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Chern rank of complex bundle

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Abstract. Motivated by the work of A. C. Naolekar and A. S. Thakur (2014) we introduce notions of upper chern rank and even cup length of a finite connected CW-complex and prove that upper chern rank is a homotopy invariant. It turns out that determination of upper chern rank of a space X sometimes helps to detect whether a generator of the top cohomology group can be realized as Euler class for some real (orientable) vector bundle over X or not. For a closed connected d-dimensional complex manifold we obtain an upper bound of its even cup length. For a finite connected even dimensional CW-complex with its upper chern rank equal to its dimension, we provide a method of computing its even cup length. Finally, we compute upper chern rank of many interesting spaces.

Keywords: Chern class; characteristic rank; cup length; chern rank

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1. Introduction

In [3] J. Korbaš introduced the idea of *characteristic rank* of a smooth closed connected manifold X of dimension d. He defined *characteristic rank* of a d-dimensional smooth closed connected manifold X as the largest integer k such that every cohomology class of $H^i(X;\mathbb{Z}_2)$, $i \le k$, can be expressed as a polynomial of the Stiefel–Whitney classes of the tangent bundle of X. In the same paper [3] J. Korbaš also used *characteristic rank* to get abound for $\mathbb{Z}_2$-cup length of a manifold X. The $\mathbb{Z}_2$-cup length, denoted by $\mathrm{Cup}(X)$ of a space X is defined to be the largest integer t such that there exist cohomology classes $x_i \in H^*(X;\mathbb{Z}_2)$, $\deg(x_i) \ge 1$, so that the cup product $x_1 x_2 \cdots x_t \ne 0$. Later in 2014, A. C. Naolekar and A. S. Thakur in [7] generalized the notion of *characteristic rank* to a real vector bundle ξ over a finite connected CW-complex X. If ξ is a real n-plane bundle over X then they defined *characteristic rank* (briefly $\operatorname{char\,rank}\xi$) of ξ over X to be the largest integer k such that every cohomology class $x \in H^i(X;\mathbb{Z}_2)$, $i \le k \le \dim X$, can be expressed as a polynomial of Stiefel–Whitney classes of ξ. They also defined *upper characteristic rank* (X) of a finite connected CW-complex X as the maximum of $\operatorname{char\,rank}\xi$ as ξ varies over all real vector bundles over X and thus by naturality of Stiefel–Whitney classes *upper characteristic rank* becomes a homotopy invariant. In [7] *characteristic rank* of real vector bundles over product of spheres $S^m \times S^n$, the real and complex projective spaces, the

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spaces $S^1 \times \mathbb{C}P^n$, the Dold manifold $P(m,n)$, the Moore space $M(\mathbb{Z}_2,n)$ and the stunted projective space $\mathbb{R}P^n/\mathbb{R}P^m$ were computed. Moreover, some general facts about *characteristic rank* of real vector bundles were also proved.

This motivates us to define *chern rank* of a complex vector bundle over X. Throughout, by a (topological) space we mean a finite connected CW-complex and $H^*(X)$ ($\widetilde{H}^*(X)$) denotes the graded (reduced) integral cohomology ring of X. We begin with the following definition.

Definition 1.1. Let ξ be a complex n-plane bundle over a finite connected CW-complex X. By $\operatorname{chern\,rank}\xi$ we mean the largest even integer $2k$, where $0 \leq 2k \leq \dim X$, such that every cohomology class $x \in H^{2i}(X)$, $i \leq k$, can be expressed as a polynomial of Chern classes of ξ. The upper chern rank (X) (in brief $\operatorname{uch\,rank}(X)$) is defined to be the maximum chern rank ξ where ξ varies over all complex vector bundles over X, that is,

$$\operatorname{uch\,rank}(X) = \max\{\operatorname{chern\,rank}\xi\colon\ \xi \text{ is a complex vector bundle over } X\}.$$

From the naturality of Chern classes it follows that if X and Y are homotopy equivalent then $\operatorname{uch\,rank}(X) = \operatorname{uch\,rank}(Y)$. We note that determining *upper chern rank* of a topological space X sometimes helps to detect whether a generator of the top cohomology group can be realized as Euler class for some real (orientable) vector bundle over X or not. If for a $2n$ dimensional closed connected smooth manifold X the only nontrivial even dimensional reduced cohomology group is $\widetilde{H}^{2n}(X)$ and $\operatorname{uch\,rank}(X) = 2n$ then clearly a generator of $\widetilde{H}^{2n}(X)$ can be realized as Euler class for some real (orientable) vector bundle over X. For example we will see that $\operatorname{uch\,rank}(S^1 \times S^3) = 4$ (cf. Corollary 3.2) and consequently a generator of $\widetilde{H}^4(S^1 \times S^3)$ can be realized as an Euler class of some real (orientable) vector bundle over $S^1 \times S^3$. Also for a finite connected CW-complex consisting of only even dimensional cells *upper chern rank* of X gives a lower bound for *upper characteristic rank* of X (cf. Lemma 2.2).

If X is a finite connected CW-complex, we denote by r_X the smallest even integer such that $\widetilde{H}^{r_X}(X) \neq 0$. For X is a CW-complex with $\widetilde{H}^{2i}(X) = 0\ \ \forall i$, we define $r_X = \dim X + 2$ if X is even dimensional and $r_X = \dim X + 1$ otherwise. Clearly, for any complex vector bundle ξ over X,

$$r_X - 2 \leq \operatorname{chern\,rank}\xi \leq \operatorname{uch\,rank}(X).$$

For a finite connected CW-complex X we define the even cup length (denoted by $\operatorname{Cup}_E(X)$) of X to be the largest integer t such that the cup product $x_1 \cdot x_2 \cdots x_t \neq 0$ where each $x_i \in H^*(X)$ is of even degree and $\deg(x_i) \geq 2$. If X consists of only even dimensional cells then clearly $1 + \operatorname{Cup}_E(X)$ is a suitable lower bound of $\operatorname{Cat}(X)$ where $\operatorname{Cat}(X)$ denotes the *Lyusternik–Shnirel'man* category. For a closed connected d-dimensional complex manifold we obtain a bound for $\operatorname{Cup}_E(X)$ using chern rank. In particular we prove the following theorem.

Theorem 1.2. *Let X be a closed connected d-dimensional complex manifold such that $H^{2i}(X)$ is a free $\mathbb{Z}$-module for all i. If ξ is a complex vector bundle over X and there exists some nonzero even integer $2k \leq \operatorname{chern rank} \xi$ such that every monomial $c_{i_1}(\xi) \cdots c_{i_r}(\xi)$, $1 \leq i_t \leq k$, of total degree $2d$ is zero then*

$$\mathrm{Cup}_E(X) \leq 1 + \frac{2(d-k-1)}{r_X}.$$

If X is a finite connected even dimensional CW-complex with

$$\operatorname{uch rank}(X) = \dim X$$

then the following theorem tells us that $\mathrm{Cup}_E(X)$ can be computed as the maximal length of nonzero product of Chern classes of a suitable complex vector bundle ξ over X.

Theorem 1.3. *Let X be an even dimensional finite connected CW-complex. If* $\operatorname{uch rank}(X) = \dim X$ *then there exists a complex vector bundle ξ such that*

$$\mathrm{Cup}_E(X) = \max\{k \colon \exists\, i_1, i_2, \ldots, i_k \geq 1 \text{ with } c_{i_1}(\xi) \cdot c_{i_2}(\xi) \cdots c_{i_k}(\xi) \neq 0\}.$$

Finally, we compute uch rank of projective spaces $\mathbb{F}P^n$ ($\mathbb{F}$ is real, complex or quaternionic). We give a full description of uch rank of product of spheres $S^m \times S^n$ where m, n are even integers and in the case where m is even and n is an odd integer. If m and n are both odd integers then we compute uch rank of $S^m \times S^n$ for some special cases. We also give computation of uch rank of X where X is wedge sum of spheres $S^m \vee S^n$, $\mathbb{R}P^n \times S^{2m}$, $\mathbb{C}P^n \times S^{2m}$, complex Stiefel manifolds $V_k(\mathbb{C}^n)$, $1 < k < n$, for $n-k$ is even or $n-k \neq 2^t - 1$, $t > 0$, and stunted complex projective spaces $\mathbb{C}P^n/\mathbb{C}P^m$.

2. Some general facts and proofs of Theorem 1.2 and Theorem 1.3

We recall that if X is a finite connected CW-complex then r_X denotes the smallest even integer such that $\widetilde{H}^{r_X}(X) \neq 0$. For any X, $\widetilde{H}^{2i}(X) = 0 \ \ \forall\, i$, we define $r_X = \dim X + 2$ if X is even dimensional and $r_X = \dim X + 1$ if X is odd dimensional CW-complex. We start with the following lemma.

Lemma 2.1. *Let ξ and η be two complex vector bundles over a finite connected CW-complex X.*

(1) *If $\overline{\xi}$ is the conjugate bundle of ξ then*

$$\operatorname{chern rank} \xi = \operatorname{chern rank} \overline{\xi}.$$

(2) *If $\omega = \mathrm{Hom}(\xi, \mathbb{C})$, the dual bundle of ξ then*

$$\operatorname{chern rank} \xi = \operatorname{chern rank} \omega.$$

(3) *If $c_{r_X}(\xi) = 0$ then* $\operatorname{chern rank} \xi = r_X - 2$.

(4) *If* $\widetilde{H}^{r_X}(X)$ *is not cyclic then* $\operatorname{uch\,rank}(X) = r_X - 2$.
(5) *If* $c(\xi) = 1$ *then* $\operatorname{chern\,rank} \xi = r_X - 2$.
(6) *If* $c(\eta) = 1$ *then* $\operatorname{chern\,rank}(\xi \oplus \eta) = \operatorname{chern\,rank} \xi$. *Moreover,*

$$\widetilde{K}(X) = 0 \quad \textit{implies} \quad \operatorname{uch\,rank}(X) = r_X - 2.$$

(7) *If* ξ *and* η *are stably isomorphic then* $\operatorname{chern\,rank} \xi = \operatorname{chern\,rank} \eta$.
(8) *There exists a complex vector bundle* ξ' *such that*

$$\operatorname{chern\,rank}(\xi \oplus \xi') = r_X - 2.$$

PROOF: (1) follows from the fact that the Chern class $c_k(\overline{\xi}) = (-1)^k c_k(\xi)$. As X is compact we may assume that ξ admits an Hermitian metric. Consequently $\omega = \mathrm{Hom}(\xi, \mathbb{C})$ becomes canonically isomorphic to $\overline{\xi}$. Hence

$$\operatorname{chern\,rank} \xi = \operatorname{chern\,rank} \omega,$$

proving (2). Assertions (3) and (4) are obvious and (5) follows from (3). Assertion (6) follows from the fact that if $c(\eta) = 1$ then $c(\xi \oplus \eta) = c(\xi)$ and again $\widetilde{K}(X) = 0$ implies $c(\eta) = 1$ for any complex vector bundle η over X. To prove the statement (7), suppose ξ and η are stably isomorphic. Then $\xi \oplus \varepsilon^m \cong \eta \oplus \varepsilon^n$ for some m and n and hence $c(\xi) = c(\eta)$. Finally, as X is compact so for any bundle ξ over X there exists a bundle ξ' over X such that $\xi \oplus \xi' \cong \varepsilon^k$ for some k. Thus (8) follows from (5). $\square$

Lemma 2.2. *If* X *is a finite connected CW-complex consisting of only even dimensional cells then upper characteristic rank* $(X) \geq \operatorname{uch\,rank}(X) + 1$.

PROOF: It is clear that the coefficient homomorphism $H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}_2)$ becomes an epimorphism as X consists of only even dimensional cells. Now it is known that if ξ is a complex vector bundle over X then the coefficient homomorphism maps the total Chern class $c(\xi)$ onto the total Stiefel–Whitney class $w(\xi_R)$, see [5], Problem 14-B. Hence the proof follows. $\square$

If $X = \Sigma Y$, where ΣY denotes the reduced suspension of Y then $\widetilde{H}^*(X) \cong \widetilde{H}^*(Y) \otimes \widetilde{H}^*(S^1)$ and consequently the cup product of two positive degree cohomology classes of $\widetilde{H}^*(X)$ becomes zero. Thus we have the following lemma.

Lemma 2.3. *Suppose* $X = \Sigma Y$ *and let* $k_X = \max\{2k\colon H^{2j}(X)$ *is cyclic*, $0 \leq j \leq k,\ 2k \leq \dim X\}$. *Then* $\operatorname{uch\,rank}(X) \leq k_X$.

In the above lemma trivial groups are considered to be cyclic. We note that if X is ordinary (nonreduced) suspension, then it is covered by two open contractible subsets, hence the cup product is trivial in this case as well and Lemma 2.3 applies.

Lemma 2.4. *Let* $f\colon X \to Y$ *be a map where* X, Y *are finite connected CW-complexes and let* $f^*\colon H^*(Y) \to H^*(X)$ *be a surjection. Then* $\operatorname{chern\,rank} f^*(\xi) \geq \min\{\operatorname{chern\,rank} \xi, \dim X - 1\}$ *for any complex vector bundle* ξ *over* Y.

Proof: If $\dim X \geq \dim Y$ then from the naturality of Chern classes it follows that $\operatorname{chern\,rank} f^*(\xi) \geq \operatorname{chern\,rank} \xi$. Let $\dim X < \dim Y$. Now if $\dim X < \operatorname{chern\,rank} \xi$ then clearly $\operatorname{chern\,rank}\ f^*(\xi) = \dim X$ if $\dim X$ is even and $\operatorname{chern\,rank} f^*(\xi) = \dim X - 1$ if $\dim X$ is odd. Again if $\dim X \geq \operatorname{chern\,rank} \xi$ then $\operatorname{chern\,rank} f^*(\xi) \geq \operatorname{chern\,rank} \xi$. Combining all the above cases we get $\operatorname{chern\,rank} f^*(\xi) \geq \min\{\operatorname{chern\,rank} \xi, \dim X - 1\}$. □

Let us consider the projective space $\mathbb{F}P^n$ where $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$, the complex or quarternionic numbers, respectively. If L and M denote the canonical (complex and quaternionic) line bundles over $\mathbb{C}P^n$ and $\mathbb{H}P^n$, respectively, then the Chern classes $c_1(L)$ and $c_2(M)$ are generators of $H^*(\mathbb{C}P^n)$ and $H^*(\mathbb{H}P^n)$, respectively. Hence we get the following theorem.

Theorem 2.5. *If $X = \mathbb{C}P^n$ or $\mathbb{H}P^n$ then* $\operatorname{uch\,rank}(X) = 2n$ *or* $4n$*, respectively.*

Now we look at the chern rank of complex vector bundles over spheres. It follows from Theorem 2.5 that there exist complex vector bundles ξ_1 (line bundle) and ξ_2 (2-plane complex bundle) over $S^2 = \mathbb{C}P^1$ and $S^4 = \mathbb{H}P^1$, respectively, such that $c_1(\xi_1)$ and $c_2(\xi_2)$ are generators of $H^2(S^2)$ and $H^4(S^4)$, respectively. Thus $\operatorname{chern\,rank} \xi_i = 2$ or 4 for $i = 1$ or 2. Consequently $\operatorname{uch\,rank}(S^{2n}) = 2n$ if $n = 1$ or 2. In this context we want to state Bott integrality theorem which will be used in the sequel.

Theorem 2.6 (Bott integrality theorem [2], Chapter 20, Corollary 9.8)**.** *Let $a \in H^{2n}(S^{2n})$ be a generator. Then for each complex vector bundle ξ over S^{2n}, the nth Chern class $c_n(\xi)$ is a multiple of* $(n-1)!\,a$*, and for each m with $m \equiv 0$* mod $(n-1)!$ *there exists a unique $\xi \in \widetilde{K}(S^{2n})$ with $c_n(\xi) = ma$.*

Now it follows from Theorem 2.6 that if ξ is any complex vector bundle over S^{2n} where $n \neq 1$ or 2 then $c_n(\xi)$ cannot be a generator of $H^{2n}(S^{2n})$ and consequently for any complex vector bundle ξ over S^{2n} ($n \neq 1$ or 2) $\operatorname{chern\,rank} \xi = 2n - 2$. We note that if n is odd then clearly $\operatorname{uch\,rank}(S^n) = n - 1$. Combining these we get the following theorem.

Theorem 2.7. *If n is odd then* $\operatorname{uch\,rank}(S^n) = n - 1$, $\operatorname{uch\,rank}(S^{2n}) = 2n$ *if $n = 1$ or 2 and* $\operatorname{uch\,rank}(S^{2n}) = 2n - 2$ *if $n \neq 1$ or 2.*

If X and Y are two closed connected smooth orientable manifolds then the following theorem tells us that under suitable conditions *upper chern rank* of the product space $X \times Y$ is strictly less than $\dim(X \times Y)$.

Theorem 2.8. *Let X and Y be closed connected smooth orientable manifolds.*

(1) *If $\widetilde{K}(X)$, $\widetilde{K}(Y)$ and $\widetilde{K}(X \wedge Y)$ are all trivial then*

$$\operatorname{uch\,rank}(X \times Y) < \dim(X \times Y).$$

(2) *If $\widetilde{KO}(X)$, $\widetilde{KO}(Y)$ and $\widetilde{KO}(X \wedge Y)$ are all trivial then*

$$\operatorname{uch\,rank}(X \times Y) < \dim(X \times Y).$$

Proof: (1) Let $\dim X = m$ and $\dim Y = n$. If $m+n$ is odd then it is trivial. So we assume $m+n$ is even. We note that as $X\times Y$ is orientable smooth manifold therefore $H_{m+n-1}(X\times Y)$ becomes torsion free and thus $H^{m+n}(X\times Y)\cong \mathbb{Z}$. We consider the inclusion followed by the quotient map $X\vee Y \hookrightarrow X\times Y \to X\wedge Y$. This yields the exact sequence $\widetilde{K}(X\wedge Y)\to \widetilde{K}(X\times Y)\to \widetilde{K}(X\vee Y)$. Now $\widetilde{K}(X\wedge Y)=0$ and again $\widetilde{K}(X)=0=\widetilde{K}(Y)$ implies $\widetilde{K}(X\vee Y)\cong \widetilde{K}(X)\oplus\widetilde{K}(Y)=0$. Thus $\widetilde{K}(X\times Y)=0$ and consequently every complex vector bundle over $X\times Y$ becomes stably trivial. Thus for any complex vector bundle ξ over $X\times Y$ the total Chern class $c(\xi)=1$ while $H^{m+n}(X\times Y)\neq 0$ and thus $\operatorname{chern\,rank}\xi < \dim(X\times Y)$.

(2) Let $m+n$ be even. As before we get $\widetilde{KO}(X\times Y)=0$ and so for any real vector bundle η over $X\times Y$ the total Stiefel–Whitney class $w(\eta)=1$. If possible $\operatorname{uch\,rank}(X\times Y)=m+n$, there exists a complex vector bundle ξ over $X\times Y$ such that $\operatorname{chern\,rank}(\xi)=m+n$. Let a be a generator of $H^{m+n}(X\times Y)\cong\mathbb{Z}$ and so a can be expressed as a polynomial of Chern classes $c_i(\xi)$. Let $a = P(c_1(\xi)\cdot c_2(\xi)\cdots c_t(\xi))$, $t\leq (m+n)/2$. Now if

$$f\colon H^*(X\times Y;\mathbb{Z})\to H^*(X\times Y;\mathbb{Z}_2)$$

be the canonical coefficient homomorphism then $f(a)$ becomes the generator of $H^{m+n}(X\times Y;\mathbb{Z}_2)\cong\mathbb{Z}_2$ and again

$$\begin{aligned} f(a) &= f(P(c_1(\xi)\cdot c_2(\xi)\cdots c_t(\xi)))\\ &= P(f(c_1(\xi))\cdot f(c_2(\xi))\cdots f(c_t(\xi)))\\ &= P(\omega_2(\xi_R)\cdot\omega_4(\xi_R)\cdots\omega_{2t}(\xi_R))=0, \end{aligned}$$

a contradiction. Thus $\operatorname{uch\,rank}(X\times Y)<\dim(X\times Y)$. □

If X is a closed connected complex manifold of complex dimension d and ξ is a complex vector bundle over X then the following theorem tells us that under certain given conditions $\operatorname{chern\,rank}\xi$ can be predicted.

Theorem 2.9. *Let X be a closed connected complex manifold of complex dimension d. If $r_X\leq d$ and $H^{r_X}(X)\cong\mathbb{Z}$ then for any complex vector bundle ξ over X,* $\operatorname{chern\,rank}\xi$ *is either less than $2d-r_X$ or $2d$.*

Proof: Every complex manifold of complex dimension d is a $2d$ dimensional smooth orientable manifold. The triviality of $H^1(X), H^2(X),\dots,H^{r_X-1}(X)$ implies $H_1(X),H_2(X),\dots,H_{r_X-2}(X)$ are all trivial and hence by Poincaré duality the cohomology groups $H^{2d-1}(X),H^{2d-2}(X),\dots,H^{2d-r_X+2}(X)$ are trivial. Let ξ be a complex vector bundle over X such that $\operatorname{chern\,rank}\xi\geq 2d-r_X$, $2d-r_X\geq r_X$. We only have to show that any cohomology class of $H^{2d}(X)\cong\mathbb{Z}$ can be expressed as a polynomial of Chern classes.

As $\operatorname{chern\,rank}(\xi)\geq 2d-r_X\geq r_X$ therefore $H^{r_X}(X)=\langle c_{r_X/2}(\xi)\rangle$. Now as X is a closed connected $\mathbb{Z}$-orientable manifold so there exists some $\beta\in H^{2d-r_X}(X)$ such that $c_{r_X/2}(\xi)\cdot\beta$ is a generator of $H^{2d}(X)$, while β can be expressed as a polynomial of Chern classes of ξ and consequently $c_{r_X/2}(\xi)\cdot\beta$ can be expressed as a polynomial of Chern classes of ξ. This completes the proof. $\square$

We recall that $\mathrm{Cup}_E(X)$, the even cup length of X is the largest integer t such that the cup product $x_1\cdot x_2\cdots x_t\neq 0$ where each x_i is an even degree cohomology class with $\deg(x_i)\geq 2$. If X is a closed connected d-dimensional complex manifold then Theorem 1.2 gives a bound for $\mathrm{Cup}_E(X)$. Proofs of Theorems 1.2 and 1.3 are similar to the proofs of Theorem 1.2 and 1.3 of [7], respectively.

Proof of Theorem 1.2: Let $\mathrm{Cup}_E(X)=t$ and $x_1\cdot x_2\cdots x_t\neq 0$ be a maximal string of nonzero cup product. We claim that $x_1\cdot x_2\cdots x_t\in H^{2d}(X)$. If not then $x_1\cdot x_2\cdots x_t\in H^{2d-2l}(X)$ for some $l>0$. Now as $H^{2i}(X)$ is torsion free for all i, therefore the cup product pairing $H^{2d-2l}(X)\times H^{2l}(X)\to\mathbb{Z}$ is nonsingular and hence there exists $y\in H^{2l}(X)$ $(y\neq 0)$ such that $x_1\cdot x_2\cdots x_t\cdot y\in H^{2d}(X)$ is a nonzero element. This contradicts the maximality of $x_1\cdot x_2\cdots x_t$.

Now we rearrange $x_1\cdot x_2\cdots x_t$ as $y_1\cdot y_2\cdots y_m\cdot z_1\cdot z_2\cdots z_n$ such that $\deg(y_i)=i$, $\deg(z_j)=j$ with $i\leq 2k$ and $j\geq 2k+2$. If possible, suppose

$$x_1\cdot x_2\cdots x_t=y_1\cdot y_2\cdots y_m.$$

As $i\leq 2k\leq \operatorname{chern\,rank}(\xi)$, therefore, $y_1\cdot y_2\cdots y_m$ is a polynomial in Chern classes $c_1(\xi),\cdots,c_k(\xi)$ laying in $H^{2d}(X)$. Hence it is a sum of monomials in Chern classes each of which is zero and thus $y_1\cdot y_2\cdots y_m=0$. Consequently, the string $z_1\cdot z_2\cdots z_n$ must exist.

Let $a=y_1\cdot y_2\cdots y_m$ and $b=z_1\cdot z_2\cdots z_n$. As $\deg(b)\geq 2k+2$ therefore $\deg(a)\leq 2d-2(k+1)$ and

$$\begin{aligned}
\mathrm{Cup}_E(X)=m+n &\leq \frac{\deg(a)}{r_X}+\frac{\deg(b)}{2k+2}=\frac{2(k+1)\deg(a)+r_X\deg(b)}{2r_X(k+1)}\\
&=\frac{2(k+1)\deg(a)+r_X(2d-\deg(a))}{2r_X(k+1)}\\
&=\frac{(2(k+1)-r_X)\deg(a)+2dr_X}{2r_X(k+1)}\\
&\leq\frac{(2(k+1)-r_X)(d-(k+1))+dr_X}{r_X(k+1)}\\
&=\frac{r_X(k+1)+2(k+1)(d-k-1)}{r_X(k+1)}=1+\frac{2(d-k-1)}{r_X}.
\end{aligned}$$

$\square$

Proof of Theorem 1.3: As $\operatorname{uch\,rank}(X)=\dim X$ therefore there exists a complex vector bundle ξ over X with $\operatorname{chern\,rank}\xi=\dim X$. Let $\mathrm{Cup}_E(X)=t$ and

$$x_1\cdot x_2\cdots x_i\cdots x_t\neq 0$$

be a maximal string of nonzero cup product. As $\operatorname{chern\,rank}\xi = \dim X$ hence x_i can be expressed as a polynomial of Chern classes of ξ and consequently $x = x_1 \cdot x_2 \cdots x_t$ can be expressed as a sum of integral multiples of monomials of Chern classes $c_1(\xi), c_2(\xi), \cdots, c_r(\xi)$, $2r \leq \max\deg(x_i)$, each of length at least t. But as monomials of Chern classes of length greater than t vanish therefore there must exists a monomial $c_{i_1}(\xi) \cdot c_{i_2}(\xi) \cdots c_{i_t}(\xi)$ of length t with $c_{i_1}(\xi) \cdot c_{i_2}(\xi) \cdots c_{i_t}(\xi) \neq 0$. □

3. Some computations

In this final section we compute uch rank of some important spaces.

Theorem 3.1. *Let $X = S^m \times S^n$.*

(1) *If m, n are even integers and $m < n$ then*

$$\operatorname{uch\,rank}(X) = \begin{cases} m-2 & \text{if } m \neq 2,4, \\ n-2 & \text{if } m = 2,4 \text{ and } n \neq 2,4, \\ m+n & \text{if } m = 2,\ n = 4. \end{cases}$$

(2) *If m, n are even integers and $m = n$ then* $\operatorname{uch\,rank}(X) = m - 2$.

(3) *If m is odd and n is even then*

$$\operatorname{uch\,rank}(X) = \begin{cases} n-2 & \text{if } n \neq 2,4, \\ m+n-1 & \text{if } n = 2,4. \end{cases}$$

(4) *If m and n are odd integers and $m+n = 2$ or 4 then* $\operatorname{uch\,rank}(X) = m+n$.

(5) *If $m, n \equiv 3 \pmod 8$ then* $\operatorname{uch\,rank}(X) = m+n-2$ *and if $n \equiv 5 \pmod 8$ then* $\operatorname{uch\,rank}(S^1 \times S^n) = n-1$.

Proof: (1) We note that $\widetilde{H}^i(S^m \times S^n)$ is nontrivial if $i = m, n$ or $m+n$. We observe that the inclusion map $i\colon S^m \hookrightarrow S^m \times S^n$ and projection $p\colon S^m \times S^n \to S^m$ induces isomorphisms on the mth cohomology groups, respectively. Thus if $m \neq 2,4$ and ξ is a complex vector bundle over $S^m \times S^n$ with $\operatorname{chern\,rank}\xi \geq m$ then $i^*(\xi)$ becomes a complex vector bundle over S^m and by naturality of Chern classes $\operatorname{chern\,rank} i^*(\xi) \geq m$ which is a contradiction as $\operatorname{uch\,rank}(S^m) = m-2$ if $m \neq 2,4$ (cf. Theorem 2.7). So it follows that $\operatorname{uch\,rank}(S^m \times S^n) = m-2$.

If $m = 2,4$ and $n \neq 2,4$ then by similar argument $\operatorname{uch\,rank}(S^m \times S^n) \leq n-2$. By Theorem 2.7, there exists a complex vector bundle γ over S^m with $\operatorname{chern\,rank}\gamma = m$. Again as $p^*\colon H^m(S^m) \to H^m(S^m \times S^n)$ is an isomorphism, it follows that $\operatorname{chern\,rank} p^*(\gamma) \geq m$. Thus $\operatorname{uch\,rank}(S^m \times S^n) = n-2$.

Finally, let $m = 2$ and $n = 4$. Note that there exist complex line bundle γ_1 and complex 2-plane bundle γ_2 over S^m and S^n, respectively, such that $\operatorname{chern\,rank}\gamma_1 = 2$ and $\operatorname{chern\,rank}\gamma_2 = 4$. Consider the projection maps $p_1\colon S^m \times S^n \to S^m$ and $p_2\colon S^m \times S^n \to S^n$. As $p_1^*\colon H^m(S^m) \to H^m(S^m \times S^n)$ and $p_2^*\colon H^n(S^n) \to H^n(S^m \times S^n)$ are isomorphisms so the total Chern class

$c(p_1^*(\gamma_1)) = 1 + a$ and $c(p_2^*(\gamma_2)) = 1 + b$ where a and b are generators of $H^m(S^m \times S^n)$ and $H^n(S^m \times S^n)$, respectively. Consider the Whitney sum $p_1^*(\gamma_1) \oplus p_2^*(\gamma_2)$ over $S^m \times S^n$ which is a 3-plane complex bundle over $S^m \times S^n$. Again $c(p_1^*(\gamma_1) \oplus p_2^*(\gamma_2)) = c(p_1^*(\gamma_1)) \cdot c(p_2^*(\gamma_2))$ and if a and b are generators of $H^m(S^m \times S^n)$ and $H^n(S^m \times S^n)$, respectively, then it follows from the cohomology ring structure of $H^*(S^m \times S^n)$ that $a \cdot b$ is a generator of $H^{m+n}(S^m \times S^n)$. Consequently it turns out that $\operatorname{chern rank}(p_1^*(\gamma_1) \oplus p_2^*(\gamma_2)) = m + n$.

(2) The first nontrivial reduced integral cohomology group of $S^m \times S^m$ is $\widetilde{H}^m(S^m \times S^m)$ which is free abelian of rank 2 and the proof follows from assertion (4) of Lemma 2.1.

(3) Here we notice that if m is odd and n is even then the only nontrivial even dimensional reduced integral cohomology group of $S^m \times S^n$ is $\widetilde{H}^n(S^m \times S^n)$ and the proof is similar to the case of (1).

(4) As $S^m \times S^n$ is a closed connected $m + n$ dimensional smooth orientable manifold hence there exists a degree 1 map $f\colon S^m \times S^n \to S^{m+n}$. Thus $f_*\colon H_{m+n}(S^m \times S^n) \to H_{m+n}(S^{m+n})$ is an isomorphism and consequently $f^*\colon \operatorname{Hom}(H_{m+n}(S^{m+n}); \mathbb{Z}) \to \operatorname{Hom}(H_{m+n}(S^m \times S^n); \mathbb{Z})$ is an isomorphism. Again as $H_{m+n-1}(S^m \times S^n)$ is torsion free (as $S^m \times S^n$ is orientable) consequently $f^*\colon H^{m+n}(S^{m+n}) \to H^{m+n}(S^m \times S^n)$ becomes an isomorphism. Now the proof follows from the fact that $\operatorname{uch rank}(S^{m+n}) = m + n$ if $m + n = 2$ or 4.

(5) If $m, n \equiv 3 \pmod 8$ then $\widetilde{KO}(S^m) = 0 = \widetilde{KO}(S^n)$ and again as $m + n \equiv 6 \pmod 8$ therefore $\widetilde{KO}(S^{m+n}) = \widetilde{KO}(S^m \wedge S^n) = 0$. By assertion (2) of Theorem 2.8 $\operatorname{uch rank}(S^m \times S^n) < m + n$ and consequently $\operatorname{uch rank}(S^m \times S^n) = m + n - 2$. If $n \equiv 5 \pmod 8$ then every orientable real vector bundle over $S^1 \times S^n$ becomes stably trivial, see [6], Lemma 3.6, therefore there cannot exist any complex vector bundle ξ over $(S^1 \times S^n)$ such that $c_{(n+1)/2}(\xi)$ is a generator of $H^{n+1}(S^1 \times S^n)$ and thus $\operatorname{uch rank}(S^1 \times S^n) = n - 1$. $\square$

We deduce the following corollary from part (4) of Theorem 3.1.

Corollary 3.2. *The upper chern rank of $S^1 \times S^1$, $S^1 \times S^3$ are* 2 *and* 4*, respectively.*

Remark. Note that $\operatorname{uch rank}(S^1 \times S^1) = 2$ also follows from the fact that the first Chern class $c_1\colon \operatorname{Vect}^1_{\mathbb{C}}(S^1 \times S^1) \to H^2(S^1 \times S^1) \cong \mathbb{Z}$ is an isomorphism ($\operatorname{Vect}^1_{\mathbb{C}}(X)$ denotes the abelian group of isomorphism classes of complex line bundles over X with respect to tensor product operations).

Theorem 3.3. *Let $X = S^m \vee S^n$.*

(1) *If m, n are even integers and $m < n$ then*

$$\operatorname{uch rank}(X) = \begin{cases} m - 2 & \text{if } m \neq 2, 4, \\ n - 2 & \text{if } m = 2 \text{ or } 4 \text{ and } n \neq 4, \\ n & \text{if } m = 2 \text{ and } n = 4. \end{cases}$$

(2) *If m is odd and n is even integer then*

$$\operatorname{uch\,rank}(X) = \begin{cases} n-2 & \text{if } m<n \text{ and } n \neq 2,4, \\ n & \text{if } m<n \text{ and } n = 2 \text{ or } 4, \\ n-2 & \text{if } m>n \text{ and } n \neq 2,4, \\ m-1 & \text{if } m>n \text{ and } n = 2 \text{ or } 4. \end{cases}$$

(3) *If m, n are even integers and $m = n$ then* $\operatorname{uch\,rank}(X) = m-2$.

Proof: (1) Let $i_1\colon S^m \hookrightarrow S^m \vee S^n$, $i_2\colon S^n \hookrightarrow S^m \vee S^n$ be the inclusions and $r_1\colon S^m \vee S^n \to S^m$, $r_2\colon S^m \vee S^n \to S^n$ be the retraction maps. We consider the sequence of maps $S^m \hookrightarrow S^m \vee S^n \to S^m$ and $S^n \hookrightarrow S^m \vee S^n \to S^n$. Clearly $i_1^*\colon H^m(S^m \vee S^n) \to H^m(S^m)$, $i_2^*\colon H^n(S^m \vee S^n) \to H^n(S^n)$ and $r_1^*\colon H^m(S^m) \to H^m(S^m \vee S^n)$, $r_2^*\colon H^n(S^n) \to H^n(S^m \vee S^n)$ are isomorphisms. Now $\operatorname{uch\,rank}(X) = m-2$ if $m \neq 2$, 4, and it is equal to $n-2$ if $m = 2$, 4 and $n \neq 4$, which follows by similar arguments as in part (1) of Theorem 3.1.

Let $m = 2$, $n = 4$ and $j\colon S^m \vee S^n \hookrightarrow S^m \times S^n$ is inclusion and p_1, p_2 are the projection maps: $p_1\colon S^m \times S^n \to S^m$, $p_2\colon S^m \times S^n \to S^n$. We consider the sequence of maps: $S^m \hookrightarrow S^m \vee S^n \hookrightarrow S^m \times S^n \to S^m$ and $S^n \hookrightarrow S^m \vee S^n \hookrightarrow S^m \times S^n \to S^n$. As $(i_k^* \circ j^*) \circ p_k^*$ is isomorphism, $k = 1$ or 2, hence $i_1^* \circ j^*\colon H^m(S^m \times S^n) \to H^m(S^m)$ and $i_2^* \circ j^*\colon H^n(S^m \times S^n) \to H^n(S^n)$ are surjections and hence isomorphisms. Again as i_k^* is an isomorphism, $k = 1$ or 2, so it follows that $j^*\colon H^m(S^m \times S^n) \to H^m(S^m \vee S^n)$ and $j^*\colon H^n(S^m \times S^n) \to H^n(S^m \vee S^n)$ are isomorphisms. Note that by part (1) of Theorem 3.1 $\operatorname{uch\,rank}(S^m \times S^n) = m+n$ and therefore there exists a complex vector bundle ξ over $S^m \times S^n$ such that $\operatorname{chern\,rank}(\xi) = m+n$. Clearly $\operatorname{chern\,rank} j^*(\xi) = n$.

(2) We note that the only even dimensional nontrivial reduced cohomology group of $S^m \vee S^n$ is $\widetilde{H}^n(S^m \vee S^n) \cong \widetilde{H}^n(S^n)$ and the arguments are similar to the first case.

Proof of (3) follows from assertion (4) of Lemma 2.1 as the only even dimensional nontrivial reduced cohomology group $\widetilde{H}^m(S^m \vee S^n)$ is free abelian of rank 2. □

Lemma 3.4. *For any complex vector bundle ξ over $\mathbb{R}P^{2k}$ (or $\mathbb{R}P^{2k+1}$),* $\operatorname{chern\,rank}\xi$ *is either* 0 *or* $2k$ *and*

$$\operatorname{uch\,rank}(\mathbb{R}P^{2k}) = 2k = \operatorname{uch\,rank}(\mathbb{R}P^{2k+1}).$$

Proof: The graded integral cohomology ring of $\mathbb{R}P^{2k}$ is given by

$$H^*(\mathbb{R}P^{2k}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}), \qquad \deg(\alpha) = 2.$$

If ξ is a complex vector bundle over $\mathbb{R}P^{2k}$ with $c_1(\xi) = 0$ then $\operatorname{chern\,rank}\xi = 0$ (for example we can take any trivial complex vector bundle) as $H^2(\mathbb{R}P^{2k}) \cong \mathbb{Z}_2$. On the contrary if $c_1(\xi) \neq 0$ then $H^{2i}(\mathbb{R}P^{2k}) = \langle (c_1(\xi))^i \rangle \cong \mathbb{Z}_2$ and consequently $\operatorname{chern\,rank}\xi = 2k$. Now as $c_1\colon \operatorname{Vect}^1_{\mathbb{C}}(\mathbb{R}P^{2k}) \to H^2(\mathbb{R}P^{2k})$ is an isomorphism therefore there exists a complex line bundle ξ over $\mathbb{R}P^{2k}$ with $c_1(\xi) \neq 0$ and thus $\operatorname{uch rank}(\mathbb{R}P^{2k}) = 2k$.

Again the graded integral cohomology ring of $\mathbb{R}P^{2k+1}$ is given by

$$H^*(\mathbb{R}P^{2k+1}) \cong \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta), \qquad \deg(\alpha) = 2, \qquad \deg(\beta) = 2k+1$$

and the proof follows in similar fashion. □

Theorem 3.5. (1) *If $X = \mathbb{R}P^n \times S^{2m}$ then*

$$\operatorname{uch rank}(X) = \begin{cases} 2(m+k) & \text{if } m = 2 \text{ and } n = 2k \text{ or } 2k+1, \\ 2(m-1) & \text{if } m \neq 2. \end{cases}$$

(2) *If $X = \mathbb{C}P^n \times S^{2m}$ then*

$$\operatorname{uch rank}(X) = \begin{cases} 2(m+n) & \text{if } m = 2, \\ 2(m-1) & \text{if } m \neq 2. \end{cases}$$

Proof: (1) Let $n = 2k$. We consider the projection maps

$$p_1 \colon \mathbb{R}P^{2k} \times S^{2m} \to \mathbb{R}P^{2k}$$

and $p_2 \colon \mathbb{R}P^{2k} \times S^{2m} \to S^{2m}$. If a and b are generators of $H^2(\mathbb{R}P^{2k})$ and $H^{2m}(S^{2m})$, respectively, then the graded integral cohomology ring $H^*(\mathbb{R}P^{2k} \times S^{2m}) \cong \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^{k+1}, \beta^2)$, $\deg(\alpha) = 2$, $\deg(\beta) = 2m$ where $\alpha = p_1^*(a)$ and $\beta = p_2^*(b)$.

If $m = 1$ then $\mathbb{R}P^{2k} \times S^2 \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and by (4) of Lemma 2.1 $\operatorname{uch rank}(X) = 0$. Let $m = 2$. Now it follows from Lemma 3.4 that there exists a complex line bundle ξ over $\mathbb{R}P^{2k}$ such that $\operatorname{chern rank} \xi = 2k$ and there exists a complex vector bundle ξ' over S^4 such that $\operatorname{chern rank} \xi' = 4$ (by Theorem 2.7). Let $a = c_1(\xi)$ and $b = c_2(\xi')$. Now we take the pull back bundles $\upsilon = p_1^*(\xi)$ and $\eta = p_2^*(\xi')$ over X and consider their Whitney sum $\upsilon \oplus \eta$. Clearly $c_1(\upsilon \oplus \eta) = c_1(\upsilon) = \alpha$ and $c_2(\upsilon \oplus \eta) = c_2(\eta) = \beta$ and consequently $\operatorname{chern rank}(\upsilon \oplus \eta) = 2(m+k)$.

Finally let $m \neq 1, 2$. We note that $\operatorname{chern rank} \upsilon \geq 2(m-1)$ and as β cannot be expressed as a product of cohomology classes of $H^*(X)$ with degree lower than $2m$ so $\operatorname{chern rank} \upsilon = 2(m-1)$. Now if $\operatorname{uch rank}(X) \geq 2m$, there exists a complex vector bundle γ over X such that $\operatorname{chern rank} \gamma \geq 2m$. Let $i \colon S^{2m} \hookrightarrow \mathbb{R}P^{2k} \times S^{2m}$ be the inclusion map. As $i^* \circ p_2^* = \mathrm{id}$ on $H^{2m}(S^{2m})$ thus it turns out that $i^*(\beta) = b$. Again as β cannot be expressed as a product of cohomology classes of $H^*(X)$ with degree lower than $2m$ therefore $c_m(\gamma)$ must be equal to β and so $c_m(i^*(\gamma)) = i^* c_m(\gamma) = i^*(\beta) = b$. Thus $\operatorname{uch rank}(S^{2m}) = 2m$; which contradicts $\operatorname{uch rank}(S^{2m}) = 2m - 2$ if $m \neq 1, 2$ (Theorem 2.7). This completes the proof for $m \neq 1, 2$.

If $n = 2k+1$ then $H^*(\mathbb{R}P^{2k+1} \times S^{2m}) \cong \mathbb{Z}[\alpha,\beta,\lambda]/(2\alpha, \alpha^{k+1}, \lambda^2, \alpha \cdot \lambda, \beta^2)$ where $\deg(\alpha) = 2$, $\deg(\beta) = 2m$, $\deg(\lambda) = 2k+1$ and the proof is similar to the case $n = 2k$.

(2) We note that the graded integral cohomology ring $H^*(\mathbb{C}P^n \times S^{2m}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^{n+1}, \beta^2)$ where $\deg(\alpha) = 2$, $\deg(\beta) = 2m$ and also if L is the canonical complex line bundle over $\mathbb{C}P^n$ then $\operatorname{chern\,rank} L = 2n$. Now the proof follows by arguments as in (1). $\square$

Now we study complex vector bundles over complex Stiefel manifolds $V_k(\mathbb{C}^n)$ which consists of the orthonormal k-frames in $\mathbb{C}^n$.

Theorem 3.6. *Let $X = V_k(\mathbb{C}^n)$, where $1 < k < n$. Then* $\operatorname{uch\,rank}(X) = 4(n-k)+2$ *if $n-k$ is even or $n-k \neq 2^t - 1$, $t > 0$.*

Proof: It is known that for any commutative ring with unit R, $H^*(V_k(\mathbb{C}^n); R) \cong \bigwedge(x_{2(n-k)+1}, x_{2(n-k)+3}, \cdots, x_{2n-1})$, that is, the exterior algebra generated by $x_{2(n-k)+1},\ x_{2(n-k)+3},\ \cdots,\ x_{2n-1}$ where $x_j \in H^j(V_k(\mathbb{C}^n); R)$, see [4], Proposition 5.11. We note that the first nontrivial even dimensional reduced cohomology group of $V_k(\mathbb{C}^n)$ with integer coefficients is $\widetilde{H}^{4(n-k)+4}(V_k(\mathbb{C}^n); \mathbb{Z}) \cong \mathbb{Z}$. Also the integral cohomology structure of $V_k(\mathbb{C}^n)$ implies that the natural coefficient homomorphism $H^{4(n-k)+4}(V_k(\mathbb{C}^n); \mathbb{Z}) \to H^{4(n-k)+4}(V_k(\mathbb{C}^n); \mathbb{Z}_2)$ is an epimorphism where $H^{4(n-k)+4}(V_k(\mathbb{C}^n); \mathbb{Z}_2) \cong \mathbb{Z}_2$. Again it is well known that for any real vector bundle ξ over a space B, if $w_m(\xi)$, $m > 0$ is the first nonzero Stiefel–Whitney class then m must be a power of 2, see [5], Problem 8-B. Now if $n-k$ (> 0) is even or $n-k \neq 2^t - 1$, $t > 0$, then $4(n-k)+4$ cannot be a power of 2 and consequently for any vector bundle ξ over $V_k(\mathbb{C}^n)$, $1 < k < n$, $w_{4(n-k)+4}(\xi) = 0$. Thus for any complex vector bundle η over $V_k(\mathbb{C}^n)$, $1 < k < n$; $c_{2(n-k)+2}(\eta)$ cannot be a generator of $H^{4(n-k)+4}(V_k(\mathbb{C}^n); \mathbb{Z})$ as under the natural coefficient homomorphism $H^{4(n-k)+4}(V_k(\mathbb{C}^n); \mathbb{Z}) \to H^{4(n-k)+4}(V_k(\mathbb{C}^n); \mathbb{Z}_2)$, which is an epimorphism, the Chern class $c_{2(n-k)+2}(\eta)$ is mapped to the Stiefel–Whitney class $w_{4(n-k)+4}(\eta_R)$ and hence $\operatorname{uch\,rank}(V_k(\mathbb{C}^n)) = 4(n-k)+2$. $\square$

Theorem 3.7. *If $X = \mathbb{C}P^n/\mathbb{C}P^m$, where $m \geq 1$, $n \geq m+2$ then*

$$\operatorname{uch\,rank}(X) = \begin{cases} 2m & \textit{if } m \neq 1, \\ 4 & \textit{if } m = 1. \end{cases}$$

Proof: First we observe that the first nontrivial cohomology group of X is $H^{2m+2}(X)$ and if $i\colon S^{2m+2} \hookrightarrow X$ is the inclusion map then $i^*\colon H^{2m+2}(X) \to H^{2m+2}(S^{2m+2})$ is an isomorphism. Now if $m \neq 1$ then $\operatorname{uch\,rank}(S^{2m+2}) = 2m$ (cf. Theorem 2.7) and consequently $\operatorname{uch\,rank}(X) = 2m$.

Next we consider the case when $m = 1$. Now $\mathbb{C}P^3/\mathbb{C}P^1 = S^4 \cup_{f_1} e^6$ where $f_1\colon S^5 \to S^4$ is the attaching map and e^6 denotes a 6-cell. It is well known that $\pi_5(S^4) \cong \mathbb{Z}_2$ and generated by $[\Sigma^2 f]$, where $\Sigma^2 f$ denotes the double suspension of the Hopf map $f\colon S^3 \to S^2$. Let α be a generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z}_2)$ where $H^*(\mathbb{C}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$. We note that the action of Steenrod square operation Sq^2 on α^2 is trivial. Let us consider the quotient map $q\colon \mathbb{C}P^\infty \to \mathbb{C}P^\infty/\mathbb{C}P^1$. Now it follows from the naturality of Steenrod squaring operation that $Sq^2(x)$ is trivial where x is the generator of $H^4(\mathbb{C}P^\infty/\mathbb{C}P^1; \mathbb{Z}_2)$. Again applying naturality property of Steenrod squares with the inclusion map $i_1\colon \mathbb{C}P^3/\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty/\mathbb{C}P^1$ it follows that the action of Sq^2 on the generator of $H^4(\mathbb{C}P^3/\mathbb{C}P^1;\mathbb{Z}_2) \cong \mathbb{Z}_2$ is trivial. Suppose now the attaching map $f_1\colon S^5 \to S^4$ was not null-homotopic. Then f_1 must be homotopic to the double suspension of the Hopf map $f\colon S^3 \to S^2$ as $\pi_5(S^4) \cong \mathbb{Z}_2$. Thus $\mathbb{C}P^3/\mathbb{C}P^1 = C_{\Sigma^2 f} = \Sigma^2 C_f$, where C_f is the associated mapping cone of $f\colon S^3 \to S^2$. Again as Steenrod square operations are invariant under suspension it follows that the action of Sq^2 on the generator of $H^4(\mathbb{C}P^3/\mathbb{C}P^1;\mathbb{Z}_2)$ is nontrivial, a contradiction. Consequently f_1 must be null-homotopic. Thus $\mathbb{C}P^3/\mathbb{C}P^1 \approx S^4 \vee S^6$.

Now by Theorem 3.3 (1), $\operatorname{uch rank}(\mathbb{C}P^3/\mathbb{C}P^1) = \operatorname{uch rank}(S^4 \vee S^6) = 4$. Again we consider the inclusion map $j\colon \mathbb{C}P^3/\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^n/\mathbb{C}P^1$. As a map $j^*\colon H^k(\mathbb{C}P^n/\mathbb{C}P^1) \to H^k(\mathbb{C}P^3/\mathbb{C}P^1)$ induces isomorphisms for $k \leq 6$ so it follows that $\operatorname{uch rank}(\mathbb{C}P^n/\mathbb{C}P^1) \leq 4$. Finally we note that a map $j^\star\colon \widetilde{K}(\mathbb{C}P^n/\mathbb{C}P^1) \to \widetilde{K}(\mathbb{C}P^3/\mathbb{C}P^1)$ induces epimorphism in reduced K-groups, see [1], Theorem 7.2, and so $\operatorname{uch rank}(\mathbb{C}P^n/\mathbb{C}P^1) = 4$. This completes the proof. □

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