phi$ such that $\dim_F(R(T))$ is the largest. If W is a subalgebra such that $W \in \Phi$ and $T \le W$, then $R(T) \le R(W)$ by above proved. On the other hand, the choice of T yields that $\dim_F(R(T)) = \dim_F(R(W))$, which proves that $R(T) = R(W)$. If S is an arbitrary subalgebra of the family Φ, then the subalgebra Y, generated by $S \cup T$, is finitely generated and includes H, so that $Y \in \Phi$. Since $T \le Y$, by above proved, $R(T) = R(Y)$. On the other hand, $R(S) \le R(Y)$, and hence $R(S) \le R(T)$. It follows that $E = R(T)$. In particular, $\dim_F(E) \le d(d+1)$.

Let F/E be an arbitrary finitely generated subalgebra of L/E, more precisely, suppose that F/E is generated by the elements $d_1 + E, \dots, d_m + E$. Denote by K the subalgebra, generated by $H \cup \{d_1, \dots, d_m\}$. Then $K \in \Phi$, and, therefore, $R(K) \le R(T) = E$. The last inclusion implies that $K/(K \cap E) \cong (K + E)/E$ is nilpotent. The choice of K yields that $F/E \le K/E$, and, therefore, F/E is also nilpotent. Hence L/E is a locally nilpotent Leibniz algebra. The fact that $L/(Z + E)$ is finite dimensional implies that $L/(Z + E)$ is nilpotent. Finally, the inclusion $(Z + E)/E \le \zeta_\infty(L/E)$ shows that L/E is hypercentral. □

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