A Q-linear automorphism of the reals with non-measurable graph

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Abstract. This note contains a proof of the existence of a one-to-one function Θ of $\mathbb{R}$ onto itself with the following properties: Θ is a rational-linear automorphism of $\mathbb{R}$, and the graph of Θ is a non-measurable subset of the plane.

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The existence of functions with non-measurable graphs was known to Sierpinski a century ago, but he apparently never published the details of a proof. However, see [2] and [1]. What is new in this note is a proof of the existence of an isomorphism Θ of $\mathbb{R}$ onto itself, as a rational vector space, which has the property that the graph of Θ has zero inner measure and its complement $\mathbb{R}^2\setminus\Theta$ also has zero inner measure. Thus, the graph of Θ is not measurable. (The characteristic function of a non-measurable set is not a measurable function, but its graph is measurable.) As in rudimentary set theory, a function is identical with its graph.

In what follows every linear combination has rational coefficients.

For any set Z let $|Z|$ be the cardinal of Z. As usual c is the cardinal of the continuum. Let m_1 be the Lebesgue measure on $\mathbb{R}$ and m_2 be the Lebesgue (2-dimensional) measure on $\mathbb{R}^2$. Let $\mathbf{K}$ be the set of compact $K \subset \mathbb{R}^2$ for which $m_2(K) > 0$. For $x \in \mathbb{R}$ let V_x be the vertical line $\{(x, y) \colon y \in \mathbb{R}\}$.

Lemma 1. *If $K \in \mathbf{K}$, then there are c elements x of $\mathbb{R}$ for which $|V_x \cap K| = c$.*

Proof: Let $0 < m_2(K) = \int m_1(V_x \cap K)\,\mathrm{d}m_1(x)$ by Fubini's theorem. So $m_1(V_x\cap K) > 0$ for a set X (of such x's) of positive measure. Therefore X has c points, and for each such point $V_x \cap K$ must contain c points, since its measure is positive. Lemma 1 is proved, along with the statement that the graph of any function has zero inner measure. (We remark that K may not contain a set $I \times J$ of positive measure.)

Now we proceed with the proof of the main statement (the title and the abstract). The set $\mathbf{K}$ has c members: Let us well order $\mathbf{K} = \{K_\alpha \colon \alpha < c\}$. We shall find a one-one map F of $2 \times c$ into $\mathbb{R}$, with these properties: the image of F is linearly independent, and defining $x_\alpha = F(0, \alpha)$ and $y_a = F(1, \alpha)$, we have $(x_\alpha, y_\alpha) \in K_\alpha$.

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Once we have the above, let $A = \{x_\alpha \colon \alpha < c\}$ and $B = \{y_\alpha \colon \alpha < c\}$, two sets whose union is linearly independent. Clearly, the linear span of A is of dimension c and of co-dimension c; the same is true for B. Extend A to a (Hamel) basis A' for $\mathbb{R}$, and extend B to B', similarly. Since $|A' \backslash A| = |B' \backslash B| = c$, the one-one map $\Theta \colon A \to B$ defined by $\Theta(x_\alpha) = y_\alpha$ can be extended to a one-one map, also called Θ from A' onto B'. Of course, this Θ extends naturally to an isomorphism Θ from $\mathbb{R}$ onto itself. The set Θ meets every $K \in \mathbf{K}$, since $\Theta(x_\alpha) = y_\alpha$ means exactly that $(x_\alpha, y_\alpha) \in \Theta$.

It follows that $\mathbb{R}^2 \backslash \Theta$ cannot contain a set of positive measure; that is, the inner measure of $\mathbb{R}^2 \backslash \Theta$ is zero. The inner measure of Θ is also zero, by Lemma 1.

In order to prove the existence of a function F with the desired properties, we shall proceed by transfinite induction. The set $\mathbf{K}$ is already well ordered. Well order $\mathbb{R} = \{z_\lambda \colon \lambda < c\}$.

By transfinite induction we shall produce a collection $\{F_\alpha \colon \alpha < c\}$ with these properties:

(1) each $F_\alpha \colon 2 \times \alpha \to \mathbb{R}$ is one-to-one;
(2) the image of each F_α is linearly independent;
(3) for $\beta < \alpha$ $F_\beta \subset F_\alpha$; and
(4) for $\beta < \alpha$ $(x_\beta, y_\beta) \in K_\beta$, where $x_\beta = F(0, \beta)$ and $y_\beta = F(1, \beta)$.

Then our desired $F = \bigcup_{\alpha < c} F_\alpha$.

Suppose we have a collection as above for all $\alpha < \gamma < c$. We are to find F_γ to satisfy the four properties. Note that for $\gamma = 0$, the properties hold vacuously. If γ is not a successor, simply take $F_\gamma = \bigcup_{\alpha < \gamma} F_\alpha$.

For $\gamma = \beta + 1$ we proceed as follows. Put L to be the linear span of the image of F_β. The span L has fewer than c points. Lemma 1 shows that there are c points z not in L for which $V_z \cap K_\beta$ contains c points. Let x_β be one of those z's, say z_η, where η is the smallest possible for definiteness. Let L' be the linear span of $L \cup \{x_\beta\}$. In a manner similar to the selection of x_β choose y_β so that y_β is not in L' but $(x_\beta, y_\beta) \in K_\beta$. Put $F(0, \beta) = x_\beta$ and $F(1, \beta) = y_\beta$, and extend F_β to $F_{\beta+1}$ accordingly. Properties (1)–(4) are evident, and the proof is complete. $\square$

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References

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[2] Kharazishvili A. B., *Nonmeasurable Sets and Functions*, North-Holland Mathematics Studies, 195, Elsevier Science B.V., Amsterdam, 2004.

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