Artinianness of formal local cohomology modules

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Abstract. Let $\mathfrak{a}$ be an ideal of Noetherian local ring $(R,\mathfrak{m})$ and M a finitely generated R-module of dimension d. In this paper we investigate the Artinianness of formal local cohomology modules under certain conditions on the local cohomology modules with respect to $\mathfrak{m}$. Also we prove that for an arbitrary local ring $(R,\mathfrak{m})$ (not necessarily complete), we have $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{MinV}(\mathrm{Ann}_R\mathfrak{F}^d_{\mathfrak{a}}(M))$.

Keywords: formal local cohomology; local cohomology

Classification: 13D45, 13E99

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and M is an R-module. Recall that the ith local cohomology module of M with respect to $\mathfrak{a}$ is denoted by $\mathrm{H}^i_{\mathfrak{a}}(M)$. For basic facts about local cohomology refer to [3]. Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module.

$$\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M) \qquad \text{for each } \ i \geq 0$$

is called the ith formal local cohomology of M with respect to $\mathfrak{a}$.

It is known that if $(R,\mathfrak{m})$ is a regular local ring, then

$$\mathfrak{F}^i_{\mathfrak{a}}(R) \simeq \mathrm{Hom}_R(\mathrm{H}^{\dim R - i}_{\mathfrak{a}}(R), \mathrm{E}_R(R/\mathfrak{m}))$$

for all $i \geq 0$, see [8, III, Proposition 2.2], also when $(R,\mathfrak{m})$ is a quotient of a local Gorenstein ring formal local cohomology modules have been studied in [11]. The basic properties of formal local cohomology modules are found in [11], [1], [4], [2], [9] and [10].

A nonzero R-module M is called secondary if its multiplication map by any element a of R is either surjective or nilpotent. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{(0 :_R S_i)} \colon i = 1, \ldots, n\}$, see [5].

DOI 10.14712/1213-7243.2019.006

Recall that $\operatorname{Assh}(M)$ denotes the set $\{\mathfrak{p} \in \operatorname{Ass}(M) \colon \dim(R/\mathfrak{p}) = \dim(M)\}$. It is well known that Artinian modules are representable and the local cohomology modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ are Artinian for all $i \geq 0$ and $\operatorname{Att}_R(\mathrm{H}^{\dim M}_{\mathfrak{m}}(M)) = \operatorname{Assh}(M)$, see [6, Theorem 2.2].

In this paper we investigate some Artinianness properties of formal local cohomology modules under certain conditions on the local cohomology modules with respect to $\mathfrak{m}$. The following theorem is one of our main results:

Theorem 1.1. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module. Let i be an integer and $\mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}M)$ be finitely generated for any ideal $\mathfrak{b} \subseteq \mathfrak{a}$. Then there exists an integer n_0 such that $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\mathfrak{a}^n \mathrm{H}^i_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Therefore $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian and $\operatorname{Att}_R(\mathfrak{F}^i_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\mathfrak{a})$.*

Recall that $\operatorname{ara}(\mathfrak{a})$, the arithmetic rank of $\mathfrak{a}$, is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{a}$. Also the finiteness dimension of M relative to $\mathfrak{a}$, denoted by $f_{\mathfrak{a}}(M)$, is the least integer i such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is not finitely generated. Here we define $Lq_{\mathfrak{a}}(M) = \inf\{i \colon \mathfrak{F}^i_{\mathfrak{a}}(M) \text{ is not Artinian}\}$ and we show that $f_{\mathfrak{m}}(M) - \operatorname{ara}(\mathfrak{a}) \leq Lq_{\mathfrak{a}}(M)$.

In [10] we showed that, if $(R, \mathfrak{m})$ is a complete local ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module of dimension d, then $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M))$. In this paper, we eliminate the complete hypothesis entirely by proving the following:

Theorem 1.2. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d. Then $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Min} \mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M))$.*

2. Main results

First we recall the following results which we will use in this paper.

Lemma 2.1. *Let M be an Artinian R-module and N a finitely generated R-module. Then $\operatorname{Att}_R(M \otimes_R N) = \operatorname{Att}_R M \cap \operatorname{Supp}_R N$.*

Proof: See [7, Proposition 5.2]. □

Theorem 2.2. *Let M be an Artinian R-module and S a multiplicative set of R. Then $\operatorname{Hom}_R(R_S, M)$ is a representable R-module and $\operatorname{Att}_R(\operatorname{Hom}_R(R_S, M)) = \{\mathfrak{p} \in \operatorname{Att}_R M \colon \mathfrak{p} \cap S = \phi\}$.*

Proof: By [7, Theorem 3.2], $\operatorname{Hom}_R(R_S, M)$ is a representable R_S-module and $\operatorname{Att}_{R_S}(\operatorname{Hom}_R(R_S, M)) = \{\mathfrak{p}R_S \colon \mathfrak{p} \in \operatorname{Att}_R M,\ \mathfrak{p} \cap S = \phi\}$. Now the result follows by [12, Lemma 4.6]. □

The following result shows that every representable formal local cohomology module with respect to $\mathfrak{a}$ is an $\mathfrak{a}$-torsion module.

Theorem 2.3. *Let M be a finitely generated module and $(R,\mathfrak{m})$ a Noetherian local ring. If $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is nonzero and representable for some integer i then $\operatorname{Att}_R \mathfrak{F}^i_{\mathfrak{a}}(M) \subseteq V(\mathfrak{a})$ and $\mathfrak{a} \subseteq \sqrt{(0 :_R \mathfrak{F}^i_{\mathfrak{a}}(M))}$.*

Proof: See [2, Theorem 2.3] and [2, Corollary 2.4]. □

We need the following lemma in the proof of the next theorem.

Lemma 2.4. *Let $(R,\mathfrak{m})$ be a local ring and M an R-module. Then the module $\operatorname{Hom}_R(R_x, M) = 0$ for all $x \in \sqrt{\operatorname{Ann}_R(M)}$.*

Proof: Since $x \in \sqrt{\operatorname{Ann}_R(M)}$, there is an integer t such that $x^tM = 0$. If $f \in \operatorname{Hom}_R(R_x, M)$ then $f(1/x^n) = x^t f(1/x^{t+n}) \in x^tM = 0$ for all $n \in \mathbb{N}$. Thus $f(1/x^n) = 0$ for all $n \in \mathbb{N}$. Therefore $f = 0$. □

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Let i be an integer and $x \in R$ be an element. If $x \in \sqrt{(0 : \mathrm{H}^{i+1}_{\mathfrak{m}}(M))}$, then there exists an integer n_0 such that $\mathfrak{F}^i_{\langle x\rangle}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Therefore $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian and $\operatorname{Att}_R(\mathfrak{F}^i_{\langle x\rangle}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\langle x\rangle)$.*

Proof: By [11, Corollary 3.16], there exists a long exact sequence

$$\cdots \to \operatorname{Hom}_R(R_x, \mathrm{H}^i_{\mathfrak{m}}(M)) \xrightarrow{\varphi} \mathrm{H}^i_{\mathfrak{m}}(M) \to \mathfrak{F}^i_{\langle x\rangle}(M) \to \operatorname{Hom}_R(R_x, \mathrm{H}^{i+1}_{\mathfrak{m}}(M)) \to \cdots .$$

If $\mathrm{H}^{i+1}_{\mathfrak{m}}(M) = 0$, then $\operatorname{Hom}_R(R_x, \mathrm{H}^{i+1}_{\mathfrak{m}}(M)) = 0$. Thus from the above long exact sequence we see that $\mathfrak{F}^i_{\langle x\rangle}(M)$ is a homomorphic image of $\mathrm{H}^i_{\mathfrak{m}}(M)$, and so $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian. Now assume that $\mathrm{H}^{i+1}_{\mathfrak{m}}(M) \neq 0$. By assumption $x \in \sqrt{(0 : \mathrm{H}^{i+1}_{\mathfrak{m}}(M))}$ and so $\operatorname{Hom}_R(R_x, \mathrm{H}^{i+1}_{\mathfrak{m}}(M)) = 0$ by Lemma 2.4. Thus from the above long exact sequence, we conclude that there exists an exact sequence

$$0 \to \operatorname{Im}(\varphi) \to \mathrm{H}^i_{\mathfrak{m}}(M) \to \mathfrak{F}^i_{\langle x\rangle}(M) \to 0.$$

Hence $\mathfrak{F}^i_{\langle x\rangle}(M)$ is homomorphic image of an Artinian module and so is Artinian. Thus there is an integer n_0 such that $\langle x\rangle^n \mathfrak{F}^i_{\langle x\rangle}(M) = 0$ for all $n \geq n_0$ by Theorem 2.3. Let $n \geq n_0$. Then from the above exact sequence we have the following exact sequence:

$$\to \frac{\operatorname{Im}\varphi}{\langle x\rangle^n \operatorname{Im}\varphi} \to \frac{\mathrm{H}^i_{\mathfrak{m}}(M)}{\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M)} \to \frac{\mathfrak{F}^i_{\langle x\rangle}(M)}{\langle x\rangle^n \mathfrak{F}^i_{\langle x\rangle}(M)} \to 0,$$

and so we have:

$$\to \frac{\operatorname{Im}\varphi}{\langle x\rangle^n \operatorname{Im}\varphi} \to \frac{\mathrm{H}^i_{\mathfrak{m}}(M)}{\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M)} \to \mathfrak{F}^i_{\langle x\rangle}(M) \to 0.$$

Since

$$\operatorname{Att}_R\Big(\frac{\operatorname{Im}\varphi}{\langle x\rangle^k \operatorname{Im}\varphi}\Big) = V(\langle x\rangle) \cap \operatorname{Att}_R(\operatorname{Im}\varphi) \subseteq V(\langle x\rangle) \cap \operatorname{Att}_R(\operatorname{Hom}_R(R_x, \mathrm{H}^i_{\mathfrak{m}}(M))$$

and by Theorem 2.2, $V(\langle x\rangle)\cap \operatorname{Att}_R(\operatorname{Hom}_R(R_x, \mathrm{H}^i_{\mathfrak{m}}(M)) = \phi$ we have $\operatorname{Att}_R\big(\operatorname{Im}\varphi/ (\langle x\rangle^n \operatorname{Im}\varphi)\big) = \phi$ and so $\operatorname{Im}\varphi/(\langle x\rangle^n \operatorname{Im}\varphi) = 0$. Now from the above exact sequence we conclude that $\mathfrak{F}^i_{\langle x\rangle}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\langle x\rangle^n \mathrm{H}^i_{\mathfrak{m}}(M))$ for all $n \geq n_0$. But $\mathrm{H}^i_{\mathfrak{m}}(M)$ is Artinian and so $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian. Now Lemma 2.1 completes the proof. □

Corollary 2.6. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Let i be an integer. If $\mathrm{H}^{i+1}_{\mathfrak{m}}(M)$ is finitely generated, then $\mathfrak{F}^i_{\langle x\rangle}(M)$ is Artinian for all $x \in R$ and $\operatorname{Att}_R(\mathfrak{F}^i_{\langle x\rangle}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\langle x\rangle)$.*

Proof: If $x \in R \setminus \mathfrak{m}$, then $\mathfrak{F}^i_{\langle x\rangle}(M) = 0$. Thus we can assume that $x \in \mathfrak{m}$. By assumption $\mathrm{H}^{i+1}_{\mathfrak{m}}(M)$ is finitely generated and so there exists $k \in \mathbb{N}$, such that $\mathfrak{m}^k \mathrm{H}^{i+1}_{\mathfrak{m}}(M) = 0$. This implies that $x \in \sqrt{(0 : \mathrm{H}^{i+1}_{\mathfrak{m}}(M))}$, the claim follows by Theorem 2.5. □

Lemma 2.7. *Let $(R,\mathfrak{m})$ be a local ring and M a finitely generated R-module. Then $\operatorname{Hom}_R(R_x, M) = 0$ for all $x \in \mathfrak{m}$.*

Proof: If $f \in \operatorname{Hom}_R(R_x, M)$ then $f(1/x^n) = x^k f(1/x^{k+n}) \in x^k M$ for all $k, n \in \mathbb{N}$. Thus $f(1/x^n) \in \bigcap_k x^k M = 0$ for all $n \in \mathbb{N}$ by Krull's theorem. Therefore $f = 0$. □

Theorem 2.8. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Let i be an integer and $\mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}M)$ be finitely generated for any ideal $\mathfrak{b}$ of R with $\mathfrak{b} \subseteq \mathfrak{a}$. Then there exists an integer n_0 such that $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathrm{H}^i_{\mathfrak{m}}(M)/(\mathfrak{a}^n \mathrm{H}^i_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Therefore $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian and $\operatorname{Att}_R(\mathfrak{F}^i_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathrm{H}^i_{\mathfrak{m}}(M)) \cap V(\mathfrak{a})$.*

Proof: Assume that $\mathfrak{a} = (a_1, \ldots, a_t)$. We use induction on t. If $t = 1$ then the result follows by Corollary 2.6. Now suppose that $t > 1$ and that the result has been proved for $t-1$. Set $\mathfrak{b} := (a_1, \ldots, a_{t-1})$. By [11, Theorem 3.15], there exists an exact sequence

$$\cdots \to \operatorname{Hom}_R(R_{a_t}, \mathfrak{F}^i_{\mathfrak{b}}(M)) \xrightarrow{\varphi} \mathfrak{F}^i_{\mathfrak{b}}(M) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \operatorname{Hom}_R(R_{a_t}, \mathfrak{F}^{i+1}_{\mathfrak{b}}(M)) \to \cdots .$$

By assumption $\mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}^n M)$ is finitely generated for all $n \in \mathbb{N}$ and so by Lemma 2.7 we have $\operatorname{Hom}_R(R_{a_t}, \mathrm{H}^{i+1}_{\mathfrak{m}}(M/\mathfrak{b}^n M)) = 0$ for all $n \in \mathbb{N}$. On the other hand

$$\begin{aligned}\operatorname{Hom}_R(R_{a_t}, \mathfrak{F}_{\mathfrak{b}}^{i+1}(M)) &\simeq \operatorname{Hom}_R\Big(R_{a_t}, \varprojlim_n \mathrm{H}_m^{i+1}\Big(\frac{M}{\mathfrak{b}^n M}\Big)\Big)\\ &\simeq \varprojlim_n \operatorname{Hom}_R\Big(R_{a_t}, \mathrm{H}_m^{i+1}\Big(\frac{M}{\mathfrak{b}^n M}\Big)\Big).\end{aligned}$$

Therefore $\operatorname{Hom}_R(R_{a_t}, \mathfrak{F}_{\mathfrak{b}}^{i+1}(M)) = 0$ and we get the following exact sequence:

$$0 \to \operatorname{Im}\varphi \to \mathfrak{F}_{\mathfrak{b}}^{i}(M) \to \mathfrak{F}_{\mathfrak{a}}^{i}(M) \to 0.$$

But by the inductive hypothesis $\mathfrak{F}_{\mathfrak{b}}^{i}(M)$ is Artinian and from the above exact sequence we conclude that $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is Artinian. Thus by Theorem 2.3, there exists an integer k_0 such that $\mathfrak{a}^k\mathfrak{F}_{\mathfrak{a}}^{i}(M) = 0$ for all $k \geq k_0$. Let $k \geq k_0$. Then from the above exact sequence we have the following exact sequence:

$$\to \frac{\operatorname{Im}\varphi}{\mathfrak{a}^k \operatorname{Im}\varphi} \to \frac{\mathfrak{F}_{\mathfrak{b}}^{i}(M)}{\mathfrak{a}^k\mathfrak{F}_{\mathfrak{b}}^{i}(M)} \to \frac{\mathfrak{F}_{\mathfrak{a}}^{i}(M)}{\mathfrak{a}^k\mathfrak{F}_{\mathfrak{a}}^{i}(M)} \to 0,$$

and so we have:

$$\to \frac{\operatorname{Im}\varphi}{\mathfrak{a}^k \operatorname{Im}\varphi} \to \frac{\mathfrak{F}_{\mathfrak{b}}^{i}(M)}{\mathfrak{a}^k\mathfrak{F}_{\mathfrak{b}}^{i}(M)} \to \mathfrak{F}_{\mathfrak{a}}^{i}(M) \to 0.$$

On the other hand,

$$\operatorname{Att}_R\Big(\frac{\operatorname{Im}\varphi}{\mathfrak{a}^k \operatorname{Im}\varphi}\Big) = V(\mathfrak{a}) \cap \operatorname{Att}_R(\operatorname{Im}\varphi) \subseteq V(\mathfrak{a}) \cap \operatorname{Att}_R(\operatorname{Hom}_R(R_{a_t}, \mathfrak{F}_{\mathfrak{b}}^{i}(M)).$$

Since $a_t \in \mathfrak{a}$ by Theorem 2.2, we have $V(\mathfrak{a}) \cap \operatorname{Att}_R(\operatorname{Hom}_R(R_{a_t}, \mathfrak{F}_{\mathfrak{b}}^{i}(M)) = \phi$. Hence $\operatorname{Att}_R(\operatorname{Im}\varphi/(\mathfrak{a}^k \operatorname{Im}\varphi)) = \phi$ and so $\operatorname{Im}\varphi/(\mathfrak{a}^k \operatorname{Im}\varphi) = 0$. Now from the above exact sequence we have $\mathfrak{F}_{\mathfrak{a}}^{i}(M) \simeq \mathfrak{F}_{\mathfrak{b}}^{i}(M)/(\mathfrak{a}^k\mathfrak{F}_{\mathfrak{b}}^{i}(M))$ for all $k \geq k_0$. But by the inductive hypothesis there exists an integer u_0 such that $\mathfrak{F}_{\mathfrak{b}}^{i}(M) \simeq \mathrm{H}_{\mathfrak{m}}^{i}(M)/(\mathfrak{b}^u \mathrm{H}_{\mathfrak{m}}^{i}(M))$ for all $u \geq u_0$. Assume that $n_0 = \max\{k_0, u_0\}$. Thus we have $\mathfrak{F}_{\mathfrak{a}}^{i}(M) \simeq \mathrm{H}_{\mathfrak{m}}^{i}(M)/(\mathfrak{b}^n \mathrm{H}_{\mathfrak{m}}^{i}(M) + \mathfrak{a}^n \mathrm{H}_{\mathfrak{m}}^{i}(M))$ for all $n \geq n_0$ and since $\mathfrak{b} \subseteq \mathfrak{a}$ we get $\mathfrak{F}_{\mathfrak{a}}^{i}(M) \simeq \mathrm{H}_{\mathfrak{m}}^{i}(M)/(\mathfrak{a}^n \mathrm{H}_{\mathfrak{m}}^{i}(M))$ for all $n \geq n_0$. Thus $\mathfrak{F}_{\mathfrak{a}}^{i}(M)$ is an Artinian module and Lemma 2.1 completes the proof. □

Corollary 2.9. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d. Let $i \geq \dim \mathfrak{a}M$ be an integer and $\mathrm{H}_{\mathfrak{m}}^{i}(M)$ be finitely generated. Then $\mathfrak{F}_{\mathfrak{a}}^{i-1}(M)$ is Artinian and $\operatorname{Att}_R(\mathfrak{F}_{\mathfrak{a}}^{i-1}(M)) = \operatorname{Att}_R(\mathrm{H}_{\mathfrak{m}}^{i-1}(M)) \cap V(\mathfrak{a})$.*

Proof: Let $\mathfrak{b} \subseteq \mathfrak{a}$ be an ideal of R. The exact sequence

$$0 \to \mathfrak{b}M \to M \to M/\mathfrak{b}M \to 0$$

induces the exact sequence

$$\cdots \to \mathrm{H}_{\mathfrak{m}}^{i}(M) \to \mathrm{H}_{\mathfrak{m}}^{i}(M/\mathfrak{b}M) \to \mathrm{H}_{\mathfrak{m}}^{i+1}(\mathfrak{b}M) \to \cdots.$$

Since $\dim \mathfrak{b}M \leq \dim \mathfrak{a}M < i+1$, by the Grothendieck's vanishing theorem [3, Theorem 6.1.2], $\mathrm{H}^{i+1}_{\mathfrak{m}}(\mathfrak{b}M) = 0$. From the above exact sequence we conclude that $\mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{b}M)$ is finitely generated. Now the result follows by Theorem 2.8. □

Now we can obtain the following corollary which is an improvement of [2, Theorem 3.1].

Corollary 2.10. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module of dimension d. Then there exists an integer n_0 such that $\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathrm{H}^d_{\mathfrak{m}}(M)/(\mathfrak{a}^n\,\mathrm{H}^d_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Thus $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian and*

$$\mathrm{Att}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M) = \mathrm{Assh}(M) \cap \mathrm{V}(\mathfrak{a}) = \Big\{\mathfrak{p} \in \mathrm{Ass}\,M \colon \dim \frac{\mathrm{R}}{\mathfrak{p}} = \dim \mathrm{M},\ \mathfrak{p} \supseteq \mathfrak{a}\Big\}.$$

Proof: By the Grothendieck's vanishing theorem [3, Theorem 6.1.2] we have $\mathrm{H}^{d+1}_{\mathfrak{m}}(M/N) = 0$ for any submodule N of M. Thus by Theorem 2.8 there exists an integer n_0 such that $\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathrm{H}^d_{\mathfrak{m}}(M)/(\mathfrak{a}^n\,\mathrm{H}^d_{\mathfrak{m}}(M))$ for all $n \geq n_0$. Since $\mathrm{H}^d_{\mathfrak{m}}(M)$ is Artinian, $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian and $\mathrm{Att}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M) = \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{m}}(M)/(\mathfrak{a}^n\,\mathrm{H}^d_{\mathfrak{m}}(M))) = \mathrm{Att}_R\,\mathrm{H}^d_{\mathfrak{m}}(M) \cap V(\mathfrak{a})$. But $\mathrm{Att}_R\,\mathrm{H}^d_{\mathfrak{m}}(M) = \{\mathfrak{p} \in \mathrm{Ass}\,M \colon \dim \mathrm{R}/\mathfrak{p} = \dim \mathrm{M}\}$ and so we have $\mathrm{Att}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \mathrm{Ass}\,M \colon \dim \mathrm{R}/\mathfrak{p} = \dim \mathrm{M}\} \cap \mathrm{V}(\mathfrak{a})$. Therefore the proof is complete. □

Theorem 2.11. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Suppose that $\mathfrak{a}$ can be generated by t elements. For every $i \geq 0$, if $\mathrm{H}^{i+1}_{\mathfrak{m}}(M), \mathrm{H}^{i+2}_{\mathfrak{m}}(M), \dots, \mathrm{H}^{i+t}_{\mathfrak{m}}(M)$ are finitely generated, then $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is Artinian.*

Proof: We use induction on t. When $t = 1$, the claim follows by Corollary 2.6. Now suppose, inductively, that $t > 1$ and the result has been proved for ideals that can be generated by fewer than t elements. Suppose that $\mathfrak{a} = \langle a_1, \dots, a_t\rangle$. Set $\mathfrak{b} := Ra_1 + \cdots + Ra_{t-1}$ and $\mathfrak{c} := Ra_t$. By [11, Theorem 5.1], there exists a long exact sequence

$$\cdots \to \mathfrak{F}^i_{\mathfrak{b}\cap\mathfrak{c}}(M) \to \mathfrak{F}^i_{\mathfrak{b}}(M) \oplus \mathfrak{F}^i_{\mathfrak{c}}(M) \to \mathfrak{F}^i_{\mathfrak{b}+\mathfrak{c}}(M) \to \mathfrak{F}^{i+1}_{\mathfrak{b}\cap\mathfrak{c}}(M) \to \cdots.$$

By the inductive hypothesis, $\mathfrak{F}^i_{\mathfrak{b}}(M)$ and $\mathfrak{F}^i_{\mathfrak{c}}(M)$ are Artinian. Since $\mathfrak{bc}$ can be generated by $t-1$ elements, it follows from the inductive hypothesis that $\mathfrak{F}^{i+1}_{\mathfrak{bc}}(M)$ is Artinian. Since the $\mathfrak{bc}$-adic and the $\mathfrak{b}\cap\mathfrak{c}$-adic topology on M are equivalent, by [11, Lemma 3.8] it follows that $\mathfrak{F}^{i+1}_{\mathfrak{b}\cap\mathfrak{c}}(M) \cong \mathfrak{F}^{i+1}_{\mathfrak{bc}}(M)$, also we have $\mathfrak{a} = \mathfrak{b} + \mathfrak{c}$. Now the above long exact sequence completes the inductive step. □

Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Recall that the finiteness dimension $f_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$ is defined by

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 \colon\ \mathrm{H}^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\}.$$

We define $Lq_{\mathfrak{a}}(M)$, the lower Artinianness dimension of M with respect to $\mathfrak{a}$, as the least integer i such that $\mathfrak{F}^i_{\mathfrak{a}}(M)$ is not Artinian. In the next result we obtain a lower bound for $Lq_{\mathfrak{a}}(M)$.

Theorem 2.12. *Let $\mathfrak{a}$ be an ideal of local ring $(R,\mathfrak{m})$ and M a finitely generated R-module. Then $f_{\mathfrak{m}}(M) - \operatorname{ara}(\mathfrak{a}) \leq Lq_{\mathfrak{a}}(M)$.*

Proof: Set $t := \operatorname{ara}(\mathfrak{a})$. Suppose that $\mathfrak{b}$ is an ideal of R such that $\mathfrak{b}$ can be generated by t elements and $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. By [11, Lemma 3.8], $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{b}}(M)$ for all $i \geq 0$. Hence $Lq_{\mathfrak{a}}(M) = Lq_{\mathfrak{b}}(M)$. Since $\mathrm{H}^0_{\mathfrak{m}}(M), \mathrm{H}^1_{\mathfrak{m}}(M), \cdots, \mathrm{H}^{f_{\mathfrak{m}}(M)-1}_{\mathfrak{m}}(M)$ are finitely generated, by Theorem 2.11, $\mathfrak{F}^i_{\mathfrak{b}}(M)$ is Artinian for all $i < f_{\mathfrak{m}}(M) - t$. Therefore $f_{\mathfrak{m}}(M) - t \leq Lq_{\mathfrak{b}}(M) = Lq_{\mathfrak{a}}(M)$, as required. □

Theorem 2.13. *Let $\mathfrak{a}$ be an ideal of local ring $(R,\mathfrak{m})$ and M a finitely generated R-module and $i \geq 0$ an integer. Then $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{a}+\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M)$.*

Proof: Let $x \in \bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$. Thus $\operatorname{Hom}_R(R_x, \mathfrak{F}^j_{\mathfrak{a}}(M)) = 0$ for all $j \geq i$ by Lemma 2.4. Hence the exact sequence

$$\cdots \to \operatorname{Hom}_R(R_x, \mathfrak{F}^i_{\mathfrak{a}}(M)) \to \mathfrak{F}^i_{\mathfrak{a}}(M) \to \mathfrak{F}^i_{\langle\mathfrak{a},x\rangle}(M) \to \operatorname{Hom}_R(R_x, \mathfrak{F}^{i+1}_{\mathfrak{a}}(M)) \to \cdots$$

implies that $\mathfrak{F}^j_{\mathfrak{a}}(M) \simeq \mathfrak{F}^j_{\langle\mathfrak{a},x\rangle}(M)$ for all $j \geq i$. For any $y \in \bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$, by replacing $\mathfrak{F}^j_{\mathfrak{a}}(M)$ with $\mathfrak{F}^j_{\langle\mathfrak{a},x\rangle}(M)$ for all $j \geq i$ in the above long exact sequence, we get $\mathfrak{F}^j_{\langle\mathfrak{a},x\rangle}(M) \simeq \mathfrak{F}^j_{\langle\mathfrak{a},x,y\rangle}(M)$. Continuing in this way completes the proof. □

Corollary 2.14. *Let $\mathfrak{a}$ be an ideal of local ring $(R,\mathfrak{m})$ and M a finitely generated R-module and $i \geq 0$ an integer. If $\mathfrak{F}^j_{\mathfrak{a}}(M)$ is Artinian for all $j \geq i$ then $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M)$.*

Proof: By [2, Theorem 2.9], $\mathfrak{a} \subseteq \sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$ for all $j \geq i$. Thus $\mathfrak{a} \subseteq \bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}$ for all $j \geq i$. Therefore Theorem 2.13 implies that

$$\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \mathfrak{F}^i_{\mathfrak{a}+\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M) \simeq \mathfrak{F}^i_{\bigcap_{j\geq i}\sqrt{\operatorname{Ann}_R(\mathfrak{F}^j_{\mathfrak{a}}(M))}}(M),$$

as required. □

The following result is an extension of [10, Corollary 2.7] for an arbitrary Noetherian local ring $(R,\mathfrak{m})$. Here R is not necessarily complete.

Corollary 2.15. *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module of dimension d. Then $\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathfrak{F}^d_{\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}(M)$.*

Proof: By [1, Lemma 2.2] the module $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian and by [11, Theorem 4.5] the module $\mathfrak{F}^i_{\mathfrak{a}}(M) = 0$ for all $i > d$. Thus by Corollary 2.14 we have

$$\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathfrak{F}^d_{\sqrt{\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}}(M) \simeq \mathfrak{F}^d_{\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}(M),$$

as required. □

In the next result we provide a generalization of [10, Theorem 2.11 (ii)] by eliminating the complete hypothesis.

Theorem 2.16. *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d. Then* $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M))$.

Proof: Since $\mathfrak{F}^d_{\mathfrak{a}}(M)$ is Artinian we have

$$\mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Min}\,\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) \subseteq \mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)).$$

On the other hand, by Corollary 2.15 and Corollary 2.10

$$\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathfrak{F}^d_{\mathrm{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))}(M)) = \mathrm{V}(\mathrm{Ann}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M)) \cap \mathrm{Assh}(M).$$

It is easy to see that, by using definition of $\mathrm{Assh}(M)$, $\mathrm{V}(\mathrm{Ann}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M)) \cap \mathrm{Assh}(M) \subseteq \mathrm{Min}\,\mathrm{V}(\mathrm{Ann}_R\,\mathfrak{F}^d_{\mathfrak{a}}(M))$ and so the proof is complete. □

Acknowledgment. The author would like to thank the referee for his/her useful suggestions.

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(*Received* December 5, 2017, *revised* December 8, 2018)