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Infinitely many weak solutions for a non-homogeneous Neumann problem in Orlicz–Sobolev spaces

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Abstract. Under a suitable oscillatory behavior either at infinity or at zero of the nonlinear term, the existence of infinitely many weak solutions for a non-homogeneous Neumann problem, in an appropriate Orlicz–Sobolev setting, is proved. The technical approach is based on variational methods.

Keywords: non-homogeneous Neumann problem; variational methods; Orlicz–Sobolev space

Classification: 35D05, 35J60, 35J20, 46N20, 58E05

1. Introduction

Our main purpose in this paper is to study the non-homogeneous Neumann problem

(1.1)
$$\begin{cases} -\operatorname{div}\big(a(x,|\nabla u(x)|)\nabla u(x)\big) + a(x,|u(x)|)u(x) = \lambda f(x,u(x)) & \text{for } x\in\Omega,\\ a(x,|\nabla u(x)|)\frac{\partial u}{\partial \nu}(x) = \mu g(\gamma(u(x))) & \text{for } x\in\partial\Omega. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 2$, with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, $g\colon\mathbb{R}\to\mathbb{R}$ is a nonnegative continuous function, λ is a positive parameter, μ is a nonnegative parameter and the functions $a(x,t)\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ and γ will be specified later.

Orlicz–Sobolev spaces have been used in the last decades to model various phenomena. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. They play a significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, nonlinear potential theory, the theory of quasi-conformal mappings, non-Newtonian fluids, image processing, differential geometry, geometric function theory and probability theory. Due to these, several authors have widely studied the existence of solutions for the eigenvalue problems involving non-homogeneous operators in the divergence form by means of variational methods and critical point theory, monotone operator methods, fixed point theory and degree theory, see for instance [2], [3], [4], [8], [6], [7], [9], [12], [16], [18], [19], [22], [25], [26], [28].

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In this paper, we will study problem (1.1) in the Orlicz–Sobolev space. Our goal is to obtain some sufficient condition to guarantee that problem (1.1) has infinitely many weak solutions. To this end, we require that the primitive F of f satisfies a suitable oscillatory behavior either at infinity or at zero, while G, the primitive of g, has an appropriate growth (see Theorems 3.1 and 3.8). Our approach is fully variational method and the main tool is a general critical point theorem contained in [5] (see Theorem 2.1 in the next section). We also refer the interested reader to the papers [3], [10], [11], [14] and references therein, in which this variational principle and its variants have been successfully used to the existence of infinitely many solutions for boundary value problems.

Our paper is organized as follows. In Section 2, some preliminaries and the abstract Orlicz–Sobolev spaces setting are presented. In Section 3, we discuss the existence of infinitely many weak solutions for problem (1.1). We also point out special cases of the results and we illustrate the results by presenting an example.

2. Preliminaries

In this section, we recall definitions and theorems used in this paper. Let $(X, \|\cdot\|)$ be a real Banach space and $J, I\colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals; put $\Gamma = J - I$ and fix $r_1, r_2 \in [-\infty, \infty]$, with $r_1 < r_2$. We say that the functional Γ satisfies the Palais–Smale condition cut off lower at r_1 and upper at r_2 (${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition) if any sequence $\{u_n\} \subset X$ such that

- ◦ $\{\Gamma(u_n)\}$ is bounded,
- ◦ $\lim_{n\to\infty} \|\Gamma'(u_n)\|_{X^*} = 0$,
- ◦ $r_1 < J(u_n) < r_2 \quad \forall n \in \mathbb{N}$,

has a convergent subsequence. If $r_1 = -\infty$ and $r_2 = \infty$, it coincides with the classical (PS)-condition, while if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(\mathrm{PS})^{[r_2]}$-condition. We shall prove our results applying the following theorem of G. Bonanno, see [5].

Theorem 2.1 (See [5, Theorem 7.4]). *Let X be a real Banach space and let $J, I\colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals with J bounded from below. For every $r > \inf_X J$, let*

$$\varphi(r) := \inf_{u\in J^{-1}(-\infty,r)} \frac{(\sup_{v\in J^{-1}(-\infty,r)} I(v)) - I(u)}{r - J(u)},$$

$$\overline{\gamma} := \liminf_{r\to\infty} \varphi(r), \qquad \textit{and} \qquad \delta := \liminf_{r\to(\inf_X J)^+} \varphi(r).$$

Then the following properties hold:

(a) *If $\overline{\gamma} < \infty$ and for every $\lambda \in (0, 1/\overline{\gamma})$ the functional $\Gamma_\lambda = J - \lambda I$ satisfies $(\mathrm{PS})^{[r]}$-condition for all $r \in \mathbb{R}$, then for each $\lambda \in (0, 1/\overline{\gamma})$, the following alternative holds: either*

(a$_1$) *Γ_λ possesses a global minimum, or* (a$_2$) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* Γ_λ *such that*

$$\lim_{n\to\infty} J(u_n) = \infty.$$

(b) *If* $\delta < \infty$, *and for every* $\lambda \in (0, 1/\delta)$, *the functional* $\Gamma_\lambda = J - \lambda I$ *satisfies* $(\mathrm{PS})^{[r]}$*-condition for all* $r > \inf_X J$ *then, for each* $\lambda \in (0, 1/\delta)$, *the following alternative holds: either*

(b$_1$) *there is a global minimum of* J *which is a local minimum of* Γ_λ, *or*

(b$_2$) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* Γ_λ *such that* $\lim_{n\to\infty} J(u_n) = \inf_X J$.

Let us first introduce Orlicz–Sobolev spaces and give just a brief review of some basic concepts and facts of their theory, for more details we refer the readers to R. A. Adams in [1], L. Diening in [15], J. Musielak in [24] and M. M. Rao and Z. D. Ren in [26].

Suppose that the function $a(x,t)\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ is such that the mapping $\phi(x,t)\colon \overline{\Omega}\times\mathbb{R}\to\mathbb{R}$, defined by

$$\phi(x,t) = \begin{cases} a(x,|t|)t & \text{for } t \neq 0,\\ 0 & \text{for } t = 0\end{cases}$$

satisfies the condition (ϕ) for all $x\in\Omega$, $\phi(x,\cdot)\colon \mathbb{R}\to\mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and

$$\Phi(x,t) = \int_0^t \phi(x,s)\,\mathrm{d}s \qquad \forall\, x\in\overline{\Omega},\ t\geq 0$$

belongs to class Φ (see [24], page 33), that is, the function Φ satisfies the following conditions:

(Φ_1) for all $x\in\Omega$, $\phi(x,\cdot)\colon [0,\infty)\to\mathbb{R}$ is a non-decreasing continuous function with $\Phi(x,0)=0$ and $\Phi(x,t)>0$ whenever $t>0$, $\lim_{t\to\infty}\Phi(x,t)=\infty$,

(Φ_2) for every $t\geq 0$, $\Phi(\cdot,t)\colon \Omega\to\mathbb{R}$ is a measurable function.

Since $\phi(x,\cdot)$ satisfies condition (ϕ), we deduce that $\Phi(x,\cdot)$ is convex and increasing from $\mathbb{R}^+$ to $\mathbb{R}^+$.

For the function Φ, we define the *generalized Orlicz class*,

$$K_\Phi(\Omega) = \left\{u\colon \Omega\to\mathbb{R}\colon u \text{ measurable}, \int_\Omega \Phi(x,|u(x)|)\,\mathrm{d}x < \infty\right\}$$

and the *generalized Orlicz space*,

$$L^\Phi(\Omega) = \left\{u\colon \Omega\to\mathbb{R}\colon u \text{ measurable}, \lim_{\lambda\to 0^+}\int_\Omega \Phi(x,\lambda|u(x)|)\,\mathrm{d}x = 0\right\}.$$

The space $L^\Phi(\Omega)$ is a Banach space endowed with the *Luxemburg norm*

$$|u|_\Phi = \inf\left\{\mu > 0\colon\ \int_\Omega \Phi\Big(x, \frac{|u(x)|}{\mu}\Big)\,\mathrm{d}x \le 1\right\}$$

or the equivalent norm (the *Orlicz norm*)

$$|u|_{(\Phi)} = \sup\left\{\left|\int_\Omega uv\,\mathrm{d}x\right|\colon\ v \in L^{\overline{\Phi}}(\Omega), \int_\Omega \overline{\Phi}(x, |v(x)|)\,\mathrm{d}x \le 1\right\},$$

where $\overline{\Phi}$ denotes the *conjugate Young* function of Φ, that is,

$$\overline{\Phi}(x,t) = \sup_{s>0}\{ts - \Phi(x,s)\colon\ s \in \mathbb{R}\} \qquad \forall\, x \in \overline{\Omega},\ t \ge 0.$$

Furthermore, for Φ and $\overline{\Phi}$ conjugate Young functions, the Hölder type inequality holds true

$$\left|\int_\Omega uv\,\mathrm{d}x\right| \le B|u|_\Phi |v|_{\overline{\Phi}} \qquad \forall\, u \in L^\Phi(\Omega),\ v \in L^{\overline{\Phi}}(\Omega)\ , \tag{2.1}$$

where B is a positive constant (see [24], Theorem 13.13).

In this paper, we assume that there exist two positive constants ϕ_0 and ϕ^0 such that

$$1 < \phi_0 \le \frac{t\phi(x,t)}{\Phi(x,t)} \le \phi^0 < \infty \qquad \forall\, x \in \overline{\Omega},\ t \ge 0. \tag{2.2}$$

The above relation implies that Φ satisfies the Δ_2-*condition*, that is,

$$\Phi(x, 2t) \le K\,\Phi(x,t) \qquad \forall\, x \in \overline{\Omega},\ t \ge 0\,, \tag{2.3}$$

where K is a positive constant (see [23, Proposition 2.3]). Relation (2.3) and Theorem 8.13 in [24] imply that $L^\Phi(\Omega) = K_\Phi(\Omega)$. Furthermore, we assume that Φ satisfies the following condition

$$\text{for each } x \in \overline{\Omega},\ \text{the function } [0,\infty) \ni t \to \Phi(x, \sqrt{t}) \text{ is convex.} \tag{2.4}$$

Relation (2.4) assures that $L^\Phi(\Omega)$ is a uniformly convex space and thus, a reflexive space (see [23, Proposition 2.2]).

On the other hand, we point out that assuming that Φ and Ψ belong to class Φ and

$$\Psi(x,t) \le K_1\,\Phi(x, K_2\,t) + h(x) \qquad \forall\, x \in \overline{\Omega},\ t \ge 0, \tag{2.5}$$

where $h \in L^1(\Omega)$, $h(x) \ge 0$ a.e. $x \in \Omega$ and K_1, K_2 are positive constants, then by Theorem 8.5 in [24] we have that there exists the continuous embedding

$L^\Phi(\Omega) \hookrightarrow L^\Psi(\Omega)$. Next, we define the *generalized Orlicz–Sobolev space*

$$W^{1,\Phi}(\Omega) = \Big\{u \in L^\Phi(\Omega) \colon \frac{\partial u}{\partial x_i} \in L^\Phi(\Omega),\ i = 1, \dots, N\Big\}.$$

On $W^{1,\Phi}(\Omega)$ we define the equivalent norms

$$\begin{aligned} \|u\|_{1,\Phi} &= |\,|\nabla u|\,|_\Phi + |u|_\Phi, \\ \|u\|_{2,\Phi} &= \max\{|\,|\nabla u|\,|_\Phi, |u|_\Phi\}, \\ \|u\| &= \inf\Big\{\mu > 0 \colon \int_\Omega \Big[\Phi\Big(x, \frac{|u(x)|}{\mu}\Big) + \Phi\Big(x, \frac{|\nabla u(x)|}{\mu}\Big)\Big]\,\mathrm{d}x \le 1\Big\}. \end{aligned}$$

More precisely, for every $u \in W^{1,\Phi}(\Omega)$, we have

$$\|u\| \le 2\|u\|_{2,\Phi} \le 2\|u\|_{1,\Phi} \le 4\|u\|, \tag{2.6}$$

(see [23, Proposition 2.4]).

The generalized Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$ endowed with one of the above norms is a reflexive Banach space.

In the following, we will use the norm $\|\cdot\|$ on $E := W^{1,\Phi}(\Omega)$ and we suppose that $\gamma \colon E \to L^\Phi(\Omega)$ is the trace operator.

Following lemma is useful in the proof of our results.

Lemma 2.2. *Let $u \in E$. Then*

$$\int_\Omega (\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|))\,\mathrm{d}x \ge \|u\|^{\phi_0} \qquad \text{if } \|u\| > 1; \tag{2.7}$$

$$\int_\Omega (\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|))\,\mathrm{d}x \ge \|u\|^{\phi^0} \qquad \text{if } \|u\| < 1. \tag{2.8}$$

For the proof of the previous result see, for instance, Lemma 2.3 of [21]. We point out that assuming that Φ and Ψ belong to class Φ, satisfying relation (2.5) and $\inf_{x\in\Omega} \Phi(x, 1) > 0$, $\inf_{x\in\Omega} \Psi(x, 1) > 0$ then there exists the continuous embedding $W^{1,\Phi}(\Omega) \hookrightarrow W^{1,\Psi}(\Omega)$.

In this paper, we study problem (1.1) in the particular case when Φ satisfies

$$M|t|^{p(x)} \le \Phi(x, t) \qquad \forall\, x \in \overline{\Omega},\ t \ge 0, \tag{2.9}$$

where $p(x) \in C(\overline{\Omega})$ with $p^- := \inf_{x\in\Omega} p(x) > N$ for all $x \in \overline{\Omega}$ and $M > 0$ is a constant.

Throughout the sequel, $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function and $g \colon \mathbb{R} \to \mathbb{R}$ is a nonnegative continuous function. We recall that $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if:

(1) the mapping $x \longmapsto f(x, \xi)$ is measurable for every $\xi \in \mathbb{R}$;
(2) the mapping $\xi \longmapsto f(x, \xi)$ is continuous for almost every $x \in \Omega$;

(3) for every $\varrho > 0$ there exists a function $l_\varrho \in L^1(\Omega)$ such that

$$\sup_{|\xi|\leq\varrho} |f(x,\xi)| \leq l_\varrho(x)$$

for almost every $x \in \Omega$.

Put

$$F(x,t) := \int_0^t f(x,\xi)\,\mathrm{d}\xi$$

for all $(x,t) \in \Omega \times \mathbb{R}$, and

$$G(t) := \int_0^t g(\xi)\,\mathrm{d}\xi \qquad \forall\, t \in \mathbb{R}.$$

By relation (2.9), we deduce that E is continuously embedded in $W^{1,p(x)}(\Omega)$ (see relation (2.5) with $\Psi(x,t) = |t|^{p(x)}$). On the other hand, as pointed out in [17] and [20], $W^{1,p(x)}(\Omega)$ is continuously embedded in $W^{1,p^-}(\Omega)$ and since $p^- > N$, we deduce that $W^{1,p^-}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, E is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \leq c\|u\| \qquad \forall\, u \in E, \tag{2.10}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$.

Finally, we say that $u \in E$ is a *weak solution* for problem (1.1) if

$$\begin{aligned}\int_\Omega a(x,|\nabla u(x)|)\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_\Omega a(x,|u(x)|)u(x)v(x)\,\mathrm{d}x\\ = \lambda\int_\Omega f(x,u(x))v(x)\,\mathrm{d}x + \mu\int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))\,\mathrm{d}\sigma\end{aligned}$$

for every $v \in E$.

3. Main results

In this section, we present our main results. Let

$$A := \liminf_{\xi\to\infty} \frac{\int_\Omega \max_{|t|\leq\xi} F(x,t)\,\mathrm{d}x}{\xi^{\phi_0}},$$

$$B := \limsup_{\xi\to\infty} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^{\phi^0}},$$

$$\lambda_1 := \frac{\int_\Omega \Phi(x,1)\,\mathrm{d}x}{B}, \qquad \lambda_2 := \frac{1}{c^{\phi_0}A},$$

and

$$\delta_1 := \frac{1}{c^{\phi_0} b(\partial\Omega) \limsup\limits_{\xi\to\infty}(G(\xi)/\xi^{\phi_0})}\big(1 - \lambda c^{\phi_0}A\big), \tag{3.1}$$

where $b(\partial\Omega) := \int_{\partial\Omega} \mathrm{d}\sigma$ and we read "$\frac{1}{0}$" $= \infty$ whenever this case occurs.

With the above notations, we formulate our main result as follows.

Theorem 3.1. *Let $f\colon \Omega\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Assume that*

(A1) $A < \frac{1}{c^{\phi_0}\int_\Omega \Phi(x,1)\,\mathrm{d}x} B$.

Then, for every $\lambda \in]\lambda_1, \lambda_2[$ and for every nonnegative continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{\xi\to\infty} \frac{G(\xi)}{\xi^{\phi_0}} < \infty, \tag{3.2}$$

there exists $\delta_1 > 0$ given by (3.1) such that for each $\mu \in [0, \delta_1[$, problem (1.1) has a sequence $\{u_n\}$ of weak solutions in E such that

$$\int_\Omega \big(\Phi(x, |\nabla u_n(x)|) + \Phi(x, |u_n(x)|)\big)\,\mathrm{d}x \to \infty.$$

Proof: Our aim is to apply Theorem 2.1 (a) to problem (1.1). To this end, fix λ and g satisfying our assumptions. Since $\lambda < \lambda_2$, we have

$$\delta_1 = \frac{1}{c^{\phi_0} b(\partial\Omega) \limsup\limits_{\xi\to\infty}(G(\xi)/\xi^{\phi_0})}\big(1 - \lambda c^{\phi_0} A\big) > 0.$$

Fix $\mu \in [0, \delta_1[$. For each $u \in E$, we let the functionals $J, I\colon E \to \mathbb{R}$ be defined by

$$J(u) := \int_\Omega \big(\Phi(x, |\nabla u(x)|) + \Phi(x, |u(x)|)\big)\,\mathrm{d}x$$

and

$$I(u) := \int_\Omega F(x, u(x))\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(u(x)))\,\mathrm{d}\sigma,$$

and put

$$\Gamma_{\lambda,\mu}(u) := J(u) - \lambda I(u), \qquad u \in E.$$

Since Φ is convex, it follows that J is a convex functional, hence J is sequentially weakly lower semi-continuous. Similar arguments as those used in [23, Lemma 4.2] imply that $J \in C^1(E, \mathbb{R})$ with the derivative given by

$$\langle J'(u), v\rangle = \int_\Omega a(x, |\nabla u(x)|)\nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_\Omega a(x, |u(x)|)u(x)v(x)\,\mathrm{d}x$$

for every $v \in E$. Also J is bounded from below. By the same argument as given in [21, Lemma 3.2] we observe that J' has a continuous inverse on E^*. Moreover, $I \in C^1(E, \mathbb{R})$ and

$$\langle I'(u), v\rangle = \int_\Omega f(x, u(x))v(x)\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} g(\gamma(u(x)))\gamma(v(x))\,\mathrm{d}\sigma$$

for every $v \in E$.

So, with standard arguments, we deduce that the critical points of the functional $\Gamma_{\lambda,\mu}$ are the weak solutions of problem (1.1).

Moreover, owing to [5, Theorem 2.1], the functional $\Gamma_{\lambda,\mu}$ satisfies the $(\mathrm{PS})^{[r]}$-condition for all $r \in \mathbb{R}$. Now, we show that $\lambda < 1/\overline{\gamma}$. For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \xi_n = \infty$ and

$$\lim_{n\to\infty} \frac{\int_\Omega \max_{|t|\le\xi_n} F(x,t)\,\mathrm{d}x}{\xi_n^{\phi_0}} = A.$$

Put

$$r_n := \left(\frac{\xi_n}{c}\right)^{\phi_0}$$

for all $n \in \mathbb{N}$. By Lemma 2.2 and this fact that $\max\{r_n^{1/\phi_0}, r_n^{1/\phi^0}\} = r_n^{1/\phi_0}$, we deduce

$$\{v \in E \colon J(v) < r_n\} \subseteq \{v \in E \colon \|v\| < r_n^{1/\phi_0}\} = \left\{v \in E \colon \|v\| < \frac{\xi_n}{c}\right\}.$$

Moreover, due to (2.10), we have

$$|v(x)| \le \|v\|_\infty \le c\|v\| \le \xi_n \qquad \forall\, x \in \overline{\Omega}.$$

Hence,

$$\left\{v \in E \colon \|v\| < \frac{\xi_n}{c}\right\} \subseteq \left\{v \in E \colon \|v\|_\infty \le \xi_n\right\}.$$

Note that $J(0) = I(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u \in J^{-1}(-\infty, r_n)} \frac{\left(\sup_{v\in J^{-1}(-\infty,r_n)} I(v)\right) - I(u)}{r_n - J(u)} \\
&\le \frac{\sup_{v\in J^{-1}(-\infty,r_n)} I(v)}{r_n} \\
&= \frac{\int_\Omega \max_{|t|\le\xi_n} F(x,t)\,\mathrm{d}x + \frac{\mu}{\lambda}\max_{|t|\le\xi_n} G(t)\,\mathrm{d}\sigma}{\left(\frac{\xi_n}{c}\right)^{\phi_0}} \\
&\le c^{\phi_0}\left[\frac{\int_\Omega \max_{|t|\le\xi_n} F(x,t)\,\mathrm{d}x}{\xi_n^{\phi_0}} + \frac{\mu}{\lambda}\frac{b(\partial\Omega)G(\xi_n)}{\xi_n^{\phi_0}}\right].
\end{aligned}$$

Therefore,

$$\overline{\gamma} \le \liminf_{n\to\infty} \varphi(r_n) \le c^{\phi_0}\left(A + \frac{\mu}{\lambda} b(\partial\Omega) \limsup_{\xi\to\infty} \frac{G(\xi)}{\xi^{\phi_0}}\right) < \infty. \tag{3.3}$$

The assumption $\mu \in (0, \delta_1)$ immediately yields

$$\overline{\gamma} \le c^{\phi_0}\left(A + \frac{\mu}{\lambda} b(\partial\Omega)\limsup_{\xi\to\infty}\frac{G(\xi)}{\xi^{\phi_0}}\right) < c^{\phi_0}A + \frac{1-\lambda c^{\phi_0}A}{\lambda}.$$

Hence,

$$\lambda = \frac{1}{c^{\phi_0}A + (1-\lambda c^{\phi_0}A)/\lambda} < \frac{1}{\overline{\gamma}}.$$

Let λ be fixed. We claim that the functional $\Gamma_{\lambda,\mu}$ is unbounded from below. Since

$$\frac{1}{\lambda} < \frac{B}{\int_\Omega \Phi(x,1)\,\mathrm{d}x},$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that

$$\lim_{n\to\infty} \eta_n = \infty$$

and

$$\frac{1}{\lambda} < \tau < \frac{\int_\Omega F(x,\eta_n)\,\mathrm{d}x}{\eta_n^{\phi^0} \int_\Omega \Phi(x,1)\,\mathrm{d}x} \tag{3.4}$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ define $w_n \in E$ by

$$w_n(x) = \eta_n, \qquad x \in \overline{\Omega}.$$

For any fixed $n \in \mathbb{N}$ large enough, due to the inequality

$$\Phi(x,\sigma t) \le \sigma^{\phi^0} \Phi(x,t) \qquad \forall\, x \in \overline{\Omega},\ t > 0,\ \sigma > 1$$

(see [23]), one has

$$J(w_n) = \int_\Omega \Phi(x,\eta_n)\,\mathrm{d}x \le \eta_n^{\phi^0} \int_\Omega \Phi(x,1)\,\mathrm{d}x. \tag{3.5}$$

On the other hand, since G is nonnegative, from the definition of I, we infer

$$\begin{aligned} I(w_n) &= \int_\Omega F(x,w_n(x))\,\mathrm{d}x + \frac{\mu}{\lambda}\int_{\partial\Omega} G(\gamma(w_n(x)))\,\mathrm{d}\sigma \\ &= \int_\Omega F(x,\eta_n)\,\mathrm{d}x + \frac{\mu}{\lambda} b(\partial\Omega) G(\eta_n). \end{aligned} \tag{3.6}$$

By (3.4), (3.5) and (3.6), we see that

$$\begin{aligned} \Gamma_{\lambda,\mu}(w_n) &\le \eta_n^{\phi^0} \int_\Omega \Phi(x,1)\,\mathrm{d}x - \lambda \int_\Omega F(x,\eta_n)\,\mathrm{d}x - \mu b(\partial\Omega) G(\eta_n) \\ &< \eta_n^{\phi^0}(1-\lambda\tau) \int_\Omega \Phi(x,1)\,\mathrm{d}x - \mu b(\partial\Omega) G(\eta_n) \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\lambda\tau > 1$ and $\lim_{n\to\infty} \eta_n = \infty$, we have

$$\lim_{n\to\infty} \Gamma_{\lambda,\mu}(w_n) = -\infty.$$

Then, the functional $\Gamma_{\lambda,\mu}$ is unbounded from below, and it follows that $\Gamma_{\lambda,\mu}$ has no global minimum. Therefore, by Theorem 2.1 (a), there exists a sequence $\{u_n\}$

of critical points of $\Gamma_{\lambda,\mu}$ such that

$$\int_\Omega (\Phi(x,|\nabla u_n(x)|)+\Phi(x,|u_n(x)|))\,\mathrm{d}x \to \infty,$$

and the conclusion is achieved. $\square$

Remark 3.2. Under the conditions $A=0$ and $B=\infty$, from Theorem 3.1 we see that for every $\lambda>0$ and for each

$$\mu \in \left[0, \frac{1}{c^{\phi_0} b(\partial\Omega) \limsup\limits_{\xi\to\infty} \frac{G(\xi)}{\xi^{\phi_0}}}\right[,$$

problem (1.1) admits a sequence of weak solutions in E. Moreover, if

$$\limsup_{\xi\to\infty} \frac{G(\xi)}{\xi^{\phi_0}} = 0,$$

the result holds for every $\lambda>0$ and $\mu\geq 0$.

Remark 3.3. If in Theorem 3.1, we assume $f(x,0)=0$ a.e. $x\in\Omega$, then the weak solutions obtained are nonnegative. Indeed, define

$$f_+(x,t) = \begin{cases} f(x,t) & \text{if } t\geq 0, \\ 0 & \text{if } t<0, \end{cases}$$

and consider the following problem

(3.7)
$$\begin{cases} -\operatorname{div}\big(a(x,|\nabla u(x)|)\nabla u(x)\big) + a(x,|u(x)|)u(x) = \lambda f_+(x,u(x)) & \text{for } x\in\Omega, \\ a(x,|\nabla u(x)|)\frac{\partial u}{\partial \nu}(x) = \mu g(\gamma(u(x))) & \text{for } x\in\partial\Omega. \end{cases}$$

Let $v_0\in E$ be one (nontrivial) weak solution of problem (3.7). Arguing by contradiction, if we assume that v_0 is negative at a point of Ω the set

$$\Omega^- := \{x\in\Omega\colon v_0(x)<0\},$$

is nonempty and open. Moreover, let us consider $v_0^\star := \min\{v_0,0\}$, one has $v_0^\star\in E$. So, taking into account that v_0 is a weak solution and by choosing $v=v_0^\star$, from our sign assumptions on the data, we have

$$\begin{aligned} &\int_{\Omega^-} a(x,|\nabla v_0(x)|)|\nabla v_0(x)|^2\,\mathrm{d}x + \int_{\Omega^-} a(x,|v_0(x)|)|v_0(x)|^2\,\mathrm{d}x \\ &\quad = \lambda \int_{\Omega^-} f(x,v_0(x))v_0(x)\,\mathrm{d}x + \mu\int_{\partial\Omega} g(\gamma(v_0(x)))\gamma(v_0(x))d\sigma \leq 0. \end{aligned}$$

Therefore,

$$\int_{\Omega^-} a(x,|\nabla v_0(x)|)|\nabla v_0(x)|^2\,\mathrm{d}x + \int_{\Omega^-} a(x,|v_0(x)|)|v_0(x)|^2\,\mathrm{d}x = 0,$$

which means

$$\int_{\Omega^-} \phi(x, |\nabla v_0(x)|)|\nabla v_0(x)|\,\mathrm{d}x + \int_{\Omega^-} \phi(x, |v_0(x)|)|v_0(x)|\,\mathrm{d}x = 0.$$

Now, from the previous relation and bearing in mind that $t\phi(x,t) \geq \Phi(x,t)$ for every $x \in \overline{\Omega}$ and $t \geq 0$, we find that

$$\int_{\Omega^-} \Phi(x, |\nabla v_0(x)|)\,\mathrm{d}x + \int_{\Omega^-} \Phi(x, |v_0(x)|)\,\mathrm{d}x = 0.$$

Hence, by Lemma 2.2 we observe that $\|v_0\|_{W^{1,\Phi}(\Omega^-)} = 0$ which is absurd. Hence, our claim is proved.

The following result is a special case of Theorem 3.1 with $\mu = 0$.

Theorem 3.4. *Assume that all the assumptions of Theorem* 3.1 *hold. Then, for each*

$$\lambda \in \left] \frac{\int_\Omega \Phi(x,1)\,\mathrm{d}x}{B}, \frac{1}{c^{\phi_0} A} \right[,$$

the problem

(3.8)

$$\begin{cases} -\mathrm{div}\big(a(x, |\nabla u(x)|)\nabla u(x)\big) + a(x, |u(x)|)u(x) = \lambda f(x, u(x)) & \text{for } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{for } x \in \partial\Omega \end{cases}$$

has a sequence of weak solutions in E.

Now, we present a consequence of Theorem 3.1.

Corollary 3.5. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function. Suppose that*

$$A < \frac{1}{c^{\phi_0}}, \qquad B > \int_\Omega \Phi(x,1)\,\mathrm{d}x.$$

Then, for every nonnegative continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{\xi \to \infty} \frac{G(\xi)}{\xi^{\phi_0}} < \infty,$$

there exists

$$\delta_2 = \frac{1}{c^{\phi_0} b(\partial\Omega) \limsup\limits_{\xi \to \infty} (G(\xi)/\xi^{\phi_0})}(1 - c^{\phi_0} A) > 0$$

such that for each $\mu \in [0, \delta_2[$, the problem

(3.9)

$$\begin{cases} -\mathrm{div}\big(a(x, |\nabla u(x)|)\nabla u(x)\big) + a(x, |u(x)|)u(x) = f(x, u(x)) & \text{for } x \in \Omega, \\ a(x, |\nabla u(x)|)\frac{\partial u}{\partial \nu}(x) = \mu g(\gamma(u(x))) & \text{for } x \in \partial\Omega \end{cases}$$

has a sequence of weak solutions in E.

Here, as an example of our main result, we state a special case of Corollary 3.5.

Corollary 3.6. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $F(\xi) := \int_0^\xi f(t)\,\mathrm{d}t$ for all $\xi \in \mathbb{R}$ and assume that*

$$\liminf_{\xi\to\infty} \frac{F(\xi)}{\xi^{\phi_0}} = 0, \qquad \limsup_{\xi\to\infty} \frac{F(\xi)}{\xi^{\phi^0}} = \infty.$$

Then, the problem

$$\begin{cases} -\mathrm{div}\big(a(x,|\nabla u(x)|)\nabla u(x)\big) + a(x,|u(x)|)u(x) = f(x,u(x)) & \text{for } x \in \Omega, \\ \frac{\partial u}{\partial \nu}(x) = 0 & \text{for } x \in \partial\Omega \end{cases}$$

has a sequence of weak solutions in E.

We next present a consequence of Theorem 3.1.

Corollary 3.7. *Let $h_1\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Put $H_1(t) := \int_0^t h_1(\xi)\,\mathrm{d}\xi$ for all $t \in \mathbb{R}$ and assume that*

(A2) $\liminf\limits_{\xi\to\infty} \frac{H_1(\xi)}{\xi^{\phi_0}} < \infty$;

(A3) $\limsup\limits_{\xi\to\infty} \frac{H_1(\xi)}{\xi^{\phi^0}} = \infty$.

Then, for every $\alpha_i \in L^1(\Omega)$ for $1 \le i \le n$, with $\min_{x\in\Omega}\{\alpha_i(x)\colon 1 \le i \le n\} \ge 0$ and with $\alpha_1 \ne 0$, and for every nonnegative continuous $h_i\colon \mathbb{R} \to \mathbb{R}$ for $2 \le i \le n$, satisfying

$$\max\Big\{\sup_{\xi\in\mathbb{R}} H_i(\xi)\colon 2 \le i \le n\Big\} \le 0$$

and

$$\min\Big\{\liminf_{\xi\to\infty} \frac{H_i(\xi)}{\xi^{\phi_0}}\colon 2 \le i \le n\Big\} > -\infty,$$

where $H_i(t) := \int_0^t h_i(\xi)\,\mathrm{d}\xi$ for all $t \in \mathbb{R}$ and $2 \le i \le n$, for each

$$\lambda \in \left]0, \frac{1}{c^{\phi_0} \liminf_{\xi\to\infty}(H_1(\xi)/\xi^{\phi_0}) \int_\Omega \alpha_1(x)\,\mathrm{d}x}\right[,$$

and for every nonnegative continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\limsup_{\xi\to\infty} \frac{G(\xi)}{\xi^{\phi_0}} < \infty,$$

there exists

$$\delta_3 = \frac{1}{c^{\phi_0} b(\partial\Omega) \limsup\limits_{\xi\to\infty}(G(\xi)/\xi^{\phi_0})} \left(1 - \lambda\left(c^{\phi_0} \int_\Omega \alpha_1(x)\,\mathrm{d}x\right) \liminf_{\xi\to\infty} \frac{H_1(\xi)}{\xi^{\phi_0}}\right) > 0$$

such that for each $\mu \in [0, \delta_3[$, the problem

$$\begin{cases} -\operatorname{div}\big(a(x,|\nabla u(x)|)\nabla u(x)\big) + a(x,|u(x)|)u(x) = \lambda \sum_{i=1}^{n} \alpha_i(x) h_i(u) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } x \in \Omega, \\ a(x,|\nabla u(x)|)\frac{\partial u}{\partial \nu}(x) = \mu g(\gamma(u(x))) & \textit{for } x \in \partial\Omega, \end{cases}$$

admits infinitely many distinct pairwise weak solutions in E.

Proof: Set $f(x,t) = \sum_{i=1}^{n} \alpha_i(x) h_i(t)$ for all $(x,t) \in \partial\Omega \times \mathbb{R}$. From the assumption (A3) and the condition

$$\min\left\{ \liminf_{\xi\to\infty} \frac{H_i(\xi)}{\xi^{\phi_0}} : 2 \le i \le n \right\} > -\infty,$$

we have

$$\limsup_{\xi\to\infty} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^{\phi^0}} = \limsup_{\xi\to\infty} \frac{\sum_{i=1}^{n}\big(H_i(\xi)\int_\Omega \alpha_i(x)\,\mathrm{d}x\big)}{\xi^{\phi^0}} = \infty.$$

Moreover, from the assumption (A2) and the condition

$$\max\left\{ \sup_{\xi\in\mathbb{R}} H_i(\xi) : 2 \le i \le n \right\} \le 0$$

we have

$$\liminf_{\xi\to\infty} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)\,\mathrm{d}x}{\xi^{\phi_0}} \le \Big(\int_\Omega \alpha_1(x)\,\mathrm{d}x\Big) \liminf_{\xi\to\infty} \frac{H_1(\xi)}{\xi^{\phi_0}} < \infty.$$

Hence, applying Theorem 3.1 the desired conclusion follows. □

Now, put

$$A' := \liminf_{\xi\to 0^+} \frac{\int_\Omega \max_{|t|\le\xi} F(x,t)\,\mathrm{d}x}{\xi^{\phi^0}},$$

$$B' := \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\,\mathrm{d}x}{\xi^{\phi_0}},$$

and

$$\lambda_3 := \frac{\int_\Omega \Phi(x,1)\,\mathrm{d}x}{B'}, \qquad \lambda_4 := \frac{1}{c^{\phi^0} A'}.$$

Using Theorem 2.1 (b) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 3.8. *Let $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ be an L^1-Carathéodory function. Suppose that*

(A4) $A' < \frac{1}{c^{\phi^0}\int_\Omega \Phi(x,1)\,\mathrm{d}x} B'$,

(A5) *there exists $\varrho > 0$ such that*

$$\Phi(x,t) \le \varrho t^{\phi_0} \qquad \forall\,(x,t) \in \Omega\times(0,1).$$

Then, for every $\lambda \in]\lambda_3, \lambda_4[$ *and for every nonnegative continuous function* $g\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$\limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^{\phi^0}} < \infty,$$

there exists

$$\delta_4 = \frac{1}{c^{\phi^0} b(\partial\Omega) \limsup\limits_{\xi \to 0^+} (G(\xi)/\xi^{\phi^0})} t(1 - \lambda c^{\phi^0} A') > 0$$

such that for each $\mu \in [0, \delta_4[$, *problem* (1.1) *has a sequence of weak solutions, which strongly converges to zero in* E.

Proof: Fix $\lambda \in]\lambda_3, \lambda_4[$ and $\mu \in]0, \delta_4[$. We take J, I and $\Gamma_{\lambda,\mu}$ as in the proof of Theorem 3.1. Now, as has been pointed out before, the functionals J and I satisfy the regularity assumptions required in Theorem 2.1. As first step, we will prove that $\lambda < 1/\delta$. Then, let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} \xi_n = 0$ and

$$\lim_{n \to \infty} \frac{\int_\Omega \max_{|t| \le \xi_n} F(x,t)\,\mathrm{d}x}{\xi_n^{\phi^0}} = A'.$$

By the fact that $\inf_E J = 0$ and the definition of δ, we have $\delta = \liminf_{r \to 0^+} \varphi(r)$. Then, as in showing (3.2) in the proof of Theorem 3.1, we can prove that $\delta < \infty$. From $\mu \in]0, \delta_4[$, the following inequalities hold

$$\delta \le c^{\phi^0} \Big(A' + \frac{\mu}{\lambda} b(\partial\Omega) \limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^{\phi^0}} \Big) < c^{\phi^0} A' + \frac{1 - \lambda c^{\phi^0} A'}{\lambda}.$$

Therefore,

$$\lambda = \frac{1}{c^{\phi^0} A' + (1 - \lambda c^{\phi^0} A')/\lambda} < \frac{1}{\delta}.$$

At this point, we will show that 0, that is the unique global minimum of J, is not a local minimum of $\Gamma_{\lambda,\mu}$. For this goal, let $\{\eta_n\}$ be a real sequence of positive numbers such that $\lim_{n\to\infty} \eta_n = 0$ and

$$\lim_{n \to \infty} \frac{\int_\Omega F(x, \eta_n)\,\mathrm{d}x}{\eta_n^{\phi_0}} = B'. \tag{3.10}$$

For all $n \in \mathbb{N}$, let $w_n \in E$ be defined by

$$w_n(x) = \eta_n, \qquad x \in \overline{\Omega}.$$

So, we have

$$J(w_n) = \int_\Omega \Phi(x, \eta_n)\,\mathrm{d}x$$

for every $n \in \mathbb{N}$. Moreover, from hypothesis (A5), taking into account that $\lim_{n\to\infty} w_n = 0$, one has that there exists $\tau > 0$ and $\nu_0 \in \mathbb{N}$ such that $w_n \in]0, \tau[$ and $\Phi(x, w_n) \le \varrho\, w_n^{\phi_0}$ for every $n \ge \nu_0$.

If $B' < \infty$, let $\varepsilon \in]\varrho \operatorname{meas}(\Omega)/\lambda B', 1[$. By (3.10) there exists ν_ε such that

$$\int_\Omega F(x, \eta_n)\,\mathrm{d}x > \varepsilon B' \eta_n^{\phi_0} \qquad \forall\, n > \nu_\varepsilon.$$

Hence,

$$\begin{aligned}\Gamma_{\lambda,\mu}(w_n) &= J(w_n) - \lambda I(w_n)\\ &\leq \varrho w_n^{\phi_0} \operatorname{meas}(\Omega) - \lambda\varepsilon B' w_n^{\phi_0} - \mu b(\partial\Omega) G(w_n)\\ &= w_n^{\phi_0}(\varrho \operatorname{meas}(\Omega) - \lambda\varepsilon B') - \mu b(\partial\Omega) G(w_n) < 0\end{aligned}$$

for every $n \geq \max\{\nu_0, \nu_\varepsilon\}$. On the other hand, if $B = \infty$, let us consider $M > \varrho \operatorname{meas}(\Omega)/\lambda$. By (3.10) there exists ν_M such that

$$\int_\Omega F(x, \eta_n)\,\mathrm{d}x > M \eta_n^{\phi_0} \qquad \forall\, n > \nu_M.$$

Moreover,

$$\begin{aligned}\Gamma_{\lambda,\mu}(w_n) &= J(w_n) - \lambda I(w_n)\\ &\leq \varrho w_n^{\phi_0} \operatorname{meas}(\Omega) - \lambda M w_n^{\phi_0} - \mu b(\partial\Omega) G(w_n)\\ &= w_n^{\phi_0}(\varrho \operatorname{meas}(\Omega) - \lambda M) - \mu b(\partial\Omega) G(w_n) < 0\end{aligned}$$

for every $n \geq \max\{\nu_0, \nu_M\}$. Hence $\Gamma_{\lambda,\mu}(w_n) < 0$ for every n sufficiently large, $\Gamma_{\lambda,\mu}(0) = J(0) - \lambda I(0) = 0$, this means that 0 is not a local minimum of $\Gamma_{\lambda,\mu}$. Then, owing to J has 0 as unique global minimum, Theorem 2.1 (b) ensures the existence of a sequence $\{u_n\}$ of critical points of the functional $\Gamma_{\lambda,\mu}$ such that $\lim_{n\to\infty} J(u_n) = 0$. By Lemma 2.2, we have $\lim_{n\to\infty} \|u_n\| = 0$. In view of the fact the embedding $E \hookrightarrow C^0(\overline{\Omega})$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 3.9. Under the conditions $A' = 0$ and $B' = \infty$, Theorem 3.8 ensures that for every $\lambda > 0$ and for each

$$\mu \in \left[0, \frac{1}{c^{\phi^0} b(\partial\Omega) \limsup\limits_{\xi\to 0^+}(G(\xi)/\xi^{\phi^0})}\right[,$$

problem (1.1) admits a sequence of weak solutions, which strongly converges to 0 in E. Moreover, if $\limsup_{\xi\to 0^+}(G(\xi)/\xi^{\phi^0}) = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Remark 3.10. We observe that the role of functions f and g can be reversed. For instance, we can study the following problem

(3.11)
$$\begin{cases} -\operatorname{div}\big(a(x, |\nabla u(x)|)\nabla u(x)\big) + a(x, |u(x)|)u(x) = \mu g(u(x)) & \text{for } x \in \Omega,\\ a(x, |\nabla u(x)|)\frac{\partial u}{\partial \nu}(x) = \lambda f(x, u(x)) & \text{for } x \in \partial\Omega,\end{cases}$$

and obtain a sequence of weak solutions providing an oscillating behavior of f for a suitable interval of parameters λ. It is enough to substitute in the proof of

Theorem 3.1 the functional I with the following

$$\tilde{I}(u) := \int_{\partial\Omega} F(x, \gamma(u(x)))\, \mathrm{d}\sigma + \frac{\mu}{\lambda} \int_{\Omega} G(\gamma(u(x)))\, \mathrm{d}x.$$

We now exhibit an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.11. Let $\Omega = \{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 < 3\}$. Define

$$\phi(x,y,t) = p(x,y) \frac{|t|^{p(x,y)-2} t}{\log(1+|t|)} \qquad \text{for} \quad t \neq 0, \quad \text{and} \quad \phi(x,y,0) = 0,$$

where $p(x,y) = x^2 + y^2 + 3$ for all $(x,y) \in \Omega$. Some simple computations imply

$$\Phi(x,y,t) = \frac{|t|^{p(x,y)}}{\log(1+|t|)} + \int_0^{|t|} \frac{s^{p(x,y)}}{(1+s)(\log(1+s))^2}\, \mathrm{d}s,$$

and relations (ϕ), (Φ_1) and (Φ_2) are verified. For each $x \in \overline{\Omega}$ fixed, by Example 3 on page 243 in [13], we have

$$p(x,y) - 1 \leq \frac{t\phi(x,y,t)}{\Phi(x,y,t)} \leq p(x,y) \qquad \forall\, t \geq 0.$$

Thus, relation (2.2) holds true with $\phi_0 = 2$ and $\phi^0 = 6$. Next, Φ satisfies condition (2.9) since

$$\Phi(x,y,t) \geq t^{p(x,y)-1} \qquad \forall\,(x,y) \in \overline{\Omega},\ t \geq 0.$$

Finally, we point out that trivial computations imply that $\frac{\mathrm{d}^2(\Phi(x,y,\sqrt{t}))}{\mathrm{d}t^2} \geq 0$ for all $(x,y) \in \overline{\Omega}$ and $t \geq 0$. Thus, relation (2.4) is satisfied. Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be the sequences defined as follows $b_1 = 2$, $b_{n+1} = (b_n)^{14}$ and $a_n = (b_n)^{12}$ for all $n \in \mathbb{N}$. Moreover, let $f\colon \mathbb{R} \to \mathbb{R}$ be a positive continuous function defined by

$$f(t) = \begin{cases} 2^7\sqrt{1-(1-t)^2} + 1 & \text{if } t \in [0,2], \\ (a_n - (b_n)^7)\sqrt{1-(a_n-1-t)^2+1} & \text{if } t \in \bigcup_{n=1}^{\infty}[a_n-2, a_n], \\ ((b_{n+1})^7 - a_n)\sqrt{1-(b_{n+1}-1-t)^2}+1 & \text{if } t \in \bigcup_{n=1}^{\infty}[b_{n+1}-2, b_{n+1}], \\ 1, & \text{otherwise.} \end{cases}$$

Put $F(\xi) = \int_0^{\xi} f(t)\, \mathrm{d}t$ for all $\xi \in \mathbb{R}$. In particular, one has $F(a_n) = a_n(\pi/2+1)$ for all $n \in \mathbb{N}$. Hence,

$$\liminf_{\xi\to\infty} \frac{F(\xi)}{\xi^2} = \lim_{n\to\infty} \frac{F(a_n)}{a_n^2} = 0,$$

and

$$\limsup_{\xi\to\infty} \frac{F(\xi)}{\xi^6} = \lim_{n\to\infty} \frac{F(b_n)}{b_n^6} = \infty.$$

Then, owing Theorem 3.1, the problem

$$\begin{cases} \operatorname{div}\Big(p(x,y)\frac{|\nabla u|^{p(x,y)-2}\nabla u}{\log(1+|\nabla u|)}\Big)+p(x,y)\frac{|u|^{p(x,y)-2}u}{\log(1+|u|)}=f(u) & \text{for } (x,y)\in\Omega,\\ p(x,y)\frac{|\nabla u|^{p(x,y)-2}}{\log(1+|\nabla u|)}\frac{\partial u}{\partial\nu}=\frac{1}{1+(\gamma(u))^2} & \text{for } (x,y)\in\partial\Omega \end{cases}$$

has a sequence of weak solutions in E.

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