On the integral representation of finely superharmonic functions

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Abstract. In the present paper we study the integral representation of nonnegative finely superharmonic functions in a fine domain subset U of a Brelot $\mathcal{P}$-harmonic space Ω with countable base of open subsets and satisfying the axiom D. When Ω satisfies the hypothesis of uniqueness, we define the Martin boundary of U and the Martin kernel K and we obtain the integral representation of invariant functions by using the kernel K. As an application of the integral representation we extend to the cone $\mathcal{S}(\mathcal{U})$ of nonnegative finely superharmonic functions in U a partition theorem of Brelot. We also establish an approximation result of invariant functions by finely harmonic functions in the case where the minimal invariant functions are finely harmonic.

Keywords: finely harmonic function; finely superharmonic function; fine potential; fine Green kernel; integral representation; Martin boundary; fine Riesz-Martin kernel

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1. Introduction

Let Ω be the space $\mathbb{R}^n$ if $n \geq 3$, or a domain of $\mathbb{R}^2$ with non-polar complement, and U a fine domain subset of Ω, that is, a domain in the sense of the fine topology on Ω (the smallest one making continuous the superharmonic functions on Ω). The problem of the integral representation of fine potentials on U was studied by B. Fuglede in [21], [22]. We denote by G_U the fine Green kernel of U defined by

$$G_U(\cdot, y) = G(\cdot, y) - \widehat{R}^{\complement U}_{G(\cdot,y)},$$

on $U \setminus \{y\}$ and extended by fine continuity at the point y, where G is the Green kernel of Ω. The result of B. Fuglede states that for any fine potential p on U, there is a unique Borel measure $\mu \geq 0$ on U as topological space endowed with the trace of the initial on it such that

$$p(x) = \int G_U(x, y)\,\mathrm{d}\mu(y)$$

for any $x \in U$.

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Let us denote by $\mathcal{S}(U)$ the cone of nonnegative finely superharmonic functions on U. A function $h \in \mathcal{S}(U)$ is said to be invariant if it belongs to the orthogonal band to the band $\mathcal{P}(U)$ of fine potentials on U, in the specific order in $\mathcal{S}(U)$ (that is, the order on $\mathcal{S}(U)$ defined by $\mathcal{S}(U)$), of the band $\mathcal{P}(U)$ of the fine potentials on U. The invariant functions are characterized by the fact that they are invariant under the sweeping on the complements of some finely open subsets V_n, $n \in \mathbb{N}$, such that $\bigcup_n V_n = U$ and $\overline{V}_n \subset U$ for any n, cf. [22, Theorem 4.4] and [21, Theorem, page 130]. In particular, an invariant function is, following [19, Theorem 11.13, page 127], finely harmonic in the complement of the polar set where it takes the value ∞.

Every nonnegative finely superharmonic function s on U can be uniquely written as the sum of an invariant function and a fine potential in U (Riesz decomposition). As a consequence, for every function $s \in \mathcal{S}(U)$ we can find a unique Borel measure $\mu \geq 0$ in U and a unique invariant function h on U such that

$$s = \int G_U(\cdot, y)\, \mathrm{d}\mu(y) + h.$$

We deduce from this fact that the invariant functions play the role of nonnegative harmonic functions in the Riesz decomposition of the nonnegative superharmonic functions in a Euclidean domain subset of $\mathbb{R}^n$.

In [12] we defined a topology on the cone $\mathcal{S}(U)$ of nonnegative finely superharmonic functions in U in order to obtain, by using Choquet's method, the result of B. Fuglede of the integral representation of fine potentials and the integral representation of invariant functions by means of extremal invariant functions, called also minimal invariant functions.

The minimal invariant functions can be finite everywhere, that is finely harmonic on U, or they may be infinite at some points (forming a polar set) of U, according to a recent result of S. J. Gardiner and W. Hansen in [24]. These authors have indeed shown the existence of a fine domain U of $\mathbb{R}^n$ of the form $U = D \cup \partial_i D$, where D is a non regular domain subset of $\mathbb{R}^n$, $n \geq 3$, and where $\partial_i D$ is the set of irregular points of ∂D, and of a minimal invariant function not finely harmonic on U. This answered in the negative two old questions posed by B. Fuglede, namely:

1. Is every invariant function on a fine domain U the sum of a sequence of nonnegative finely harmonic functions on U (or, equivalently, the upper envelope of its nonnegative minorants finely harmonic)?
2. If D is a non regular Green domain subset of $\mathbb{R}^n$ and if s is a minimal harmonic function in D with fine limit $\text{f-lim}_{x\to z} h(x) = \infty$ at a non regular point z of ∂D, do we have $h = \alpha G_U(\cdot, z)$ for some $\alpha \geq 0$, where $U = D \cup \partial_i D$ is the smallest finely regular domain containing D?

In the present work we shall consider the question of the integral representation in $\mathcal{S}(U)$ in the more general framework of a fine domain U of a Brelot space with countable base of open sets, satisfying the axiom D, admitting a Green kernel and a base of completely determining open subsets whose adjoint harmonic space satisfies the axiom D and in which the adjoint fine topology is finer than the fine topology and the adjoint base of a set contains the fine interior of that set. These hypothesis are of course weaker than those of [22]. The method is based as in [12] on Choquet's theorem of integral representation in the cones with compact base. We define in particular the Martin boundary $\Delta(U)$ of U, the corresponding Martin kernel on $\Delta(U)\times U$ and obtain the integral representation of the invariant functions on U.

As an application of the integral representation of invariant functions, we show that if any minimal invariant function is finely harmonic, then every invariant function can be approximated in the sense of the topology of the cone $\mathcal{S}(U)$ (the natural topology) by nonnegative finely harmonic functions in U. Finally, using a simple property of the extremal functions of $\mathcal{S}(U)$ to be selfreduced on A or on $U \setminus A$ for any subset A of U (Lemma 6.1 below), we extend to the nonnegative finely superharmonic functions a partition theorem of M. Brelot in [6].

The interest in the general context of a Brelot space is to obtain general results that may apply to an open subset of the more general potential theory defined by an elliptic operator on an open subset of $\mathbb{R}^n$ or on a Riemannian manifold (with potential greater than 0).

The Sections 2 and 3 and a part of Section 4 are trivial generalizations of the same results obtained in the classical case by M. El Kadiri and B. Fuglede in [14], [15], [16]. However, the extension of the crucial Corollary 3.5 of Theorem 3.2 does not seem to be evident. There are also some new results on the fine Green kernel in Section 4.

Notation and definitions: If Ω is a Brelot harmonic space and U is a fine domain of Ω, we denote by $\mathcal{S}(U)$ the convex cone of nonnegative finely superharmonic functions on U in the sense of [17]. The cone of fine potentials on U (that is the functions of $\mathcal{S}(U)$ without nonnegative finely subharmonic minorant is denoted by $\mathcal{P}(U)$, it is a band of $\mathcal{S}(U)$. The topology of Ω and that induced by this topology on U will be called the initial topology. We denote by $\mathcal{B}_+(U)$ the cone of Borel functions (relative to the initial topology) on U (taking values in $[0,\infty]$). The coarsest topology on Ω which is finer than the initial topology and which makes continuous all functions of $\mathcal{S}(\Omega)$ will be called the fine topology. The induced topology on U by the fine topology on Ω is also called the fine topology on U. If $f\colon U \to [0,\infty]$ we denote, without distinction, by R_f or by $\widehat{R}_f$, respectively, the reduced or the swept out function, respectively, of f on A with respect to U or Ω (in the case of U cf. [17, Section 11] for this notion). If $u \in \mathcal{S}(U)$ and $A \subset U$ we can write $\widehat{R}_u^A$ for $\widehat{R}_f$ with $f := 1_A u$. However, we also often denote it by ${}^U\widehat{R}_u^A$ to distinguish it from the swept out of u in Ω. For any subset $A \subset \Omega$ we denote by $\widetilde{A}$ the fine closure of A in Ω, and $b(A)$ or $b^*(A)$, respectively, the base or the adjoint base, respectively, when it is defined, of A in Ω, that is, the set of points of Ω where A is not thin or not adjoint thin, respectively.

If E is a locally convex topological vector space (l.c.t.v.s. in abbreviated form), and if K is a convex compact of E and μ is a probability measure on K, we denote by $b(\mu)$ the barycenter of μ and we write $b(\mu) = \int_K k \,\mathrm{d}\mu(k)$, which means that for any continuous affine form l on K, we have $l(b(\mu)) = \int l(k)\,\mathrm{d}\mu(k)$; the set of extreme points of K is denoted by $\mathrm{Ext}(K)$.

2. Construction of a resolvent associated with the cone of nonnegative finely superharmonic functions in a fine domain

Let Ω be a Brelot harmonic space with countable base of open subsets, satisfying the domination axiom D and admitting a potential greater than 0, and U a fine domain of Ω, that is, a domain subset in the sense of fine topology of Ω. Recall that the fine topology on Ω is the coarsest topology that makes continuous the superharmonic functions in Ω. For the main properties of this topology we refer to [17, Chapter 1, Section 3]. We assume that the constant functions are superharmonic in Ω (this is not really a restriction because we can always reduce the study of superharmonic and finely superharmonic functions to this case by considering the f-harmonic functions, where f is a suitable continuous finite function and greater than 0 on Ω). We denote by $\mathcal{S}(U)$ the cone of nonnegative finely superharmonic functions in U and $\mathcal{U}_+(U) = \mathcal{S}(U) \cup \{\infty\}$ the cone of nonnegative finely hyperharmonic functions in U.

The set $\partial_i U$ of all irregular points of the fine boundary $\partial_f U$ of U is polar and the set $r(U) = U \cup \partial_i U$ is a fine domain subset of Ω, it is the smallest regular fine domain of Ω which contains U, see [17, page 10]. Furthermore, any function of $\mathcal{U}_+(U)$ has a unique extension to a function of $\mathcal{U}_+(r(U))$ according to [17, Theorem 9.14, page 96]. This allows us to assume throughout this paper that U is regular.

Let p be a strict continuous and bounded potential greater than 0 on Ω. Then it is well known from [29, Theorem 2, page 362] that there exists a unique Borel kernel V on Ω such that

1. $V1 = p$.
2. For every continuous function $\varphi \in \mathcal{C}_c^+(\Omega)$, $V\varphi$ is a finite and continuous potential and harmonic in the complement of the support of φ.

The kernel V is associated with a resolvent $(V_\lambda)_{\lambda>0}$ of Borel kernels on Ω of which the cone of excessive functions is $\mathcal{U}_+(\Omega)$. Since the excessive functions of (V_λ) are l.s.c., it follows from [10, Chapter XII, no. 41] that there is a bounded Radon measure $\tau \geq 0$ on Ω such that the resolvent (V_λ) is absolutely continuous with respect to τ (that is for each $x \in \Omega$ and $\lambda > 0$, the measure $V_\lambda(x, \cdot)$ is absolutely continuous with respect to τ). The measure τ does not charge the polar sets, and the cone $\mathcal{S}(\Omega)$ is exactly the cone of excessive functions of the resolvent (V_λ) which are finite τ-a.e. (we also say that they are finite (V_λ)-a.e.). Moreover, τ charges all nonempty finely open subsets. Indeed, if ω is a nonempty finely open subset of Ω such that $\tau(\omega) = 0$, then $\tau(r(\omega)) = 0$ since $r(\omega) \setminus \omega$ is polar, hence τ-negligible, where $r(\omega)=\omega\cup\partial_i\omega$. Then we have $\widehat{R}_p^{\complement\omega}=p$ τ-a.e. Therefore $\lambda V_\lambda(\widehat{R}_p^{\complement\omega})=\lambda V_\lambda(p)$ for any $\lambda>0$. By letting $\lambda\to\infty$, we obtain $\widehat{R}_p^{\complement\omega}=p$ everywhere, which is absurd because p is strict and ω is not empty. We deduce in particular that two superharmonic functions equal τ-a.e. are equal everywhere.

According to [9, Proposition 10.2.2, page 248], the cone of excessive functions of the resolvent (V_λ) is the cone $\mathcal{U}_+(\Omega)=\mathcal{S}(\Omega)\cup\{\infty\}$ (and hence $\mathcal{S}(\Omega)$ is the cone of excessive functions of (V_λ) which are finite (V_λ)-a.e.). It follows then from [5, Theorem 4.4.6, page 136] that $\mathcal{S}(\Omega)$ is a H-cone standard of functions. Define a kernel W on U by

$$Wf=V\overline{f}-\widehat{R}^{\complement U}_{V\overline{f}}$$

(restricted to U) for any Borel measurable function $f\geq 0$ on U, where $\overline{f}$ denotes the extension of f to Ω equal to 0 in $\complement U$. Then by [4, Theorem 2.5] there exists a unique resolvent family (W_λ) of Borel measurable kernels with the potential kernel W.

We now proceed to determine the excessive functions of the resolvent (W_λ). We need to recall the following approximation theorem [20, Theorem 3, page 68]:

Theorem 2.1. *Let $s\in\mathcal{S}(U)$. Then we can find a sequence (s_n) of nonnegative superharmonic functions on Ω, such that the sequence $\big(s_n-\widehat{R}^{\complement U}_{s_n}\big)$ is increasing and*

$$s=\sup_n\big(s_n-\widehat{R}^{\complement U}_{s_n}\big).$$

Lemma 2.2. *For any function $s\in\mathcal{S}(\Omega)$, the function $s-\widehat{R}^{\complement U}_s$ is excessive for the resolvent (W_λ).*

Proof: The function s is excessive for the resolvent (V_λ), then according to [10, Theorem 17, page 11], we can find an increasing sequence (f_n) of bounded Borel functions on Ω such that $s=\sup_n V(f_n)$. Then we have $s-\widehat{R}^{\complement U}_s=\sup_n W(g_n)$ where g_n is the restriction of f_n to U. Hence $s-\widehat{R}^{\complement U}_s$ is excessive for (W_λ). □

Theorem 2.3. *The cone of excessive functions of the resolvent (W_λ) is identical to the cone $\mathcal{U}_+(U)$.*

Proof: Let $\mathcal{E}(U)$ be the cone of excessive functions of the resolvent (W_λ). The inclusion $\mathcal{S}(U)\subset\mathcal{E}(U)$ follows easily from Theorem 2.1 and Lemma 2.2. Since every function $u\in\mathcal{U}_+(U)$ is the supremum of an increasing sequence (s_n) of functions of $\mathcal{S}(U)$ (it suffices to take $s_n=s\wedge n$ for any integer n), it follows that $\mathcal{U}_+(U)\subset\mathcal{E}(U)$. Let us prove the opposite inclusion. Let $s\in\mathcal{E}(U)$, then, according to [10, Theorem 17, page 11], s is the supremum of an increasing sequence $(W(f_n))$, where (f_n) is an increasing sequence of bounded Borel functions nonnegative in U. For any integer n we have $W(f_n)=V(\overline{f}_n)-\widehat{R}^{\complement U}_{V(\overline{f}_n)}$. As $V(\overline{f}_n)$ is finite and continuous on U, the function $\widehat{R}^{CU}_{V(\overline{f}_n)}$ is finely harmonic on U by [17, Theorem 11.13], and hence $W(f_n)\in\mathcal{S}(U)$. Consequently, we have $s\in\mathcal{U}_+(U)$. It follows that $\mathcal{E}(U)\subset\mathcal{U}_+(U)$, and hence $\mathcal{E}(U)=\mathcal{U}_+(U)$. □

Remark 2.4. The excessive functions in the sense of [5, page 16] of (W_λ) are the excessive functions of (W_λ) that are finite (W_λ)-a.e. In this sense the cone of excessive functions of (W_λ) is the cone $\mathcal{S}(U)$.

As the resolvent (W_λ) is subordinated to the resolvent (V_λ), then it is basic and transient (cf. [10, Chapter XII]). The finely open U is assumed to be regular, then it is a Radonian space, that is, embeddable in a compact metric space as a universally measurable subspace, which enables us to apply the results of Chapter 12 of [10, pages 74–75].

Corollary 2.5. *The cone $\mathcal{S}(U)$ is a standard H-cone of functions on U.*

PROOF: The corollary follows from the previous theorem and [5, Theorem 4.4.6]. $\square$

3. Topology of cone $\mathcal{S}(U)$ and integral representation of nonnegative finely superharmonic functions

In order to simplify the notations, we shall use the same notations to denote the reduced and the swept out in the cones $\mathcal{S}(\Omega)$ and $\mathcal{S}(U)$ and introduce the necessary details if there is a risk of confusion.

Following [5, Section 4.5, page 141], we endow $\mathcal{S}(U)$ by the natural topology. This topology is induced on $\mathcal{S}(U)$ by that of a locally convex vector space in which $\mathcal{S}(U)$ is a well capped convex cone (that is, $\mathcal{S}(U)$ is the union of its caps in the Choquet sense). This will be sufficient to the study of the integral representation of finely superharmonic functions nonnegative, however, we show a stronger result, namely that the cone $\mathcal{S}(U)$ has a compact base. This result is very important because it allows us, in the case of proportionality of potentials of the same punctual carrier (support) to define the Martin boundary $\Delta(U)$ of U and the integral representation of invariant functions by means of a Martin kernel K on $U \times \Delta(U)$ (see [14] and Section 5 of the present paper).

Since $\mathcal{S}(U)$ is a standard H-cone, we can find an increasingly dense countable set $D = \{s_n \in \mathcal{S}(U) \colon n \in \mathbb{N}\}$ in $\mathcal{S}(U)$. That is, any element s of $\mathcal{S}(U)$ (and also any element of $\mathcal{U}_+(U)$) is the upper envelope of an (increasing) sequence of elements of D. Thus for any $s \in \mathcal{S}(U)$, we have $s = \sup\{t \in D \colon t \leq s\}$.

Lemma 3.1. *For any $x \in U$, there is a fine neighborhood K_x of x, compact in the initial topology, such that the restriction of any function $s \in \mathcal{S}(U)$ to K_x is l.s.c. in initial topology.*

PROOF: According to [19, Lemma, page 114], any point x of U has a finely compact neighborhood K_x in initial topology such that the restriction of any function s_n of D to K_x is continuous in the initial topology. Since any function $s \in \mathcal{S}(U)$ is the upper envelope of an increasing sequence of elements of D, then its restriction to K_x is l.s.c. (in initial topology). $\square$

Theorem 3.2. *There is a sequence (K_n) of compacts (in the initial topology of Ω) contained in U and a polar set P such that*

1. *$U = \bigcup_n K'_n \cup P$, where K'_n denotes the fine interior of K_n.*
2. *For any n, the restriction to K_n of any function of $\mathcal{S}(U)$ is l.s.c. in the initial topology.*

Proof: According to Lemma 3.1, any point x of U has a fine neighborhood K_x, compact in the initial topology, such that the restriction of any function $t \in \mathcal{S}(U)$ to K_x is l.s.c. On the other hand, it follows from the quasi-Lindelöf property of the fine topology that we can find a sequence (x_j) of points of U and a polar set P such that $U = \bigcup_j K'_{x_j} \cup P$. The compact subsets $K_j = K_{x_j}$ satisfy the conditions of the theorem. □

Remark 3.3. The existence of a sequence (K_j) of compact subsets of U and a polar set P with $U = P \cup \bigcup_j K_j$ such that a given finely continuous function (in particular a superharmonic function) is continuous relative to each K_j follows from the pioneering work of Le Jan [26], [27], [28], which applies more generally to the excessive functions of the resolvent associated with the Hunt process. The weaker form of condition 1. in our Theorem 3.2 in which $U = P \cup \bigcup K'_j$ is replaced by the condition $U = P \cup \bigcup K_j$, is a consequence of [3, Corollaire 1.6] together with the existence of a family of universally continuous elements which is increasingly dense in $\mathcal{S}(U)$. In the present case our Theorem 3.2 is stronger than that of L. Beznea and N. Boboc. In fact, our result is not a consequence of that of L. Beznea and N. Boboc because for a nest (K_j) of U the set $\bigcup_j K_j \setminus \bigcup_j K'_j$ is not necessarily polar, as it is seen by the following example in [14, Remark 2.10]:

Example. Let A be a compact non-polar subset of Ω with empty fine interior (for example A can be a compact ball in some hyperplane in $\mathbb{R}^n$ such that $A \subset \Omega$). Let $\Omega_1 = \Omega \setminus A$. Then Ω_1 is open and there exists an increasing sequence (B_j) of open subsets of Ω_1 such that $\overline{B}_j \subset \Omega_1$ for every j ($\overline{B}_j$ denotes the Euclidean closure of B_j) and that $\bigcup_j B_j = \Omega_1$. For any j write $K_j = \overline{B}_j \cup A$. Clearly, (K_j) is an increasing sequence of compact subsets of Ω with $\bigcup_j K_j = \bigcup_j \overline{B}_j \cup A = \Omega_1 \cup A = \Omega$. It suffices to show that $K'_j \subset \overline{B}_j$ for every j, for then $\bigcup_j K'_j \subset \bigcup_j \overline{B}_j = \Omega_1 = \Omega \setminus A$ with A non-polar. Let $x \in K'_j$. If $x \in A$ then $V := \Omega \setminus \overline{B}_j$ is an open neighborhood of x and $V \cap \overline{B}_j = \emptyset$. On the other hand, $W := K'_j$ is a fine neighborhood of x contained in K_j. Then $W \cap V \subset K_j$ and $(W \cap V) \cap \overline{B}_j = \emptyset$, hence $x \in W \cap V \subset A$. But $W \cap V$ is finely open and $A' = \emptyset$, so actually $x \notin A$, and since $x \in K'_j \subset K_j = \overline{B}_j \cup A$ we have $x \in \overline{B}_j$. Because this holds for every $x \in K'_j$ we indeed have $K'_j \subset \overline{B}_j$.

Remark 3.4. The finely open subsets K'_n are regular because for any integer n the set K_n is compact (in initial topology).

Corollary 3.5. *There is a sequence (H_n) of compact subsets of U, each is non-thin at any of its points, and a polar set P such that*

1. $U = \bigcup_n H_n \cup P$.
2. *For any integer n, the restriction of any function of $\mathcal{S}(U)$ to H_n is l.s.c. in initial topology.*

Proof: One may write $U = \bigcup_n K'_n \cup P$ as in Theorem 3.2. For any integer n, let (U_n^m) be the sequence of fine components of K'_n. The finely open subsets U_n^m are necessarily regular according to Remark 3.4. Let $p > 0$ be a strict, finite and continuous potential on Ω (cf. [9, page 166]). Then according to [9, Proposition 7.2.2] we have for any pair of integers (m, n),

$$b(\complement U_n^m) = \big\{x \in \Omega\colon \widehat{R}_p^{\complement U_n^m}(x) = p(x)\big\}.$$

Since U_n^m is regular then we have

$$U_n^m = \big\{x \in \Omega\colon \widehat{R}_p^{\complement U_n^m}(x) < p(x)\big\}.$$

For any pair of integers (m, n) and for any integer $l > 0$, put

$$H_{n,m,l} = \Big\{x \in U_n^m : p(x) - \widehat{R}_p^{\complement U_n^m}(x) \geq \frac{1}{l}\Big\}.$$

Then the sets $H_{n,m,l}$ are compact in the initial topology and each one of them is non-thin at any of its points and we have $U = \bigcup_{n,m,l} H_{n,m,l} \cup P$. The restriction of any function $s \in \mathcal{S}(U)$ to $H_{n,m,l}$ is indeed l.s.c. in the initial topology. □

We shall use the sequence (H_n) of the previous corollary to define by analogy with [30] a locally compact topology on the cone $\mathcal{S}(U)$. For any n we denote by $\mathcal{C}_l(H_n)$ the cone of l.s.c. functions on H_n with values in $\overline{\mathbb{R}}_+$, and we endow it with the topology of the convergence in graph (cf. [30]). It is known that $\mathcal{C}_l(H_n)$ is a compact and metrizable space for this topology. Let d_n be a distance compatible with this topology. We define a distance d on $\mathcal{U}_+(U)$ by putting

$$d(u; v) = \sum_n \frac{1}{2^n \delta(\mathcal{C}_i(H_n))}\, d_n(u_{|H_n}, v_{|H_n})$$

for any pair (u, v) of functions of $\mathcal{U}_+(U)$, where $\delta(\mathcal{C}_l(H_n))$ denotes the diameter of $\mathcal{C}_l(H_n)$. Since two finely hyperharmonic functions are identical if they coincide τ-a.e., it follows that d is a distance on $\mathcal{U}_+(U)$. We denote by $\mathcal{T}$ the topology defined on $\mathcal{U}(U)$ by the distance d.

Let $\mathcal{F}$ be a filter on $\mathcal{U}_+(U)$, put

$$\liminf\widehat{\ }_{\mathcal{F}} = \sup_{M \in \mathcal{F}} \widehat{\inf_{u \in M}}.$$

Theorem 3.6. *The cone $\mathcal{U}_+(U)$ of finely hyperharmonic functions nonnegative on U is compact in the topology $\mathcal{T}$. For any ultrafilter $\mathcal{G}$ on $\mathcal{U}_+(U)$ we have*

$$\lim_{\mathcal{G}} = \lim \widehat{\inf}_{\mathcal{G}}.$$

Proof: We will copy word by word (with necessary modifications) the proof of the same result in [12], corresponding to the case where Ω is a Green open subset of $\mathbb{R}^n$. Let $\mathcal{G}$ be an ultrafilter on $\mathcal{U}_+(U)$. For any $M \in \mathcal{G}$, put $u_M = \inf_{u\in M} u$. Then the bases of ultrafilters $\mathcal{G}_n$, images of $\mathcal{G}$ by restrictions to the compacts H_n, converge, in the compact spaces $\mathcal{C}_l(H_n)$, to the functions $u_n = \sup_{M\in\mathcal{U}} \widehat{u_M}^n$, where $\widehat{u}^n$ denotes the l.s.c. regularization of the restriction of u to H_n. Let $M \in \mathcal{G}$ and n be an integer, then the finely l.s.c. regularized $\widehat{u_M}$ of u_M is l.s.c. on H_n according to Theorem 3.2 and minorizes u_M, then $\widehat{u_M} \le \widehat{u_M}^n$ in H_n. On the other hand, according to the axiom D, there is a polar subset A of Ω such that $u_M = \widehat{u_M}$ in $U \setminus A$, then $\widehat{u_M}^n \le \widehat{u_M}$ on $H_n \setminus A$. Otherwise, if $x \in A$, we have $\widehat{u_M}^n(x) \le \widehat{u_M}(x)$ because $\widehat{u_M}$ is finely continuous on U and x belongs to the fine closure of $H_n \setminus A$ since H_n is not thin at the point x. Hence $u_n = \widehat{\liminf}_\mathcal{G}$ in H_n for any n, and since the function $u = \widehat{\liminf}_\mathcal{G}$ belongs to $\mathcal{U}_+(U)$ by [17, Theorem 12.9], we see that the ultrafilter $\mathcal{G}$ converges to u with respect to the topology $\mathcal{T}$. Hence $\mathcal{U}_+(U)$ is compact in the topology $\mathcal{T}$. □

Corollary 3.7. *The topology of the convergence in graph coincides with the natural topology on $\mathcal{S}(U)$.*

Proof: According to Theorem 3.6 and Theorem 4.5.8 of [5], the natural topology on $\mathcal{S}(U)$ is coarser than the topology of convergence in graph on $\mathcal{S}(U)$. Let $\mathcal{G}$ be an ultrafilter on $\mathcal{S}(U)$ which converges with respect to the natural topology to $s \in \mathcal{S}(U)$, then, always according to the previous theorem, $\mathcal{G}$ converges in graph to s. We deduce that the topology in graph on $\mathcal{S}(U)$ is coarser than the natural topology of $\mathcal{S}(U)$. So both topologies are identical on $\mathcal{S}(U)$. □

Corollary 3.8. *Let $\mathcal{F}$ be a filter on $\mathcal{S}(U)$, which converges with respect to the topology of the convergence in graph. Then we have $\lim_\mathcal{F} = \widehat{\liminf}_\mathcal{F}$.*

Proof: The corollary follows from Corollary 3.7 and from [5, Theorem 4.5.2]. □

Corollary 3.9. *The cone $\mathcal{S}(U)$ endowed with the natural topology has a compact base.*

Proof: The natural topology on $\mathcal{S}(U)$ is locally compact, then it follows from a theorem by Klee, see [1, Theorem II.2.6], that $\mathcal{S}(U)$ has a compact base. □

Corollary 3.10. *For any $x \in U$ and for any subset A of U, the functions $u \mapsto u(x)$ and $u \mapsto \widehat{R}_u^A(x)$, with values in $[0,\infty]$, are affine and l.s.c. with respect to the natural topology on $\mathcal{S}(U)$.*

Proof: It is clear that the map $u \mapsto \widehat{R}_u^A(x)$ is affine for fixed point $x \in U$. Let (u_j) be a sequence in $\mathcal{S}(U)$ which converges naturally (i.e. in natural topology) to $u \in \mathcal{S}(U)$. For any integer k we have

$$\widehat{\inf_{j\ge k}}\, \widehat{R}^A_{u_j}(x) \ge \widehat{R}^A_{\widehat{\inf}_{j\ge k} u_j}(x).$$

Both members of this inequality increase with k and we have when $k \to \infty$,

$$\liminf_k \widehat{R}^A_{u_k}(x) \geq \lim_k \widehat{\inf}\, \widehat{R}^A_{u_k}(x) \geq \widehat{R}^A_{\lim \widehat{\inf} u_k}(x) \geq \widehat{R}^A_u(x),$$

and then the function $u \mapsto \widehat{R}^A_u(x)$ is l.s.c. on $\mathcal{S}(U)$. For the function $u \mapsto u(x)$ it suffices to take $A = U$. $\square$

Let B be a compact base of the cone $\mathcal{S}(U)$ and μ a probability measure on B. For any $x \in U$ and any subset A of U, the integrals $\int p(x)\,\mathrm{d}\mu(p)$ and $\int_B \widehat{R}^A_p(x)\,\mathrm{d}\mu(p)$ are well defined because the functions $u \mapsto \widehat{R}^A_u(x)$ are Borel functions and nonnegative on B according to the previous corollary. We denote by $\int p\,\mathrm{d}\mu(p)$ and $\int_B \widehat{R}^A_p\,\mathrm{d}\mu(p)$ the functions u and v defined on U by $u(x) = \int_B p(x)\,\mathrm{d}\mu(p)$ and $v(x) = \int_B \widehat{R}^A_p(x)\,\mathrm{d}\mu(p)$ for every $x \in U$. Let s be the barycenter of μ, $A \subset U$ and $x \in U$. The functions $p \mapsto p(x)$ and $p \mapsto \widehat{R}^A_p(x)$ are nonnegative affine functions and l.s.c. on B, then, according to [1, Corollary I.1.4], there are increasing sequences (f_n) and (g_n) of continuous affine forms on B such that $p(x) = \sup_n f_n(p)$ and $\widehat{R}^A_p(x) = \sup_n g_n(p)$ for any $p \in B$. Thus we have $s(x) = \sup_n f_n(s) = \sup_n \int_B f_n(p)\,\mathrm{d}\mu(p) = \int_B p(x)\,\mathrm{d}\mu(p)$ and $\widehat{R}^A_s(x) = \sup_n g_n(s) = \sup_n \int_B g_n(p)\,\mathrm{d}\mu(p) = \int_B \widehat{R}^A_p(x)\,\mathrm{d}\mu(p)$ by monotone convergence theorem. We deduce that $u = s$ and $v = \widehat{R}^A_s$ and consequently, $u, v \in \mathcal{S}(U)$. Thus, we have proved the following theorem:

Theorem 3.11. *Let B be a compact base of $\mathcal{S}(U)$, μ a Radon measure on B and A a subset of U, and let $s = \int_B p\,\mathrm{d}\mu(p)$. Then s is a finely superharmonic function in U and $\widehat{R}^A_s = \int_B \widehat{R}^A_p\,\mathrm{d}\mu(p)$. In particular the function $\int_B \widehat{R}^A_p\,\mathrm{d}\mu(p)$ is finely superharmonic in U.*

Theorem 3.12. *Let B be a compact base of $\mathcal{S}(U)$ and $u \in \mathcal{S}(U)$. Then there exists a unique Radon measure nonnegative μ on B supported by the set* $\mathrm{Ext}(B)$ *of extreme elements of B such that $u = \int_B p\,\mathrm{d}\mu(p)$.*

Proof: We may suppose that $u \neq 0$ ($\mu = 0$ is the unique measure corresponding to the case where $u = 0$), so that there is a real $\alpha > 0$ such that $\alpha u \in B$. Since the cone $\mathcal{S}(U)$ is a lattice in its own order (the specific order) according to [17, 11.15 a), page 131], then it follows from Choquet's theorem of integral representation that there exists a unique probability measure ν on B, carried by $\mathrm{Ext}(B)$ (which is a G_δ set by a result of Choquet) and of barycenter ν. Then we have $\alpha u = \int_B p\,\mathrm{d}\nu(p)$. The measure $\mu = (1/\alpha)\nu$ satisfies the conditions of the theorem. $\square$

The measure μ associated to $u \in \mathcal{S}(U)$ in the previous theorem will be called the maximal measure representing u.

4. Fine Green kernel and integral representation of fine potentials and invariant functions

From now on we assume that Ω is a $\mathcal{P}$-Brelot harmonic space with countable base satisfying the domination axiom and the uniqueness hypothesis, that is, the hypothesis of proportionality of the potentials of the same support reduced to one point. According to [25, Theorem 18.1 and Proposition 18.1], Ω has a Green kernel G, that is, a function $G\colon \Omega\times\Omega \to \overline{\mathbb{R}}_+$ such that

1. For any $y \in \Omega$, the function $G(\cdot, y)$ is a potential greater than 0 in Ω and harmonic on $\Omega \setminus \{y\}$.
2. Function G is l.s.c. on $\Omega\times\Omega$ and continuous outside the diagonal of $\Omega\times\Omega$.

Moreover, every potential p on Ω admits an integral representation $p = G\mu := \int G(\cdot, y)\,\mathrm{d}\mu(y)$, where μ is a (nonnegative) Radon measure on Ω (see [25, Théorème 18.2, page 481]).

Assume further that the topology of Ω has a base formed by completely determining open subsets (cf. [25, Definition, page 451]). According to R.-M. Hervé [25, Chapter VI] we can define on Ω a structure of (Brelot) adjoint harmonic space, in which the function G^* defined by $G^*(x, y) = G(y, x)$ is a Green kernel. We also assume that the adjoint harmonic space satisfies the axiom of domination, so that we can use the notions related to the adjoint fine potential theory on adjoint finely open subsets of Ω.

To show the existence of a fine Green kernel greater than 0 on a fine domain U of Ω in [18], B. Fuglede assumes that the fine topology and the adjoint fine topology are identical. In the following we place ourselves in a somewhat more general framework where we only assume that the fine topology is coarser than the adjoint fine topology, or just only that U is an adjoint finely open set (that is, an open set relatively to the adjoint fine topology). The results of the preceding sections apply to this framework. We shall also show that the hypothesis that U is an adjoint finely open set is necessary (and sufficient modulo an additional condition (see Theorem 4.5 below)) for the existence of a fine Green kernel greater than 0 in U. This situation is a bit more general than the one considered by B. Fuglede in [18].

Let U be a fine domain of Ω and we suppose, without loss of generality, that U is regular. For any $y \in U$, the function $\widehat{R}^{\complement U}_{G(\cdot,y)}$ is finely harmonic on U if $\{y\}$ is not polar, and finely harmonic in $U \setminus \{y\}$ if $\{y\}$ is polar in view of [17, Theorem 11.13, page 127]. We denote by $G_U(\cdot, y)$ the finely superharmonic function in U, defined on $U \setminus \{y\}$ by

$$G_U(x, y) = G(x, y) - \widehat{R}^{\complement U}_{G(\cdot,y)}(x),$$

and eventually extended by fine continuity at the point y if $\{y\}$ is a polar set (cf. [5, Theorem 9.15, page 98]).

Lemma 4.1. *Let $p = \int G(\cdot, y)\,\mathrm{d}\mu(y)$ be a potential on Ω, harmonic outside a compact subset K of Ω, where μ is a Radon measure on Ω. Then the measure μ is supported by K.*

Proof: Let ω be a relatively compact open subset of Ω such that $\overline{\omega} \subset \Omega \setminus K$. We have $\int G(\cdot, y)\,\mathrm{d}\mu(y) = \int_\omega G(\cdot, y)\,\mathrm{d}\mu(y) + \int_{\Omega\smallsetminus\omega} G(\cdot, y)\,\mathrm{d}\mu(y)$, and then the function $\int_\omega G(\cdot, y)\,\mathrm{d}\mu(y)$ is harmonic on the complement of K. On the other hand, it is well known that the function $\int_\omega G(\cdot, y)\,\mathrm{d}\mu(y)$ is harmonic on $\Omega \smallsetminus \overline{\omega}$, and therefore on a neighborhood of K. It follows that the function $\int_\omega G(\cdot, y)\,\mathrm{d}\mu(y)$ is harmonic in Ω. Hence it is null because it minorizes the potential p. It follows then that $\mu(\omega) = 0$. Since the open set ω is arbitrary and the space Ω has a countable base, then $\mu(\Omega \smallsetminus K) = 0$, that is, μ is supported by K. $\square$

Lemma 4.2. *The set $A = \{y \in U \colon G_U(\cdot, y) = 0\}$ is polar.*

Proof: Let us first prove that A is a Borel subset of Ω. In fact, let $x \in U$, then we have $A \smallsetminus \{x\} = \big\{y \in \Omega \smallsetminus \{x\} \colon G(x, y) = \widehat{R}^{\complement U}_{G(\cdot,y)}(x)\big\}$. Thus $A \smallsetminus \{x\}$ is a Borel subset of $\Omega \smallsetminus \{x\}$ because the functions $y \mapsto G(x, y)$ and $y \mapsto \widehat{R}^{\complement U}_{G(\cdot,y)}(x)$ are Borel measurable (the first being continuous and the second is l.s.c. on Ω) and U is a Borel subset of Ω since it is supposed regular. We deduce that $A \smallsetminus \{x\}$ is a Borel subset of $\Omega \smallsetminus \{x\}$. Hence A is a Borel subset of Ω. Let K be a compact subset of A and p a finite, continuous and strict potential in Ω. According to [25, Théorème 18.2, page 481], there is a measure $\mu \geq 0$ such that $\widehat{R}^K_p = G\mu = \int G(\cdot, y)\,\mathrm{d}\mu(y)$. The function $\widehat{R}^K_p$ is harmonic in $\Omega \smallsetminus K$, then the measure μ is supported by K by the previous lemma. On the other hand, we have $\widehat{R}^{\complement U}_{\widehat{R}^K_p} = \int \widehat{R}^{\complement U}_{G_{(\cdot,y)}}\,\mathrm{d}\mu(y)$ by [25, Théorème 22.4, page 508] and the monotone convergence theorem since $\complement U$ is an F_σ. The measure μ is supported by K and for any $y \in K$ we have $\widehat{R}^{\complement U}_{G(\cdot,y)} = G(\cdot, y)$ because U is assumed regular, then $\widehat{R}^K_p = \widehat{R}^{\complement U}_{\widehat{R}^K_p}$. The function $\widehat{R}^K_p$ is harmonic in $\Omega \smallsetminus K$ and the function $\widehat{R}^{\complement U}_{\widehat{R}^K_p}$ is finely harmonic in U, we deduce from the above equality that $\widehat{R}^K_p$ is finely harmonic in Ω, and hence harmonic in Ω according to [17, Theorem 9.8, page 87] because it is locally bounded. Since p is a potential and $\widehat{R}^K_p \leq p$, it follows that $\widehat{R}^K_p = 0$, so that K is polar according to the polarity criterion of [25, page 434]. Thus any compact subset contained in A is polar. It follows from Choquet's capacitability theorem that A is polar because it is a Borel set, then analytic in the space with countable base Ω. $\square$

Corollary 4.3. *Assume that the adjoint potentials on Ω of the same punctual support are proportional. Then the set $A = \{x \in U \colon G_U(x, \cdot) = 0\}$ is polar.*

Proof: It suffices to apply the previous lemma to the kernel G^* and use the relation between the balayage and adjoint balayage (cf. [25, page 550]), and the fact that the polar sets and the adjoint polar sets in Ω are identical by [25, Theorem 32.1]. $\square$

Proposition 4.4. *Suppose in addition that U is [also] an adjoint finely open subset of Ω. Then we have $G_U(x, y) > 0$ for any pair $(x, y) \in U^2$.*

Proof: Let $x \in U$. According to Lemma 4.2, the set $A = \{z \in U\colon G_U(x,z) = 0\} = \{z \in U\colon G_U(\cdot,z) = 0\}$ is polar. Since the polar sets and the adjoint polar sets are identical by [25, Theorem 32.1], it follows that A is adjoint polar, then the interior of A with respect to the adjoint fine topology is empty. Let ω be an adjoint finely connected component of U, then $\omega \setminus A \neq \emptyset$, and hence there exists $z \in \omega$ such that $G_U(x,z) > 0$. The function $y \mapsto G_U(x,y)$ is adjoint finely superharmonic and nonnegative on $U \setminus \{x\}$ according to [25, Lemma 30.1 and Theorem 31.1] and [17, Theorem 11.13, page 127]. It is not identically zero on $\omega \setminus \{x\}$, and hence $G_U(x,y) > 0$ for any $y \in \omega \setminus \{x\}$ according to [17, Theorem 12.6, page 150]. We deduce that $G_U(x,y) > 0$ for any $y \in U$. The proposition is proved. $\square$

The following theorem can then be proved exactly as [18, Theorem, page 203]:

Theorem 4.5. *Suppose that U is an adjoint finely open subset of U and, moreover, that for any subset A of U, $A' \subset b^*(A)$. Then for any $y \in U$, the function $x \mapsto G_U(x,y)$ is a fine potential in U finely harmonic on $U \setminus \{y\}$. Every fine potential verifying this condition is of the form $\alpha G_U(\cdot,y)$ for some $\alpha > 0$.*

Remark 4.6. Without the additional condition that for every $A \subset \Omega$, one has $A' \subset b^*(A)$, we can only prove that $G_U(\cdot,y)$ is a fine potential in U (finely harmonic in $U \setminus \{y\}$ by definition of G_U). Indeed, let $y \in U$ and let s be a finely subharmonic function in U such that $s \leq G_U(\cdot,y)$, and q a finite potential in Ω. Then for any integer n we have for any $z \in \partial_f U$, $s \leq G(\cdot,y) - \widehat{R}^{\complement U}_{G(\cdot,y)\wedge nq} = G(\cdot,y) - \overline{H}^U_{G(\cdot,y)\wedge nq}$ on U, so that $\limsup_{x\in U\to z} s(x) \leq G(z,y) - \liminf_{x\in U, x\to z} \overline{H}^U_{G(\cdot,y)\wedge nq}(x) = G(z,y) - G(z,y) \wedge nq(z)$, where the last equality follows from [17, Theorem 14.7]. We get $\limsup_{x\in U, x\to z} s(x) \leq 0$ by letting $n \to \infty$. It follows that $s \leq 0$ by the minimum principle [17, Theorem 10.8]. Hence $G_U(\cdot,y)$ is a fine potential.

Remark 4.7. Similarly, if again U is an adjoint finely open subset of Ω, then for any $x \in U$, the function $y \mapsto G_U(x,y)$ is an adjoint fine potential in U.

Definition 4.8. The function $G_U\colon (x,y) \mapsto G_U(x,y)$ defined on U^2 is called the fine Green kernel of U.

Proposition 4.9. *The following statements are equivalent:*

1. *The function $G_U(\cdot,y) > 0$ for any $y \in U$.*
2. *The set U is an adjoint finely open subset of Ω.*

Proof: The implication 2. $\Rightarrow$ 1. was established in the proof of Proposition 4.4. Let us prove the opposite implication. Suppose that $G_U(\cdot,y) > 0$ for any $y \in U$. Let us put $U_x = \left\{y \in \Omega \setminus \{x\}\colon G(x,y) > \widehat{R}^{\complement U}_{G(\cdot,y)}(x)\right\}$ for any $x \in U$. Then U_x is an adjoint finely open subset of Ω because the functions $y \mapsto G(x,y)$ and $y \mapsto \widehat{R}^{\complement U}_{G(\cdot,y)}(x)$ are adjoint superharmonic by [25, Proposition 30.1], then continuous in adjoint fine topology. Let $x_1, x_2 \in U$ such that $x_1 \neq x_2$, then we have for example $y \neq x_1$ and hence $y \in U_{x_1}$. Anyway $U = U_{x_1} \cup U_{x_2}$, and hence U is an adjoint finely open subset of Ω. □

A function $h \in \mathcal{S}(U)$ is said to be invariant if it is orthogonal for the specific order (order defined on the cone $\mathcal{S}(U)$) to the band $\mathcal{P}(U)$ of all fine potentials on U. The set of invariant functions of $\mathcal{S}(U)$ is denoted by $\mathcal{H}_i(U)$. It is a convex cone and a band of $\mathcal{S}(U)$. Every function $u \in \mathcal{S}(U)$ has a unique decomposition of the form $u = p + h$, where p is a finely potential and h is an invariant function on U. This decomposition is called the Riesz decomposition of nonnegative finely superharmonic functions. Then the invariant functions play (for the nonnegative finely superharmonic functions) the role of nonnegative harmonic functions in the Riesz decomposition of nonnegative superharmonic functions on an open subset with respect to the initial topology of Ω.

We shall say that a function $u \in \mathcal{S}(U)$ is extremal if it belongs to an extreme ray of the cone $\mathcal{S}(U)$. It follows from the Riesz decomposition of finely superharmonic functions that every extremal function of $\mathcal{S}(U)$ is either an invariant function or a fine potential.

Proposition 4.10. *Under the hypotheses of Theorem 4.5, the function $G_U(\cdot, y)$ is extremal for any $y \in U$.*

Proof: Let $u_1, u_2 \in \mathcal{S}(U)$ such that $u_1 + u_2 = G_U(\cdot, y)$, then u_1 and u_2 are fine potentials finely harmonic in $U \setminus \{y\}$. Hence, according to Theorem 4.5, u_1 and u_2 are proportional to $G_U(\cdot, y)$, thus $G_U(\cdot, y)$ is extremal. □

The Lemma 2.1 in [12] contains, as well as its proof, some imperfections, and the proofs of Theorem 2.3 and Theorem 2.4 in [12] are also incomplete and need the following correct version of this lemma

Proposition 4.11. *Let $u, s \in \mathcal{S}(U)$ such that s is finely harmonic outside a polar set. Then $s \leq u$ if and only if $s \prec u$.*

Proof: Suppose that $s \leq u$. There is a polar set $E \subset U$ such that s is finely harmonic on $U \setminus E$, and hence $u - s$ is finely superharmonic nonnegative on $U \setminus E$. According to [17, Theorem 9.14], $u - s$ extends by fine continuity to a function $t \in \mathcal{S}(U)$ such that $t + s = u$ on $U \setminus E$ and hence on all of U, whence $h \prec u$. The converse is obvious. □

Corollary 4.12. *Let $u \in \mathcal{S}(U)$ and h be an invariant function. Then $h \leq u$ if and only if $h \prec u$.*

Proof: By [17, Theorem 10.10] there is a polar set $E \subset U$ such that h is finely harmonic in $U \setminus E$, hence $h \prec u$ by Proposition 4.11. □

Remark 4.13. Let $u, v \in \mathcal{S}(U)$ such that $u \prec v$, that is $v = u + w$ for some $w \in \mathcal{S}(U)$. Then the function w is unique, we denote it by $v - u$. Furthermore we have $w(x) = u(x) - v(x)$ for every $x \in \{v < \infty\}$.

Proposition 4.11 applies in particular when $u = s_{|U}$ and $v = \widehat{R}_s^{\complement U}{}_{|U}$, where $s \in \mathcal{S}(\Omega)$, because, in view of [17, Theorem 11.13, page 127], $\widehat{R}_s^{\complement U}$ is finely harmonic outside the polar set where it takes the value ∞. The function $s - \widehat{R}_s^{\complement U}$ is also denoted by s_U. It follows from fine minimum principle (cf. [17, Theorem 10.8, page 106]) that s_U is a fine potential in U if s is a potential in Ω.

Lemma 4.14. *Let $x_0 \in U$ and $A \subset U$ relatively compact in Ω such that $\inf\{G_U(x_0, y)\colon y \in A\} = c > 0$ and $\widetilde{A} \subset U$. Then for any $s \in \mathcal{S}(U)$ such that $s(x_0) < \infty$, there exists a measure μ on Ω with support contained in $\overline{A}$ such that $s \leq G\mu$ on U.*

Proof: Let p be a finite potential greater than 0 on Ω. By Lemma 1.3 in [20] and its proof, there is a sequence of potentials p_n in Ω, harmonic in $\Omega \setminus \overline{A}$ such that $(p_n)_U \prec {}^U\widehat{R}^A_{s\wedge nq} \leq p_n$. By [25, Théorème 18.2, page 481], for each $n \in \mathbb{N}$, there is a (Radon) measure μ_n on Ω such that $p_n = G\mu_n$. The measures μ_n are carried by $\widetilde{A}$ and we have $|\mu_n| \leq (1/c)G_U\mu_n(x_0) \leq s(x_0)$ for every $n \in \mathbb{N}$. Hence there is a subsequence (μ_{n_k}) of (μ_n) which converges weakly to a measure μ supported by $\overline{A}$. By letting $n \to \infty$ in the inequality ${}^U\widehat{R}^A_{s\wedge nq} \leq G\mu_n$ we obtain ${}^U\widehat{R}^A_s \leq \liminf G\mu_{n_k} \leq G\mu$ on U and the proof is complete. $\square$

Let $s \in \mathcal{S}(U)$, $s > 0$ and $x_0 \in U$ such that $s(x_0) < \infty$. Consider a sequence of relatively compact open subsets ω_n of Ω, $n \geq 1$, such that $\bigcup_n \omega_n = \Omega$ and let us put $E = \{x \in U\colon s(x) = \infty\}$. For each integer $n > 0$ we put $U_n = \omega_n \cap \{y \in U\colon G_U(x_0, y) > 1/n\}$, $e_n = U_n \cap E$ and $s_n = \widehat{\inf}\{{}^U\widehat{R}^V_s : e_n \subset V \subset U_n : V \text{ finely open set}\}$. Suppose that the function $y \mapsto G_U(x_0, y)$ is finely continuous on U. Then the sets U_n are finely open and for every $n \geq 1$ $\widetilde{U_n} \subset U$. The functions s_n are finely harmonic outside polar subsets of U and we have $s_n \leq s$ and then $s_n \prec s$ by Proposition 4.11. Let us also remark that by Lemma 4.14, each function s_n is majorized on U by a potential in Ω.

Lemma 4.15. *For any potential p on Ω and any finely open set $V \subset \widetilde{V} \subset U$, one has:*

1. ${}^U\widehat{R}^{U\setminus V}_p = \widehat{R}^{\complement V}_p$ *on V.*
2. ${}^U\widehat{R}^{U\setminus V}_{\widehat{R}^{\complement U}_p} = \widehat{R}^{\complement U}_p$ *on U.*

Proof: Without loss of generality we may suppose that the constants are harmonic on Ω.

1. Let $s \in \mathcal{S}(\Omega)$ be such that $s \geq p$ on $\complement V$, then $s \geq p$ on $U \setminus V$ and hence $s \geq {}^U\widehat{R}^{U\setminus V}_p$, whence $\widehat{R}^{\complement V}_p \geq {}^U\widehat{R}^{U\setminus V}_p$. For the opposite inequality let k be a real number greater than 0 and $s \in \mathcal{S}(U)$ such that $s \geq p$ on $U \setminus V$. The function $\widehat{R}^{\complement V}_{p\wedge k}$ is finely harmonic in V by [17, Corollary of Lemma 9.7, page 86], thus the function $s - \widehat{R}^{\complement V}_{p\wedge k}$ is finely superharmonic in V. For any $z \in \partial_f V$, we have $\liminf_{x\in V, x\to z}(s(x) - \widehat{R}^{\complement V}_{p\wedge k}(x)) = \liminf_{x\in U, x\to z}(s(x) - \widehat{R}^{\complement V}_{p\wedge k}(x)) = s(z) - p \wedge k(z) \geq 0$. By the fine minimum principle [17, Theorem 10.8, page 106], we deduce

that $s \geq \widehat{R}^{\complement V}_{p\wedge k}$ and conclude that ${}^U\widehat{R}^{U\setminus V}_p \geq \widehat{R}^{\complement V}_{p\wedge k}$ on V. By letting $k\to\infty$, we obtain ${}^U\widehat{R}^{U\setminus V}_p \geq \widehat{R}^{\complement V}_p$ on V. The requested equality is now proved.

2. It suffices to apply the first equality to the potential $\widehat{R}^{\complement U}_p$ to obtain the equality in V and use the property that $\widehat{R}^{\complement V}_{\widehat{R}^{\complement U}_p} = \widehat{R}^{\complement U}_p$ q.e. on $\complement V$. $\square$

Proposition 4.16. *In addition to the hypotheses of Theorem* 4.5, *suppose that the adjoint fine topology on Ω is finer than the fine topology. Let (U_n) be an increasing sequence of finely open subsets of U such that $\widetilde{U_n} \subset U$ for every integer n and $\bigcup_n U_n = U$. Then for any $y\in U$ we have $\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} = \widehat{R}^{\complement U}_{G(\cdot,y)}$.*

Proof: Let y be a fixed point in U. According to Proposition 4.11 and [17, Theorem 11.13, page 127], we have $\widehat{R}^{\complement U}_{G(\cdot,y)} \prec \widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)}$ and then $\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} - \widehat{R}^{\complement U}_{G(\cdot,y)} \leq G_U(\cdot,y)$. It follows by Remark 4.6 that $\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} - \widehat{R}^{\complement U}_{G(\cdot,y)}$ is a fine potential on U, finely harmonic on $U\setminus\{y\}$. In view of Theorem 4.5 there is a real $\alpha\in[0,1]$ such that $\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} - \widehat{R}^{\complement U}_{G(\cdot,y)} = \alpha G_U(\cdot,y)$ and hence

$$\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} = \alpha G(\cdot,y) + (1-\alpha)\widehat{R}^{\complement U}_{G(\cdot,y)}. \tag{4.1}$$

For any integer m and any $x\in U_m$, $x\neq y$, x outside a polar subset e of U,

$$\widehat{R}^{\complement U_m}_{\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)}} = \int \widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)}\, \mathrm{d}\varepsilon_x^{\complement U_m} = \widehat{\inf}_n \int \widehat{R}^{\complement U_n}_{G(\cdot,y)}\, \mathrm{d}\varepsilon_x^{\complement U_m}$$

by Lebesgues convergence theorem. On the other hand we have

$$\int \widehat{R}^{\complement U_n}_{G(\cdot,y)}\, \mathrm{d}\varepsilon_x^{\complement U_m} = \widehat{R}^{\complement U_m}_{\widehat{R}^{\complement U_n}_{G(\cdot,y)}}(x) = \widehat{R}^{\complement U_n}_{G(\cdot,y)}(x)$$

for any $n\geq m$. Hence

$$\widehat{R}^{\complement U_n}_{\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)}} = \widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)}$$

on $U\setminus e$ and therefore everywhere by fine continuity. Then it follows from (4.1) that

$$\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} = \alpha \widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} + (1-\alpha)\widehat{R}^{\complement U}_{G(\cdot,y)},$$

and hence

$$(1-\alpha)\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} = (1-\alpha)\widehat{R}^{\complement U}_{G(\cdot,y)}. \tag{4.2}$$

According to (4.1) we have $\alpha<1$ because otherwise we would have

$$G(\cdot,y) \geq \widehat{R}^{\complement U_n}_{G(\cdot,y)} \geq \widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} = G(\cdot,y)$$

and hence $G(\cdot,y) = \widehat{R}^{\complement U_n}_{G(\cdot,y)}$ for any n such that $y\in U_n$, which is absurd by Proposition 4.9. Thus it follows from (4.2) that $\widehat{\inf}_n \widehat{R}^{\complement U_n}_{G(\cdot,y)} = \widehat{R}^{\complement U}_{G(\cdot,y)}$. $\square$

Corollary 4.17. *For any nonnegative Borel measure μ on U such that $q = G\mu \neq \infty$, we have $\inf_n {}^U\widehat{R}_{q_U}^{U\setminus U_n} = 0$ q.e. on U.*

Proof: We have $q = q_U + \widehat{R}_q^{\complement U}$ on U and hence ${}^U\widehat{R}_q^{U\setminus U_n} ={}^U \widehat{R}_{q_U}^{U\setminus U_n} +{}^U \widehat{R}_{\widehat{R}_q^{\complement U}}^{U\setminus U_n}$. By using Lemma 4.15 this yields $\widehat{R}_q^{\complement U_n} ={}^U \widehat{R}_{q_U}^{U\setminus U_n} +{}^U \widehat{R}_q^{\complement U}$ on U_n for each n and therefore $\widehat{\inf}_n \widehat{R}_q^{\complement U_n} = \widehat{\inf}_n {}^U\widehat{R}_{q_U}^{U\setminus U_n} +{}^U \widehat{R}_q^{\complement U}$ on U. Following Proposition 4.16, we have $\widehat{\inf}_n \widehat{R}_q^{\complement U_n} ={}^U \widehat{R}_q^{\complement U}$, hence $\widehat{\inf}_n^U \widehat{R}_{q_U}^{U\setminus U_n} = 0$ and, by the convergence theorem for finely hyperharmonic functions, $\inf_n {}^U\widehat{R}_{q_U}^{U\setminus U_n} = 0$ q.e. on U. □

Theorem 4.18. *Let $h \in \mathcal{S}(U)$, $x_0 \in U$ be such that $h(x_0) < \infty$ and $(\omega_n)_{n\geq 1}$ an increasing sequence of relatively compact subsets of Ω such that $\Omega = \bigcup_n \omega_n$. For any integer $n \geq 1$, put $U_n = \{y \in U : G_U(x_0, y) > 1/n\} \cap \omega_n$. Suppose that for any $x \in U$, the function $y \mapsto G_U(x_0, y)$ is finely continuous on U, and that the adjoint fine topology is finer than the fine topology on Ω. Then h is invariant if and only if ${}^U\widehat{R}_h^{U\setminus U_n} = h$ for every n.*

Proof: Assume that h is invariant and let $n \in \mathbb{N}^*$. We have $h \leq {}^U\widehat{R}_h^{U_n} +{}^U\widehat{R}_h^{U\setminus U_n}$. By the Riesz decomposition property there exist $h_1, h_2 \in \mathcal{S}(U)$ such that $h = h_1 + h_2$ and $h_1 \leq^U \widehat{R}_h^{U_n}$, $h_2 \leq {}^U\widehat{R}_h^{U\setminus U_n}$. Since h is invariant we infer that h_1 and h_2 are invariant and, by Lemma 4.14, h_1 is majorized by a potential in Ω. Let $q = \widehat{R}_{h_1}^{U_n}$ in $\mathcal{S}(\Omega)$. Then q is a potential in Ω and hence it is of the form $q = G\lambda$ for some measure λ on Ω carried by $\widetilde{U_n}$ according to [25, Corollaire 2, page 552] and [17, 4.8, page 37] (indeed, λ is carried by the adjoint fine closure of U_n which in turn is contained in $\widetilde{U_n}$ because the adjoint fine topology is supposed to be finer than the fine topology). On the other hand, we have $q = q_U + \widehat{R}_q^{\complement U}$ on U and q_U is a fine potential in U. Hence $h_1 = t_1 + t_2$ where $t_1, t_2 \in \mathcal{S}(U)$ are such that $t_1 \leq q_U$ and $t_2 \leq \widehat{R}_q^{\complement U}$. The function h_1 is invariant, then t_1 is invariant and thus $t_1 \prec q_U$ following Corollary 4.12. Since q_U is a fine potential, we deduce that $t_1 = 0$. It follows that $h_1 \leq \widehat{R}_q^{\complement U}$ and hence $\widehat{R}_q^{\complement U} = q = G\lambda$ and the measure λ, carried by $\widetilde{U_n}$, is also carried by $\complement U$, so that $\lambda = 0$ and $q = 0$ and therefore $h = h_2 \leq^U \widehat{R}_h^{U\setminus U_n}$, whence $h =^U \widehat{R}_h^{U\setminus U_n}$.

Conversely, suppose that ${}^U\widehat{R}_h^{U\setminus U_n} = h$ for every $n \geq 1$. By the Riesz decomposition we have $h = p + k$ where p is a fine potential and k an invariant function on U. It follows that ${}^U\widehat{R}_p^{U\setminus U_n} = p$ for every integer $n \geq 1$ and then p is finely harmonic outside a polar subset of U by [17, Theorem 10.2]. Let p_n, $n \geq 1$, the potentials associated with $s = p$, using the notations following the proof of Lemma 4.14. Suppose that $p_k \neq 0$ for some $k \geq 1$: By Lemma 4.14, there is a measure μ carried by $\widetilde{U_k}$ such that $p_k \leq q = G\mu$. We have $p_k \prec p$, so that ${}^U\widehat{R}_{p_k}^{U\setminus U_n} = p_k$ for any $n \geq 1$. On the other hand, we have $p_k \leq q_U + \widehat{R}_q^{\complement U}$, and hence, by Corollary 4.17, $p_k \leq {}^U\widehat{R}_{q_U}^{U\setminus U_n} + \widehat{R}_q^{\complement U}$ for every integer $n \geq 1$. This implies that $q_U = 0$ because $\widehat{\inf}_n^U \widehat{R}_q^{U\setminus U_n} = 0$ according to Corollary 4.17. Hence

$\widehat{R}^{\complement U}_{G\mu} = G\mu$, which implies that μ is supported by $\complement U$, so that $\mu = 0$ since μ is also supported by $\widetilde{U_k} \subset U$, which is a contradiction with $p_k \neq 0$. It follows that $p_n = 0$ for every $n \geq 1$. Then there exists $x_1 \in U$ such that for any $n \geq 1$, there is a finely open set V_n such that $e_n \subset V_n \subset U_n$ and ${}^U\widehat{R}^{V_n}_p(x_1) < 1/2^n$. The function $t = \sum_n {}^U\widehat{R}^{V_n}_{p_n}$ is a fine potential in U by [17, Remark, page 105] and we have $p \leq t$ in $\bigcup_n V_n$, and hence in $\widetilde{\bigcup V_n} \cap U$ by fine continuity. The inequality $p \leq t$ in $U \setminus \widetilde{\bigcup V_n}$ follows by the fine minimum principle applied to the finely superharmonic function $t - p$ in $U \setminus \widetilde{\bigcup V_n}$. Putting $q = \widehat{R}^U_p$, we have $p \leq q_U + \widehat{R}^{\complement U}_q$ and hence, again by Lemma 4.15 and Corollary 4.17, $p =^U \widehat{R}^{U \setminus U_n}_p \leq {}^U\widehat{R}^{U \setminus U_n}_{q_U} + \widehat{R}^{\complement U}_q$ for every integer $n \geq 1$. By proceeding as above for p_k, we show that $q = 0$ and therefore $p = 0$, so that $h = k$ is invariant. □

Proposition 4.19. *Let p be an extremal fine potential in U, majorized on U by a potential in Ω. Then p is of the form $\alpha G_U(\cdot, y)$ for some $\alpha \geq 0$ and $y \in U$.*

Proof: We may suppose $p > 0$. By the hypothesis the function $P = \widehat{R}^U_p$ is a potential on Ω and we have $p \leq P$ on U. Let $P_1, P_2 \in \mathcal{S}(\Omega)$ be such that $P = P_1 + P_2$ and $P_1 \neq P_2$. By the Riesz decomposition property there are two potentials p_1, p_2 on U such that $p = p_1 + p_2$ and $p_1 \leq P_1$ and $p_2 \leq P_2$. Since p is extremal, there is a real $\beta \in [0,1]$ such that $p_1 = \beta p$ and $p_2 = (1-\beta)p$. Thus we have $P \leq \widehat{R}^U_{p_1} + \widehat{R}^U_{p_2} \leq P_1 + P_2 = P$ an hence $P_1 = \beta P$ and $P_2 = (1-\beta)P$ because $P_1 \geq \widehat{R}^U_{p_1} = \beta P$ and $P_2 \geq \widehat{R}^U_{p_2} = (1-\beta)P$. It follows that P is extremal in $\mathcal{S}(\Omega)$ and thus it is of the form $\alpha G(\cdot, y)$ for some $\alpha > 0$ and $y \in \Omega$. By the Riesz decomposition property applied to the inequality $p \leq \alpha G_U(\cdot, y) + \alpha \widehat{R}_{G(\cdot,y)}$, we get $y \in U$ and $p = \alpha G(\cdot, y)_U = \alpha G_U(\cdot, y)$ because p is a fine potential on U. □

Theorem 4.20. *Suppose that, in addition of the hypotheses of Theorem 4.5, for every $x \in U$, the function $y \mapsto G_U(x, y)$ is finely continuous on U. Then a fine potential p on U is extremal if and only if it is of the form $\alpha G_U(\cdot, y)$ for some $y \in U$ and a real $\alpha \geq 0$.*

Proof: Suppose that p is not majorized on U by a potential in Ω and let $x_0 \in U$ such that $p(x_0) < \infty$, and $(U_n)_{n \geq 1}$ the sequence of finely open subsets of U from Theorem 4.18. For any integer $n \geq 1$ we have $p \leq^U \widehat{R}^{U_n}_p +^U \widehat{R}^{U \setminus U_n}_p$, and by the Riesz decomposition property there are $p_1, p_2 \in \mathcal{S}(U)$ such that $p = p_1 + p_2$ and $p_1 \leq {}^U\widehat{R}^{U_n}_p$ and $p_2 \leq {}^U\widehat{R}^{U \setminus U_n}_p$. Since p is extremal there is a real $\alpha \in [0,1]$ such that $p_1 = \alpha p$ and $p_2 = (1-\alpha)p$. We have necessarily $\alpha = 0$ because otherwise p would be majorized on U by a potential in Ω according to Lemma 4.14, and hence $p = {}^U\widehat{R}^{U \setminus U_n}_p$. But it follows from Theorem 4.18 that p is invariant, which is a contradiction. Hence p is majorized on U by a potential in Ω and the theorem follows from Proposition 4.19. □

Definition 4.21. An invariant function $h \in \mathcal{S}(U)$ is termed minimal if it is extremal, that is h belongs to an extreme ray of the cone $\mathcal{S}(U)$.

5. Martin boundary of a fine domain and integral representation of invariant functions

All the results of this section were obtained in the classical case in [14] and they are stated in this section in the general framework considered in this paper with necessary adaptations. We assume that the adjoint fine topology is finer than the fine topology on Ω (in particular U is also an adjoint finely open set), we denote by G_U the Green kernel of U and we also assume that for any $x \in U$ the function $y \mapsto G_U(x,y)$ is finely continuous on U.

Let B be a compact base of the cone $\mathcal{S}(U)$ and Φ be a nonnegative continuous affine form on $\mathcal{S}(U)$ such that

$$B = \{u \in \mathcal{S}(U) \colon \Phi(u) = 1\}.$$

Then $\Phi(u) > 0$ except for $u = 0$. Consider the mapping $\varphi \colon U \longrightarrow B$ defined by

$$\varphi(y) = P_y = \frac{G_U(\cdot, y)}{\Phi(G_U(\cdot, y))},$$

and identify $y \in U$ with $\varphi(y) = P_y \in B$ and then U with $\varphi(U)$. The topology induced on U by that of B will be called the natural topology.

We denote by $\overline{U}$ the closure of U in B (with respect to the natural topology), and put $\Delta(U) = \overline{U} \setminus U$. Then $\overline{U}$ is compact in B, we will call it the Martin compactification of U, and $\Delta(U)$ will be called the Martin boundary of U.

If B and B' are two compact bases of $\mathcal{S}(U)$, the Martin compactifications of U relative to B and B' are clearly homeomorphic.

Throughout the rest of this article, we fix a base B of the cone $\mathcal{S}(U)$ and a continuous affine form $\Phi \colon \mathcal{S}(U) \longrightarrow [0, \infty[$ defining this base, that is, such that $B = \{u \in \mathcal{S}(U) \colon \Phi(u) = 1\}$. The Martin compactification $\overline{U} \subset B$ and the Martin boundary $\Delta(U) = \overline{U} \setminus U$ of U will be considered with respect to the base B.

We shall say that a nonnegative measure on B is carried (or supported) by a Borel subset A of B if $\mu(B \setminus A) = 0$.

We denote by $\mathrm{Ext}(B)$ the set of extreme elements of B and we put $\mathrm{Ext}_p(B) = \mathcal{P}(U) \cap \mathrm{Ext}(B)$ and $\mathrm{Ext}_i(B) = \mathcal{H}_i(U) \cap \mathrm{Ext}(B)$. Let us recall that since B is metrizable, then by a result of G. Choquet, $\mathrm{Ext}(B)$ is a G_δ of B.

Remark 5.1. According to Theorem 4.20, we have $\mathrm{Ext}_p(B) = U$.

Proposition 5.2. *Let (A_n) be an increasing sequence of compact subsets of Ω such that $\bigcup_n A_n = \Omega$. For any real $\alpha > 0$ and any integer l, the set $A_{\alpha,l} = \{y \in U \colon \Phi(G_U(\cdot, y)) \geq \alpha\} \cap A_l$ is compact with respect to the natural topology.*

Proof: Let (y_n) be a sequence of points of $A_{\alpha,l}$. According to the compactness of $\mathcal{S}(U) \cup \{\infty\}$ and A_l, we can find a subsequence (y_{n_k}) of (y_n) which converges to a point y of A_l such that the sequences $(G_U(\cdot, y_{n_k}))$ and $(\widehat{R}^{\complement U}_{G(\cdot, y_{n_k})})$ converge, respectively, in $\mathcal{U}_+(U) = \mathcal{S}(U) \cup \{\infty\}$ to a function s and $\widehat{\liminf} \widehat{R}^{\complement U}_{G(\cdot, y_{n_k})}$. Both these functions belong to $\mathcal{S}(U)$ and we have

$$s + \widehat{\liminf} \widehat{R}^{\complement U}_{G(\cdot, y_{n_k})} = G(\cdot, y).$$

Since we have $\widehat{\liminf} \widehat{R}^{\complement U}_{G(\cdot, y_{n_k})} \geq \widehat{R}^{\complement U}_{G(\cdot, y)}$, then $s \leq G(\cdot, y) - \widehat{R}^{\complement U}_{G(\cdot, y)} = G_U(\cdot, y)$. Hence the function s is a fine potential on U, finely harmonic on $U \setminus \{y\}$ because $s \prec G(\cdot, y)_{|U}$. We deduce by Theorem 4.5 that $s = \gamma G_U(\cdot, y)$ for some $\gamma \in [0, 1]$. Furthermore, we have $\Phi(s) \geq \alpha$, then $s > 0$ and consequently $\gamma > 0$, and we have $\Phi(G_U(\cdot, y)) = (1/\gamma)\Phi(s) \geq \alpha$. Hence $y \in A_{\alpha,l}$. It follows then that $A_{\alpha,l}$ is compact. □

Proposition 5.3. *The set $U = \mathrm{Ext}_p(B)$ is a K_σ-set and $\mathrm{Ext}_i(B)$ is a G_δ set of B.*

Proof: Indeed we have $\mathrm{Ext}_p(B) = U = \bigcup_{k \in \mathbb{N}^*, l \in \mathbb{N}} A_{1/k,l}$ according to Theorem 4.20, where $A_{\alpha,l}$ are the compact sets of Proposition 5.2. Then $\mathrm{Ext}_p(B)$ is a K_σ. On the other hand, we have $\mathrm{Ext}_i(B) = \mathrm{Ext}(B) \setminus \mathrm{Ext}_p(B)$ and since B is metrizable $\mathrm{Ext}(B)$ is a G_δ-set, thus $\mathrm{Ext}_i(B)$ is a G_δ set of B. □

Proposition 5.4. *Let μ be a probability measure on B carried by $\mathrm{Ext}(B)$ and s the barycenter of μ. Then s is a fine potential or an invariant function if and only if μ is supported by $\mathrm{Ext}_p(B)$ or $\mathrm{Ext}_i(B)$, respectively.*

Proof: We shall prove only that s is an invariant function if and only if μ is supported by $\mathrm{Ext}_i(B)$. Let $x_0 \in U$ and (U_n) be the sequence of finely open subsets of Theorem 4.18. We have $s = \int_B u \, \mathrm{d}\mu(u) = \int_{\mathrm{Ext}_p(B)} p \, \mathrm{d}\mu(p) + \int_{\mathrm{Ext}_i(B)} k \, \mathrm{d}\mu(k)$. If μ is supported by $\mathrm{Ext}_i(B)$, we have $s = \int_{\mathrm{Ext}_i(B)} k \, \mathrm{d}\mu(k)$, and then for any $n > 0$, we have, according to Theorem 4.18 and Theorem 3.11, $\widehat{R}^{U \setminus V_n}_s = \int_{\mathrm{Ext}_i(B)} \widehat{R}^{U \setminus V_n}_k = \int k \, \mathrm{d}\mu(k) = s$, and consequently, s is invariant according to Theorem 4.18. Conversely, if s is invariant, we have $\widehat{R}^{U \setminus V_n}_s = s$ and $\widehat{R}^{U \setminus V_n}_k = k$ for any integer n and any function $k \in \mathrm{Ext}_i(B)$. Hence $\int \widehat{R}^{U \setminus V_n}_u \, \mathrm{d}\mu(u) = \int u \, \mathrm{d}\mu(u)$, whence for every n, $\widehat{R}^{U \setminus V_n}_u = u$ μ-a.e. It follows that $\widehat{R}^{U \setminus V_n}_u = u$ μ-a.e. for all n, and therefore μ is supported by $\mathrm{Ext}_i(B)$ by virtue of Theorem 4.18. □

Definition 5.5. A point $Y \in \Delta(U)$ is termed minimal if the function $K(\cdot, Y)$ is minimal, that is, it belongs to an extreme ray of the cone $\mathcal{S}(U)$.

We have $\mathrm{Ext}(B) \subset \overline{U}$ and, by Theorem 4.20, $\mathrm{Ext}_p(B) = U$. It follows that $\mathrm{Ext}_i(B) \subset \overline{U} \setminus U = \Delta(U)$. Put $\Delta_1(U) = \mathrm{Ext}_i(B) \cap \Delta(U)$. The set $\Delta_1(U)$ is called the minimal Martin boundary of U.

Corollary 5.6. *The sets $\Delta(U)$ and $\Delta_1(U)$ are G_δ-sets of $\overline{U}$.*

Proof: Indeed we have $\Delta_1(U) = \mathrm{Ext}_i(B)$ and $\Delta(U) = B \setminus \mathrm{Ext}_p(U)$. Both these sets are G_δ-set by Proposition 5.3. □

Theorem 5.7. *Let $s \in \mathcal{P}(U)$ or $s \in \mathcal{H}_i(U)$. Then there exists a unique measure $\mu \geq 0$ on B supported by $\mathrm{Ext}_p(B)$, or $\mathrm{Ext}_i(B)$, respectively, such that $s = \int_B u \, \mathrm{d}\mu(u)$.*

PROOF: Let $s \in \mathcal{S}(U)$. According to Theorem 3.12, there exists a unique Radon measure $\mu \geq 0$ on B supported by $\mathrm{Ext}(B)$ such that $s = \int_B u \, \mathrm{d}\mu(u) = p+h$, where $p = \int_{\mathrm{Ext}_p(B)} u \, \mathrm{d}\mu(u)$ and $h = \int_{\mathrm{Ext}_i(B)} u \, \mathrm{d}\mu(u)$. It follows from Proposition 5.4 that p is a fine potential and h is an invariant function. Moreover we have $p \prec s$ and $h \prec s$. Suppose that s is a fine potential, then $h = 0$ and hence $s = \int_{\mathrm{Ext}_p(B)} u \, \mathrm{d}\mu(u)$. By the uniqueness of integral representation, we necessarily have $\mu = 1_{\mathrm{Ext}_p(B)}\mu$ and consequently, μ is supported by $\mathrm{Ext}_p(B)$. Similarly, if s is an invariant function, then $p = 0$ and μ is supported by $\mathrm{Ext}_i(B)$. $\square$

Proposition 5.8. *With the notations from Proposition* 5.2 *for any $\alpha > 0$ and any integer l, the function $g\colon A_{\alpha,l} \longrightarrow \mathcal{S}(U)$ defined by $g(y) = G_U(\cdot, y)$ is continuous with respect to the initial topology.*

PROOF: Let (y_n) be a sequence of points of $A_{\alpha,l}$ which converges with respect to the initial topology to $y \in A_{\alpha,l}$. Since $A_{\alpha,l}$ is compact with respect to the natural topology, for any cluster value z of (y_n) in the closure of $A_{\alpha,l}$ with respect to the natural topology, one can extract from (y_n) a subsequence (y'_n) which converges with respect to the natural topology to the point z. Reasoning as in the proof of Proposition 5.2, it follows that $y = z$ and consequently, $\lim_n G_U(\cdot, y_n) = G_U(\cdot, y)$ in $\mathcal{S}(U)$. $\square$

Corollary 5.9. *Let $x \in U$. Then $U \ni y \mapsto G_U(x, y)$ and $U \ni y \mapsto \Phi(G_U(\cdot, y))$ are Borel functions.*

Corollary 5.10. *Any Borel subset with respect to the natural topology of U is a Borel subset of U with respect to the initial topology.*

As a consequence of Theorem 5.7, we have the integral representation theorem of Fuglede given in the classical case in [22]:

Theorem 5.11. *Let p be a fine potential on U. Then there exists a unique positive Borel measure μ on U such that*

$$p(x) = \int G_U(x, y) \, \mathrm{d}\mu(y) \qquad \forall \, x \in U.$$

PROOF: According to Theorem 5.7 there exists a unique measure ν on B supported by $\mathrm{Ext}_p(U)$ such that

$$p = \int_{\mathrm{Ext}_p(B)} q \, \mathrm{d}\nu(q) = \int_U \frac{G_U(\cdot, y)}{\Phi(G(\cdot, y))} \, \mathrm{d}\nu(y),$$

the second equality by Theorem 4.18. For any $x \in U$ the functions $G_U(x, y)$ and $\Phi(G_U(\cdot, y))$ are Borel functions on U with respect to the natural topology, then with respect to the initial topology according to Corollary 5.10. The measure $\mu = \big(1/\Phi(G(\cdot,y))\big)\nu$ satisfies the conditions of the theorem. □

For any $Y \in \overline{U}$ consider the function $K(\cdot,Y) \in B \subset \mathcal{S}(U) \setminus \{0\}$ defined on U by $K(x,Y) = \varphi(Y)(x)$ if $Y \in U$ and $K(\cdot,Y) = Y$ if $Y \in \Delta(U)$. It is clear that the mapping $Y \longmapsto K(\cdot,Y)$ is a bijection of $\overline{U}$ on itself.

Definition 5.12. The function $K\colon U \times \overline{U} \longrightarrow]0,\infty]$ defined by $K(x,Y) = K(\cdot,Y)(x)$ is called the (fine) Riesz-Martin kernel of U, and its restriction to $U \times \Delta(U)$ is called the (fine) Martin kernel of U.

Proposition 5.13. *The (fine) Riesz-Martin kernel $K\colon U \times \overline{U} \longrightarrow]0,\infty]$ has the following properties, $\overline{U}$ being endowed with the natural topology:*

(i) *For any $x \in U$, $K(x,\cdot)$ is l.s.c. on $\overline{U}$.*
(ii) *For any $Y \in \overline{U}$, $K(\cdot,Y) \in \mathcal{S}(U)$ is finely continuous on U.*
(iii) *The kernel K is l.s.c. on $U \times \overline{U}$ when U is endowed with the fine topology and $\overline{U}$ is endowed with the natural topology.*

Proof: The property (i) follows from Corollary 3.10 applied to $u = K(\cdot,Y)$, where $K(\cdot,Y)$ is identified to Y. The property (ii) is obvious. Let us prove the property (iii). Let $x_0 \in U$, $Z \in \overline{U}$, and (V_j) be a fundamental system of neighborhoods of Z in $\overline{U}$ such that $V_{j+1} \subset V_j$ for any j. Given some constant $c > 0$, consider the increasing sequence of functions

$$k_j := \inf_{Y\in V_j} K(\cdot,Y) \wedge c$$

and their l.s.c. regularizatione $\widehat{k}_j \in \mathcal{S}(U)$. According to the Brelot property, cf. [19, Lemma, page 114], there is a fine neighborhood H of x_0 in U such that H is compact with respect to the initial topology and the restrictions of functions $\widehat{k}_j \in \mathcal{S}(U)$ and $K(\cdot,Z) \wedge c \in \mathcal{S}(U)$ to H are continuous on H (with respect to the initial topology). From the property (i) we have on U

$$K(\cdot,Z) \wedge c = \liminf_{Y\to Z} K(\cdot,Y) \wedge c = \sup_j \inf_{Y\in V_j} K(\cdot,Y) \wedge c,$$

which is quasi-everywhere, and then everywhere on U equal to $\sup_j \widehat{\inf_{Y\in V_j}} K(\cdot,Y) \wedge c \in \mathcal{S}(U)$. According to Corollary 3.8 and the Dini theorem, there exists for any $\varepsilon > 0$ an integer $j_0 > 0$ such that

$$K(\cdot,Z) \wedge c = \sup_j \widehat{\inf_{Y\in V_j}} K(\cdot,Y) \wedge c = \sup_j \widehat{k}_j < \widehat{k}_i + \varepsilon$$

on H for any $i \geq j_0$. For any fine neighborhood W of x_0 such that $W \subset H$ we have

$$\inf_{x\in W,\, Y\in V_j} K(x,Y)\wedge c = \inf_{x\in W} k_j(x) \geq \inf_{x\in W} \widehat{k}_j(x)$$
$$\geq \inf_{x\in W} K(x,Z)\wedge c - \varepsilon \geq K(x_0,Z)\wedge c - 2\varepsilon$$

for $j \geq j_0$. The assertion (iii) follows by taking $\varepsilon \to 0$ and $c \to \infty$. $\square$

Remark 5.14. As in the classical case, a set $A \subset U$ is said to be an quasi-Borel subset if it differs only by a polar set (with respect to the initial topology) from a Borel subset of U. We denote by $\mathcal{B}(U)$, or $\mathcal{B}^*(U)$, the σ-algebra (with respect to the initial topology) of Borel subsets of U or quasi-Borel subsets of U, respectively. Any finely open subset $V \subset U$ is an quasi-Borel subset because its regularization $r(V)$ is an F_σ (with respect to the initial topology) and $r(V) \setminus V$ is a polar set. It follows that any open subset W of $U \times \overline{U}$, where U is endowed with the fine topology and $\overline{U}$ with the natural topology, belongs to the σ-algebra $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$ generated by the sets $A_1 \times A_2$ where $A_1 \in \mathcal{B}^*(U)$ and $A_2 \in \mathcal{B}(\overline{U})$, that is, A_2 is a Borel subset of $\overline{U}$ with respect to the natural topology. According to Proposition 5.13 (iii), any subset of the form $\{(x,Y) \in U \times \overline{U} \colon K(x,Y) > \alpha\}$, $\alpha \in \mathbb{R}$, is an open subset of $U \times \overline{U}$, and then belongs to $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$. This means that the Riesz-Martin kernel K is measurable relatively to $\mathcal{B}^*(U) \times \mathcal{B}(\overline{U})$.

The following theorem and its corollary are easy consequences of Theorem 5.7:

Theorem 5.15. *For any invariant function $u \in \mathcal{S}(U)$, there exists a unique Radon measure μ on $\overline{U}$ supported by $\Delta_1(U)$ such that $u = \int K(\cdot,Y)\,\mathrm{d}\mu(Y)$.*

Corollary 5.16. *For any function $u \in \mathcal{S}(U)$, there exists a unique Radon measure μ on $\overline{U}$ supported by $U \cup \Delta_1(U)$ such that $u = \int K(\cdot,Y)\,\mathrm{d}\mu(Y)$.*

6. Brelot decomposition of nonnegative finely superharmonic functions

In [7], M. Brelot proved that if $u \in \mathcal{S}(\Omega)$ and $A \subset \Omega$, then u has a decomposition $u = u_1 + u_2$, where $\widehat{R}^A_{u_1} = u_1$ and $\widehat{R}^{\complement A}_{u_2} = u_2$, with uniqueness of the decomposition if we take for u_2 the greatest specific minorant v of u which is self-reduced on $\complement A$, that is, $\widehat{R}^{\complement A}_v = v$. As an application of the integral representation we shall extend this result to nonnegative finely superharmonic functions.

Lemma 6.1. *Let u be an extremal element of $\mathcal{S}(U)$ and $A \subset U$. Then we have $u = \widehat{R}^A_u$ or $u = \widehat{R}^{U\setminus A}_u$.*

Proof: Suppose that $u \neq \widehat{R}^A_u$ (and then in particular $u \neq 0$) and let $f = u - \widehat{R}^A_u$ (understood as 0 at points where $\widehat{R}^A_u(x) = \infty$). Then $\widehat{R}_f > 0$ and we have $\widehat{R}_f \prec u$ according to the lemma of [17, page 129]. Since u is extremal, we have $u = \alpha\widehat{R}_f$, with $\alpha > 0$. On the other hand, since $f = 0$ q.e. on A then $\widehat{R}_f = \widehat{R}_f^{U \setminus A}$ and consequently, $\widehat{R}_f = \widehat{R}_{\widehat{R}_f}^{U \setminus A}$. Hence $u = \widehat{R}_u^{U \setminus A}$. □

Proposition 6.2. *Let B be a compact base of the cone $\mathcal{S}(U)$ and $A \subset U$. Then the set $\mathrm{Ext}_A(B) = \{u \in \mathrm{Ext}(B) \colon \widehat{R}_u^A = u\}$ is a Borel subset of B.*

PROOF: We may suppose that the constants are superharmonic. Let τ be the measure of the base of the resolvent (V_λ) of Section 2. Without loss of generality we may suppose that the constants are τ-integrable. Since two positive superharmonic functions equal τ-a.e. are necessarily equal everywhere, we have $\mathrm{Ext}_A(B) = \bigcap_n C_n$, where for any integer n, $C_n = \{u \in B \colon \int u \wedge n \, \mathrm{d}\tau = \int \widehat{R}_u^A \wedge n \, \mathrm{d}\tau\}$. Then it suffices to show that for any n the set C_n is a Borel subset of B. But this follows from the fact that the functions $u \mapsto \int u \wedge n \, \mathrm{d}\tau$ and $u \mapsto \int \widehat{R}_u^A \wedge n \, \mathrm{d}\tau$ are l.s.c. on B as easily shown by application of Fatou lemma and Corollary 3.10. □

We say that u is selfreduced on $A \subset U$ if $\widehat{R}_u^A = A$.

Theorem 6.3. *Let $u \in \mathcal{S}(U)$ and $A \subset U$. Then there exists a decomposition $u = u_1 + u_2$ of u in $\mathcal{S}(U)$ such that*

1. *The function u_1 is selfreduced on A.*
2. *The function u_2 is selfreduced on $U \setminus A$.*

PROOF: Let $u \in \mathcal{S}(U)$ and μ be the maximal measure on B representing u. We have $u = \int_{\mathrm{Ext}(B)} p \, \mathrm{d}\mu(p) = \int_{\mathrm{Ext}_A(B)} p \, \mathrm{d}\mu(p) + \int_{\mathrm{Ext}(B) \setminus \mathrm{Ext}_A(B)} p \, \mathrm{d}\mu(p)$ in view of Proposition 6.2. According to Lemma 6.1 we have $\mathrm{Ext}(B) = \mathrm{Ext}_A(B) \cup \mathrm{Ext}_{U \setminus A}(B)$, and for any $p \in \mathrm{Ext}(B) \setminus \mathrm{Ext}_A(B)$, we have $\widehat{R}_p^{U \setminus A} = p$. Put $u_1 = \int_{\mathrm{Ext}_A(B)} p \, \mathrm{d}\mu(p)$ and $u_2 = \int_{\mathrm{Ext}(B) \setminus \mathrm{Ext}_A(B)}$ (these integrals are well defined according to Proposition 6.2). Then we have $u = u_1 + u_2$ and, by Theorem 3.11, $\widehat{R}_{u_1}^A = \int_{\mathrm{Ext}_A(B)} \widehat{R}_p^A \, \mathrm{d}\mu(p) = \int_{\mathrm{Ext}_A(B)} p \, \mathrm{d}\mu(p) = u_1$ and $\widehat{R}_{u_2}^{U \setminus A} = \int_{\mathrm{Ext}(B) \setminus \mathrm{Ext}_A(B)} \widehat{R}_p^{U \setminus A} \, \mathrm{d}\mu(p) = \int_{\mathrm{Ext}(B) \setminus \mathrm{Ext}_A(B)} p \, \mathrm{d}\mu(p) = u_2$. □

Remark 6.4. We have uniqueness in the decomposition of u in the preceding theorem if we impose on u_2 (or u_1) to be the specific greatest minorant of u which is selfreduced on $U \setminus A$ (or A, respectively).

7. Approximation of invariant functions by finely harmonic functions

If $U = D \cup \partial_i D$, where D is a non regular domain of $\mathbb{R}^2$, then any minimal invariant function u on $U = D \cup \partial_i D$ is finely harmonic according to a theorem of M. Brelot (cf. [6, Section 7]). Indeed, the restriction h of u to D is invariant by [14, Theorem 2.6 (a)], and hence harmonic since D is open in the initial topology. In fact, the positive finely superharmonic functions on D (or the fine potentials on D), are the same as the usual positive superharmonic functions (the usual potentials, respectively), according to [17, Teorems 9.8 and 10.13 (and

Section 10.4)]; hence the invariant functions on D are the same as the positive harmonic functions there. Next, h is minimal harmonic on D. In fact, let $h = h_1 + h_2$ with h_1, h_2 harmonic nonnegative on D. Since $\partial_i D$ is polar, h_1 and h_2 extend by the removable singularities theorem [17, Theorem 9.14] to positive finely superharmonic u_1 and u_2 on U, specifically majorized by u and hence likewise invariant on U. By minimality of u we conclude that $u_1 = (1-\alpha)u$ and $u_2 = \alpha u$ for some $\alpha \in [0,1]$, and hence $h_1 = (1-\alpha)h$ and $h_2 = \alpha h$, showing that indeed h is minimal harmonic in D.

For any point $x_0 \in \partial_i D$ it now follows from M. Brelot [6, Section 7] (see also [24, Quesion 2]) that if $\operatorname{fine\,lim}_{x\to x_0} h(x) = \infty$, that is $u(x_0) = \infty$, then u is positive constant multiple of $G_U(\cdot, x_0)$, which is a fine potential on U and hence vanishes there because u is also invariant. But this contradicts the fact that $u(x_0) = \infty$. Hence $u(x_0) < \infty$, and therefore u is bounded on some fine neighborhood of x_0, then u is finely harmonic on the finely open set $D \cup \{x_0\}$ according to the removable singularity theorem for finely harmonic functions [17, Theorem 9.15]. We deduce that u is finely harmonic on some finely open set containing x_0, and hence on all of U, by varying x_0 and recalling that h is harmonic on D.

Returning to an arbitrary fine domain U in the setting of a $\mathcal{P}$-Brelot space satisfying the axiom D, the above example suggests to ask the following question: If any minimal invariant function on U is finely harmonic, is then any invariant function on U the sum of a sequence of nonnegative finely harmonic functions on U (equivalently: is any invariant function the pointwise limit of an increasing sequence of nonnegative finely harmonic functions on U)?

In this section we give a partial answer to this question (Theorem 7.2). More precisely, we shall show, under the hypotheses of Section 5, that if any minimal invariant function on U is finely harmonic on U, then any invariant function on U is approachable in the natural topology by nonnegative finely harmonic functions nonnegative on U.

In this section we assume that the adjoint fine topology is finer than the fine topology on Ω (in particular U is also an adjoint finely open set), we denote by G_U the Green kernel of U and we also assume that for any $x \in U$ the function $y \mapsto G_U(x,y)$ is finely continuous.

Proposition 7.1. *Let $K \subset \overline{U}$ be compact with respect to the natural topology such that $K \subset \operatorname{Ext}_i(B)$ and $\mathcal{H}_K(U)$ the set of invariant functions of the form $\int k\,\mathrm{d}\mu(k)$, where μ is a probability measure on K. Then $\mathcal{H}_K(U)$ is a compact convex subset of B and $\operatorname{Ext}(\mathcal{H}_K(U)) = \operatorname{Ext}(B) \cap \mathcal{H}_K(U)$.*

Proof: It is clear that $\mathcal{H}_K(U)$ is convex. It remains to prove that it is compact. Let (μ_j) be a sequence of probability measures on K. We can extract from the sequence (μ_j) a subsequence ν_j which converges vaguely to a probability measure μ on K. For any continuous affine form l on B, we have $l\big(\int_K k\,\mathrm{d}\mu(k)\big) = \int_K l(k)\,\mathrm{d}\mu(k) = \lim_j l\big(\int_K k\,\mathrm{d}\nu_j(k)\big) = l\big(\lim_j \int_K k\,\mathrm{d}\nu_j(k)\big)$, and therefore it follows $\int_K k\,\mathrm{d}\mu(k) = \lim_j \int_K k\,\mathrm{d}\nu_j(k) \in \mathcal{H}_K(U)$. It hence follows that $\mathcal{H}_K(U)$ is compact. The inclusion $\mathrm{Ext}(B) \cap H_K(U) \subset \mathrm{Ext}(\mathcal{H}_K(U))$ is obvious. Let us prove the opposite inclusion. Let $h \in \mathrm{Ext}(\mathcal{H}_K(U))$, and $u, v \in \mathcal{S}(U)$ such that $h = u + v$. We can find two finite measures σ and τ on B, supported by $\mathrm{Ext}_i(B)$ such that $u = \int_B k\, \mathrm{d}\sigma(k)$ and $v = \int_B k\, \mathrm{d}\tau(k)$, and a measure μ on K such that $h = \int_K \mathrm{d}\mu(k)$. According to the uniqueness of the integral representation in the Choquet's theorem, we have $\mu = \sigma + \tau$, and then σ and τ are supported by K, and consequently, $u, v \in \mathcal{H}_K(U)$. Since $h \in \mathrm{Ext}(\mathcal{H}_K(U))$, we deduce that u and v are proportional to h and then $h \in \mathrm{Ext}(B)$. This proves the inclusion $\mathrm{Ext}(\mathcal{H}_K(U)) \subset \mathrm{Ext}(B) \cap \mathcal{H}_K(U)$, and hence the required equality holds. □

Theorem 7.2. *Suppose that any minimal invariant function on U is finely harmonic, then any invariant function on U is limit (in the natural topology) of a sequence of finely harmonic functions on U.*

Proof: Let h be an invariant function, $h > 0$, and μ the measure on B supported by $\mathrm{Ext}_i(B)$ which represents h (Theorem 5.7). We can find a sequence $(K_n)_{n\in J}$, $J \subset \mathbb{N}$ possibly finite, of compact pairwise disjoint subsets of B, contained in $\mathrm{Ext}_i(B)$ and such that $\mu(B) = \mu\big(\bigcup_n K_n\big)$ and $\mu(K_n) > 0$ for any integer $n \in J$. Thus we have $h = \sum_{n\in J} \int_{K_n} k\, \mathrm{d}\mu(k) = \sum_{n\in J} \mu(K_n) b(\mu_n)$, where μ_n is the probability measure $(1/\mu(K_n))1_{K_n} \cdot \mu$ and $b(\mu_n)$ its barycenter (cf. [1, page 12]). For any integer n we have $b(\mu_n) \in \mathcal{H}_{K_n}(U)$ because $\mathcal{H}_{K_n}(U)$ is convex. According to the Krein–Milman theorem, the function $h_n = b(\mu_n)$ is a limit of a sequence (h_n^i) of affine combinations of extreme elements of $\mathcal{H}_{K_n}(U)$, which are finely harmonic functions according to the preceding proposition and the hypothesis of the theorem. It follows that $h = \sum_n \mu(K_n) h_n$ is the limit (in the natural topology) of a sequence of finely harmonic functions on U. □

Remark 7.3. Theorem 7.2 is not a direct consequence of the Krein–Milman theorem, because $B \cap \mathcal{H}_i(U)$ is not compact if U is not an open set relative to initial topology.

Corollary 7.4. *Let D be a non regular bounded open subset of $\mathbb{R}^2$, and $U = D \cup \partial_i(D)$. Then any invariant function on U is the limit (in the natural topology) of a sequence of finely harmonic functions on U.*

Proof: In fact, as explained in the beginning of the present section, any minimal invariant function on U is finely harmonic, and the result follows immediately from the preceding theorem. □

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