# Periodic solutions of a class of third-order differential equations with two delays depending on time and state

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*Abstract.* The goal of the present paper is to establish some new results on the existence, uniqueness and stability of periodic solutions for a class of third order functional differential equations with state and time-varying delays. By Krasnoselskii's fixed point theorem, we prove the existence of periodic solutions and under certain sufficient conditions, the Banach contraction principle ensures the uniqueness of this solution. The results obtained in this paper are illustrated by an example.

*Keywords:* periodic solution; iterative differential equation; fixed point theorem; Green's function

*Classification:* 39B12, 39B82

## 1. Introduction

Iterative functional differential equation which is a relation between an unknown function, its derivatives and its iterates arises as a model including state-dependent deviating arguments. This type of equations is often used to modelize a wide range of phenomena appearing in different field of sciences such as electrodynamics, mechanics, epidemiology, economics, biology and other numerous areas see [3], [4], [9].

Iterative equations have received increasing attention during the past few decades. They have been studied first by C. Babbage in circa 1815 in his paper [1]. In 1967, K. Cooke in [3] introduced the equation

$$u'(t) + au\big(t - r(u(t))\big) = 0,$$

to model a genetic phenomenon in population dynamics or infection models. After one year, A. Pelczyr in [10] used Picard's successive approximation for studding the equation

$$x'(t) = f\big(t, x(t), x(x(t))\big),$$

where 0 is the left endpoint of the domain.

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Later on, in 1984, by virtue of contraction principle, E. Eder in [5] obtained some existence results on the unique monotone solution for the following equation

$$x'(t) = x(x(t)),$$

with $x(t_0) = t_0$, $t_0 \in [-1, 1]$.

The equation

$$x'(t) = f\big(x(x(t))\big),$$

with $x(a) = a$, where $a$ is an endpoint of the well-defined interval was studied in 1990 by K. Wang, see [13], and by M. Fečkan in 1993, see [6], where $x(0) = 0$. In 1997, under the assumption that $x(t_0) = x_0$ on a given compact interval, where the endpoints of the interval are two adjacent null points of $f$, the same equation attracted the attention of W. Ge and Y. Mo in [7].

While, V. Berinde in 2010, see [2], applied the nonexpansive operators to investigate the last initial problem.

One year later, M. Lauran in [8] treated the nonautonomous equation

$$x'(t) = f\big(t, x(t), x(x(t))\big)$$

with $x(t_0) = x_0$.

Recently, sizable researches have focused on this type of equations by using different tools, such as Krasnoselskii fixed point theorem, Banach contraction mapping principle, Schauder's fixed point theorem and so on. In [14], H. Y. Zhao and J. Liu applied Krasnoselskii and Banach's fixed point theorems for establishing the existence, uniqueness and stability of periodic solutions of the following first order iterative differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = c_1(t)x^{[1]}(t) + c_2(t)x^{[2]}(t) + \cdots + c_n(t)x^{[n]}(t) + F(t).$$

In [15], H. Y. Zhao and M. Fečkan used Schauder's fixed point theorem to show the existence and the stability of periodic solutions of the following equation

$$x'(t) = \sum_{m=1}^{k}\sum_{l=1}^{\infty} C_{l,m}(x^{[m]}(t))^l + G(t).$$

Motivated by the preceding works, we consider the following functional differential equations depending on both the state variable $x$ and the time $t$

$$x'''(t) + 3\varrho x''(t) + 3\varrho^2 x'(t) + \varrho^3 x(t) = \frac{\mathrm{d}}{\mathrm{d}t} f\big(t, x(t-\tau(t))\big) + \sum_{k=0}^{n} a_k(t)x^{[k]}(t), \tag{1.1}$$

and

$$x'''(t) - 3\varrho x''(t) + 3\varrho^2 x'(t) - \varrho^3 x(t) = \frac{\mathrm{d}}{\mathrm{d}t} f\big(t, x(t-\tau(t))\big) + \sum_{k=0}^{n} a_k(t)x^{[k]}(t), \tag{1.2}$$

where $\varrho > 0$, $\tau \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ is a $T$-periodic function, $x^{[m]}(t)$ are the $m$th iterates of the function $x$, $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $f(t, x)$ is $T$-periodic in $t$.

By a technique based on the combination of the Krasnoselskii and Banach fixed point theorems and some useful properties of the explicit forms of the Green's functions obtained in [11], we will establish sufficient conditions to guarantee the existence, uniqueness and stability of periodic solutions of these two equations. The rest of the paper is organized as follows: in Section 2, we give some preliminary results. Existence, uniqueness and stability of periodic solutions of (1.1) and (1.2) are established in the last section. In addition, an illustrative example is given to show the applicability of our results.

## 2. Preliminaries

For $T > 0$ and $L$, $M \geq 0$, let

$$P_T = \{x \in \mathcal{C}(\mathbb{R}, \mathbb{R}) \colon x(t) = x(t+T)\},$$

equipped with the norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0,T]} |x(t)|,$$

and

$$P_T(L, M) = \{x \in P_T \colon \|x\| \leq L,\ |x(t_2) - x(t_1)| \leq M|t_2 - t_1|,\ \forall\, t_1, t_2 \in \mathbb{R}\},$$

then $(P_T, \|\cdot\|)$ is a Banach space and $P_T(L, M)$ is a closed convex and bounded subset of $P_T$.

It is convenient to suppose that the function $f$ is globally Lipschitz in $x$, that is, there exists a positive constant $K$ such that

$$|f(t, x) - f(t, y)| \leq K\|x - y\|. \tag{2.1}$$

**Lemma 1** ([11]). *For $\varrho > 0$ and $h \in P_T$ the equation*

$$\begin{cases} x'''(t) + 3\varrho x''(t) + 3\varrho^2 x'(t) + \varrho^3 x(t) = h(t), \\ x(0) = x(T), \quad x'(0) = x'(T), \quad x''(0) = x''(T), \end{cases}$$

*has a unique $T$-periodic solution*

$$x(t) = \int_0^T G_1(t, s) h(s)\, \mathrm{d}s,$$

*where*

$$G_1(t, s) = \begin{cases} \frac{[(s-t)\,\mathrm{e}^{-\varrho T} + T - s + t]^2 + T^2\,\mathrm{e}^{-\varrho T}}{2(1 - \mathrm{e}^{-\varrho T})^3} \\ \quad \times \mathrm{e}^{-\varrho(t+T-s)} \qquad \text{if } 0 \leq t \leq s \leq T, \\ \frac{[(s-t+T)\,\mathrm{e}^{-\varrho T} - s + t]^2 + T^2\,\mathrm{e}^{-\varrho T}}{2(1 - \mathrm{e}^{-\varrho T})^3} \\ \quad \times \mathrm{e}^{-\varrho(t-s)} \qquad \text{if } 0 \leq s \leq t \leq T. \end{cases} \tag{2.2}$$

**Lemma 2** ([11]). *For $\varrho > 0$ and $h \in P_T$ the equation*

$$\begin{cases} x'''(t) - 3\varrho x''(t) + 3\varrho^2 x'(t) - \varrho^3 x(t) = h(t), \\ x(0) = x(T), \quad x'(0) = x'(T), \quad x''(0) = x''(T), \end{cases}$$

*has a unique $T$-periodic solution*

$$x(t) = \int_0^T G_2(t,s)(-h(s))\,\mathrm{d}s,$$

*where*

$$G_2(t,s) = \begin{cases} \frac{[(s-t)\,\mathrm{e}^{\varrho T} + T - s + t]^2 + T^2\,\mathrm{e}^{\varrho T}}{2(\,\mathrm{e}^{\varrho T} - 1)^3} \\ \quad \times\,\mathrm{e}^{\varrho(t+T-s)} \qquad \text{if } \ 0 \le t \le s \le T, \\ \frac{[(s-t+T)\,\mathrm{e}^{\varrho T} - s + t]^2 + T^2\,\mathrm{e}^{\varrho T}}{2(\,\mathrm{e}^{\varrho T} - 1)^3} \\ \quad \times\,\mathrm{e}^{\varrho(t-s)} \qquad \text{if } \ 0 \le s \le t \le T. \end{cases} \tag{2.3}$$

**Lemma 3** ([11]). *Let*

$$A = \frac{T^2(1 + \,\mathrm{e}^{\varrho T})}{2\varrho^2(\,\mathrm{e}^{\varrho T} - 1)}, \qquad B = \frac{T^2\,\mathrm{e}^{2\varrho T}(1 + \,\mathrm{e}^{\varrho T})}{2(\,\mathrm{e}^{\varrho T} - 1)^3}, \tag{2.4}$$

*then*

$$0 < A \le G_1(t,s) \le B \qquad \text{and} \qquad 0 < A \le G_2(t,s) \le B$$

*for all $t \in [0,T]$ and $s \in [0,T]$.*

**Lemma 4** ([14]). *For any $\varphi, \psi \in P_T(L,M)$,*

$$\|\varphi^{[m]} - \psi^{[m]}\| \le \sum_{j=0}^{m-1} M^j \|\varphi - \psi\|. \tag{2.5}$$

**Lemma 5** ([15]). *We have*

$$P_T(L,M) = \{x \in P_T \colon \|x\| \le L,\ |x(t_2) - x(t_1)| \le M|t_2 - t_1|,\ \forall\, t_1, t_2 \in [0,T]\}.$$

**Theorem 1** ([12]). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) *for $x, y \in \mathbb{M}$, we have $\mathcal{A}x + \mathcal{B}y \in \mathbb{M}$;*
(ii) *mapping $\mathcal{A}$ is compact and continuous;*
(iii) *mapping $\mathcal{B}$ is a contraction mapping.*

*Then there exists $z \in \mathbb{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.*

## 3. Main results

**Lemma 6.** *Let*

$$
\begin{aligned}
H_{1,1}(t,s) &= \frac{[(s-t+T)\,\mathrm{e}^{-\varrho T} - s + t]^2 + T^2\,\mathrm{e}^{-\varrho T}}{2(1-\,\mathrm{e}^{-\varrho T})^3}\,\mathrm{e}^{-\varrho(t-s)},\\
H_{1,2}(t,s) &= \frac{[(s-t)\,\mathrm{e}^{-\varrho T} + T - s + t]^2 + T^2\,\mathrm{e}^{-\varrho T}}{2(1-\,\mathrm{e}^{-\varrho T})^3}\,\mathrm{e}^{-\varrho(t+T-s)},\\
H_{2,1}(t,s) &= \frac{[(s-t+T)\,\mathrm{e}^{\varrho T} - s + t]^2 + T^2\,\mathrm{e}^{\varrho T}}{2(\,\mathrm{e}^{\varrho T}-1)^3}\,\mathrm{e}^{\varrho(t-s)},\\
H_{2,2}(t,s) &= \frac{[(s-t)\,\mathrm{e}^{\varrho T} + T - s + t]^2 + T^2\,\mathrm{e}^{\varrho T}}{2(\,\mathrm{e}^{\varrho T}-1)^3}\,\mathrm{e}^{\varrho(t+T-s)},
\end{aligned}
$$

*then*

$$
\begin{aligned}
\frac{\partial}{\partial s}H_{1,1}(t,s) &= (s-t)\frac{\mathrm{e}^{-\varrho(t-s)}}{1-\,\mathrm{e}^{-\varrho T}} - T\,\mathrm{e}^{-\varrho(t+T-s)}(1-\,\mathrm{e}^{-\varrho T})^2 + \varrho H_{1,1}(t,s),\\
\frac{\partial}{\partial s}H_{1,2}(t,s) &= (s-t)\frac{\mathrm{e}^{-\varrho(t-s)}}{1-\,\mathrm{e}^{-\varrho T}}\,\mathrm{e}^{-\varrho(T)} - T\frac{\mathrm{e}^{-\varrho(t+T-s)}}{(1-\,\mathrm{e}^{-\varrho T})^2} + \varrho H_{1,2}(t,s),\\
\frac{\partial}{\partial s}H_{2,1}(t,s) &= (s-t)\frac{\mathrm{e}^{\varrho(t-s)}}{(\,\mathrm{e}^{\varrho T}-1)} + T\frac{\mathrm{e}^{\varrho(t+T-s)}}{(\,\mathrm{e}^{\varrho T}-1)^2} - \varrho H_{2,1}(t,s),\\
\frac{\partial}{\partial s}H_{2,2}(t,s) &= (s-t)\frac{\mathrm{e}^{\varrho(t-s)}}{(\,\mathrm{e}^{\varrho T}-1)}\,\mathrm{e}^{\varrho T} + T\frac{\mathrm{e}^{\varrho(t+T-s)}}{(\,\mathrm{e}^{\varrho T}-1)^2} - \varrho H_{2,2}(t,s).
\end{aligned}
$$

**Lemma 7.** *If $x \in P_T(L,M)$, then $x$ is a solution of* (1.1) *if and only if*

$$
\begin{aligned}
x(t) = {} & \sum_{k=0}^{n}\int_0^t H_{1,1}(t,s)a_k(s)x^{[k]}(s)\,\mathrm{d}s + \sum_{k=0}^{n}\int_t^T H_{1,2}(t,s)a_k(s)x^{[k]}(s)\,\mathrm{d}s\\
& - \alpha\int_0^t (s-t)f\big(s,x(s-\tau(s))\big)\,\mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s\\
& - \alpha\,\mathrm{e}^{-\varrho(T)}\int_t^T (s-t)f\big(s,x(s-\tau(s))\big)\,\mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s\\
& + T\alpha^2\int_0^T f\big(s,x(s-\tau(s))\big)\,\mathrm{e}^{-\varrho(t+T-s)}\,\mathrm{d}s\\
& - \varrho\int_0^t f\big(s,x(s-\tau(s))\big)H_{1,1}(t,s)\,\mathrm{d}s - \varrho\int_t^T f\big(s,x(s-\tau(s))\big)H_{1,2}(t,s)\,\mathrm{d}s,
\end{aligned}
$$

*where*

$$
\alpha = \frac{1}{1-\,\mathrm{e}^{-\varrho T}}.
$$

Proof: From Lemma 1,

$$\begin{aligned} x(t) = {} & \int_0^t H_{1,1}(t,s)\Big[\frac{\mathrm{d}}{\mathrm{d}s} f\big(s, x(s-\tau(s))\big) + \sum_{k=0}^{n} a_k(s) x^{[k]}(s)\Big]\,\mathrm{d}s \\ & + \int_t^T H_{1,2}(t,s)\Big[\frac{\mathrm{d}}{\mathrm{d}s} f\big(s, x(s-\tau(s))\big) + \sum_{k=0}^{n} a_k(s) x^{[k]}(s)\Big]\,\mathrm{d}s. \end{aligned}$$

The integration by parts gives

$$\begin{aligned} & \int_0^t H_{1,1}(t,s)\frac{\mathrm{d}}{\mathrm{d}t} f\big(s, x(s-\tau(s))\big)\,\mathrm{d}s \\ & \qquad = \big[f\big(s, x(s-\tau(s))\big) H_{1,1}(t,s)\big]_0^t - \int_0^t f\big(s, x(s-\tau(s))\big)\frac{\partial}{\partial s} H_{1,1}(t,s)\,\mathrm{d}s, \end{aligned}$$

and

$$\begin{aligned} & \int_t^T H_{1,2}(t,s)\frac{\mathrm{d}}{\mathrm{d}t} f\big(s, x(s-\tau(s))\big)\,\mathrm{d}s \\ & \qquad = \big[f\big(s, x(s-\tau(s))\big) H_{1,2}(t,s)\big]_t^T - \int_t^T f\big(s, x(s-\tau(s))\big)\frac{\partial}{\partial s} H_{1,2}(t,s)\,\mathrm{d}s. \end{aligned}$$

Since

$$\begin{aligned} & \big[f\big(s, x(s-\tau(s))\big) H_{1,1}(t,s)\big]_0^t \\ & \qquad = f\big(t, x(t-\tau(t))\big)\frac{T^2\,\mathrm{e}^{-T\varrho}(\,\mathrm{e}^{-T\varrho}+1)}{2(1-\,\mathrm{e}^{-\varrho T})^3} \\ & \qquad\quad - f\big(0, x(-\tau(0))\big)\frac{[(T-t)\,\mathrm{e}^{-\varrho T}+t]^2 + T^2\,\mathrm{e}^{-\varrho T}}{2(1-\,\mathrm{e}^{-\varrho T})^3}\,\mathrm{e}^{-t\varrho}, \end{aligned}$$

and

$$\begin{aligned} & \big[f\big(s, x(s-\tau(s))\big) H_{1,2}(t,s)\big]_t^T \\ & \qquad = f\big(T, x(T-\tau(T))\big)\frac{[(T-t)\,\mathrm{e}^{-\varrho T}+t]^2 + T^2\,\mathrm{e}^{-\varrho T}}{2(1-\,\mathrm{e}^{-\varrho T})^3}\,\mathrm{e}^{-\varrho t} \\ & \qquad\quad - f\big(t, x(t-\tau(t))\big)\frac{T^2(\,\mathrm{e}^{-T\varrho}+1)}{2(1-\,\mathrm{e}^{-\varrho T})^3}\,\mathrm{e}^{-T\varrho} \\ & \qquad = f\big(0, x(-\tau(0))\big)\frac{[(T-t)\,\mathrm{e}^{-\varrho T}+t]^2 + T^2\,\mathrm{e}^{-\varrho T}}{2(1-\,\mathrm{e}^{-\varrho T})^3}\,\mathrm{e}^{-\varrho t} \\ & \qquad\quad - f\big(t, x(t-\tau(t))\big)\frac{T^2(\,\mathrm{e}^{-T\varrho}+1)}{2(1-\,\mathrm{e}^{-\varrho T})^3}\,\mathrm{e}^{-T\varrho}, \end{aligned}$$

then

$$\big[f\big(s, x(s-\tau(s))\big) H_{1,1}(t,s)\big]_0^t + \big[f\big(s, x(s-\tau(s))\big) H_{1,2}(t,s)\big]_0^T = 0.$$

So, we have

$$
\begin{aligned}
x(t) = {} & \sum_{k=0}^{n} \int_0^t H_{1,1}(t,s) a_k(s) x^{[k]}(s) \,\mathrm{d}s + \sum_{k=0}^{n} \int_t^T H_{1,2}(t,s) a_k(s) x^{[k]}(s) \,\mathrm{d}s \\
& - \int_0^t f\big(s, x(s-\tau(s))\big) \frac{\partial}{\partial s} H_{1,1}(t,s) \,\mathrm{d}s \\
& - \int_t^T f\big(s, x(s-\tau(s))\big) \frac{\partial}{\partial s} H_{1,2}(t,s) \,\mathrm{d}s.
\end{aligned} \tag{3.1}
$$

The substitution of $\frac{\partial}{\partial s} H_{1,1}(t,s)$ and $\frac{\partial}{\partial s} H_{1,2}(t,s)$ in (3.1) complete the proof. $\square$

Similarly to the above proof we can prove the following lemma.

**Lemma 8.** *If $x \in P_T(L, M)$, then $x$ is a solution of (1.2) if and only if*

$$
\begin{aligned}
x(t) = {} & \sum_{k=0}^{n} \int_0^t H_{2,1}(t,s) a_k(s) x^{[k]}(s) \,\mathrm{d}s + \sum_{k=0}^{n} \int_t^T H_{2,2}(t,s) a_k(s) x^{[k]}(s) \,\mathrm{d}s \\
& + \alpha \int_0^t (s-t) f\big(s, x(s-\tau(s))\big) \,\mathrm{e}^{\varrho(t-s)} \,\mathrm{d}s \\
& + \alpha \,\mathrm{e}^{\varrho T} \int_t^T (s-t) f\big(s, x(s-\tau(s))\big) \,\mathrm{e}^{\varrho(t-s)} \,\mathrm{d}s \\
& - T\alpha^2 \int_0^T f\big(s, x(s-\tau(s))\big) \,\mathrm{e}^{\varrho(t+T-s))} \,\mathrm{d}s \\
& + \varrho \int_0^t f\big(s, x(s-\tau(s))\big) H_{2,1}(t,s) \,\mathrm{d}s + \varrho \int_t^T f\big(s, x(s-\tau(s))\big) H_{2,2}(t,s) \,\mathrm{d}s.
\end{aligned}
$$

**3.1 Existence of periodic solutions.** For the application of Krasnoselskii's fixed point theorem we need to define two compact operators and two contractions mappings. For this end, and by using Lemmas 7 and 8, we define the following mappings.

Let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4 \colon P_T(L, M) \to P_T$ such that

$$
\begin{aligned}
(\mathcal{N}_1 \varphi)(t) = {} & \sum_{k=0}^{n} \int_0^t H_{1,1}(t,s) a_k(s) \varphi^{[k]}(s) \,\mathrm{d}s \\
& + \sum_{k=0}^{n} \int_t^T H_{1,2}(t,s) a_k(s) \varphi^{[k]}(s) \,\mathrm{d}s,
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
(\mathcal{N}_2\varphi)(t) = T\alpha^2 \int_0^T & f\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t+T-s)}\, \mathrm{d}s \\
& - \alpha \int_0^t (s-t) f\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t-s)}\, \mathrm{d}s \\
& - \alpha\, \mathrm{e}^{-\varrho T} \int_t^T (s-t) f\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t-s)}\, \mathrm{d}s \\
& - \varrho \int_0^t f\big(s, \varphi(s-\tau(s))\big) H_{1,1}(t,s)\, \mathrm{d}s \\
& - \varrho \int_t^T f\big(s, \varphi(s-\tau(s))\big) H_{1,2}(t,s)\, \mathrm{d}s,
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
(\mathcal{N}_3\varphi)(t) = & \sum_{k=0}^{n} \int_0^t H_{2,1}(t,s) a_k(s) \varphi^{[k]}(s)\, \mathrm{d}s \\
& + \sum_{k=0}^{n} \int_t^T H_{2,2}(t,s) a_k(s) \varphi^{[k]}(s)\, \mathrm{d}s,
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
(\mathcal{N}_4\varphi)(t) = & -T\alpha^2 \int_0^T f\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{\varrho(t+T-s)}\, \mathrm{d}s \\
& + \alpha \int_0^t (s-t) f\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{\varrho(t-s)}\, \mathrm{d}s \\
& + \alpha\, \mathrm{e}^{\varrho T} \int_t^T (s-t) f\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{\varrho(t-s)}\, \mathrm{d}s \\
& + \varrho \int_0^t f\big(s, \varphi(s-\tau(s))\big) H_{2,1}(t,s)\, \mathrm{d}s \\
& + \varrho \int_t^T f\big(s, \varphi(s-\tau(s))\big) H_{2,2}(t,s)\, \mathrm{d}s.
\end{aligned} \tag{3.5}
$$

**Lemma 9.** *Suppose that $a_k \in P_T(L_{a_k}, M_{a_k})$, $k = \overline{0,n}$, then the operator $\mathcal{N}_1$ defined by* (3.2) *is continuous and compact on $P_T(L, M)$.*

Proof: For $\varphi, \psi \in P_T(L, M)$, $a_k \in P_T(L_{a_k}, M_{a_k})$, $k = 0, 1, \ldots, n$ and from Lemma 3, we have

$$
\begin{aligned}
|(\mathcal{N}_1\varphi)(t) - (\mathcal{N}_1\psi)(t)| \le & \sum_{k=0}^{n} \int_0^t |H_{1,1}| |a_k(s)| |\varphi^{[k]}(s) - \psi^{[k]}(s)|\, \mathrm{d}s \\
& + \sum_{k=0}^{n} \int_t^T |H_{1,2}| |a_k(s)| |\varphi^{[k]}(s) - \psi^{[k]}(s)|\, \mathrm{d}s
\end{aligned}
$$

$$\leq B\sum_{k=0}^{n}\int_0^t |a_k(s)||\varphi^{[k]}(s)-\psi^{[k]}(s)|\,\mathrm{d}s$$
$$+ B\sum_{k=0}^{n}\int_t^T |a_k(s)||\varphi^{[k]}(s)-\psi^{[k]}(s)|\,\mathrm{d}s$$
$$= B\sum_{k=0}^{n}\int_0^T |a_k(s)||\varphi^{[k]}(s)-\psi^{[k]}(s)|\,\mathrm{d}s.$$

Using Lemma 4, we obtain

$$|(\mathcal{N}_1\varphi)(t)-(\mathcal{N}_1\psi)(t)| \leq BT\sum_{k=1}^{n}\sum_{j=0}^{k-1} L_{a_k}M^j\|\varphi-\psi\|.$$

This proves the continuity of $\mathcal{N}_1$.

By the definition of $P_T(L,M)$, clearly this part of $P_T$ is uniformly bounded and equicontinuous subset of the space of the continuous functions on the compact $[0,T]$. So, we can apply the Ascoli–Arzelà theorem to guarantee that $P_T(L,M)$ is a compact subset from this space. Since $\mathcal{N}_1$ is a continuous operator and since any continuous operator maps compact sets into compact sets, $\mathcal{N}_1(P_T(L,M))$ is also a compact set in $P_T$. Consequently, $\mathcal{N}_1$ is a compact operator. □

We replace $\mathcal{N}_1$ by $\mathcal{N}_3$, we repeat the same steps of the previous proof and we obtain the following lemma.

**Lemma 10.** *Suppose that $a_k \in P_T(L_{a_k}, M_{a_k})$, $k=\overline{0,n}$, then the operator $\mathcal{N}_3$ defined by* (3.4) *is continuous and compact on $P_T(L,M)$.*

**Lemma 11.** *If $\mathcal{N}_2$ is given by* (3.3) *with*

$$KT(B\varrho + T\alpha + T\alpha^2) < 1, \tag{3.6}$$

*then $\mathcal{N}_2$ is a contraction mapping on $P_T(L,M)$.*

Proof: For $\varphi,\psi \in P_T(L,M)$, we have

$$|(\mathcal{N}_2\varphi)(t)-(\mathcal{N}_2\psi)(t)|$$
$$\leq T\alpha^2\int_0^T \big|f\big(s,\varphi(s-\tau(s))\big)-f\big(s,\psi(s-\tau(s))\big)\big|\,\mathrm{e}^{-\varrho(t+T-s)}\,\mathrm{d}s$$
$$+\alpha\int_0^t (t-s)\big|f\big(s,\varphi(s-\tau(s))\big)-f\big(s,\psi(s-\tau(s))\big)\big|\,\mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s$$

$$
\begin{aligned}
&\quad + \alpha\, \mathrm{e}^{-\varrho(T)} \int_t^T (s-t)\big|f\big(s,\varphi(s-\tau(s))\big) \\
&\qquad\qquad\qquad - f\big(s,\psi(s-\tau(s))\big)\big|\, \mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s \\
&\quad + \varrho \int_0^t \big|\big(f(s,\varphi\big(s-\tau(s))\big) - f\big(s,\psi(s-\tau(s))\big)\big)\big| |H_{1,1}(t,s)|\,\mathrm{d}s \\
&\quad + \varrho \int_0^t \big|\big(f\big(s,\varphi(s-\tau(s))\big) - f\big(s,\psi\big(s-\tau(s))\big)\big)\big| |H_{1,2}(t,s)|\,\mathrm{d}s \\
&\le T^2\alpha^2 K\|\varphi-\psi\| + \frac{1}{2}K\alpha t^2\|\varphi-\psi\| + \frac{1}{2}K\alpha(T-t)^2\|\varphi-\psi\| \\
&\quad + \varrho B \int_0^T \big|f\big(s,\varphi(s-\tau(s))\big) - f\big(s,\psi(s-\tau(s))\big)\big|\,\mathrm{d}s \\
&\le T^2\alpha^2 K\|\varphi-\psi\| + \frac{1}{2}KT^2\alpha\|\varphi-\psi\| + \frac{1}{2}KT^2\alpha\|\varphi-\psi\| + \varrho BTK\|\varphi-\psi\| \\
&= KT(B\varrho + T\alpha + T\alpha^2)\|\varphi-\psi\|.
\end{aligned}
$$

By using (3.6), we conclude that $\mathcal{N}_2$ is a contraction mapping on $P_T(L,M)$. □

We replace $\mathcal{N}_2$ by $\mathcal{N}_4$ and we repeat the same steps of the previous proof we obtain the following lemma.

**Lemma 12.** *If $\mathcal{N}_4$ is given by* (3.5) *and* (3.6) *is satisfied, then $\mathcal{N}_4$ is a contraction mapping on $P_T(L,M)$.*

**Lemma 13.** *For any $\varphi,\psi \in P_T(L,M)$, $a_k \in P_T(L_{a_k},M_{a_k})$, $k=\overline{0,n}$ and if*

$$
T(\delta+KL)(T\alpha^2+T\alpha+B\varrho) + BTL\sum_{k=0}^{n} L_{a_k} \le L, \tag{3.7}
$$

*then*

$$
|(\mathcal{N}_1\varphi)(t) + (\mathcal{N}_2\psi)(t)| \le L.
$$

Proof: We have

$$
|(\mathcal{N}_1\varphi)(t) + (\mathcal{N}_2\psi)(t)| \le |(\mathcal{N}_1\varphi)(t)| + |(\mathcal{N}_2\psi)|.
$$

From (2.1), we arrive at

$$
\begin{aligned}
|f(s,x)| &= |f(s,x) - f(s,0) + f(s,0)| \\
&\le |f(s,x) - f(s,0)| + |f(s,0)| \\
&\le K\|x\| + \delta,
\end{aligned}
$$

where $\delta = \max_{[0,T]} |f(t,0)|$. Since

$$\begin{aligned}
|(\mathcal{N}_1\varphi)(t)| \leq & \sum_{k=0}^{n} \int_0^t |H_{1,1}(t,s)||a_k(s)\varphi^{[k]}(s)| \,\mathrm{d}s \\
& + \sum_{k=0}^{n} \int_t^T |H_{1,2}(t,s)||a_k(s)\varphi^{[k]}(s)| \,\mathrm{d}s \\
& \leq B \sum_{k=0}^{n} \int_0^T |a_k(s)||\varphi^{[k]}(s)| \,\mathrm{d}s \\
& \leq BTL \sum_{k=0}^{n} L_{a_k},
\end{aligned}$$

and

$$\begin{aligned}
|(\mathcal{N}_2\psi)(t)| \leq & \, T\alpha^2 LK \int_0^T \mathrm{d}s + T\alpha^2\delta \int_0^T \mathrm{d}s + \alpha LK \int_0^t (t-s)\,\mathrm{d}s \\
& + \alpha\delta \int_0^t (t-s)\,\mathrm{d}s + \alpha LK \int_t^T (s-t)\,\mathrm{d}s + \alpha\delta \int_t^T (s-t)\,\mathrm{d}s \\
& + \varrho BLK \int_0^T \mathrm{d}s + B\varrho\delta \int_0^T \mathrm{d}s \\
\leq & \, T^2\alpha^2\delta + KLT^2\alpha^2 + \frac{1}{2}t^2\alpha\delta + \frac{1}{2}KLt^2\alpha + \frac{1}{2}\alpha\delta(T-t)^2 \\
& + \frac{1}{2}KL\alpha(T-t)^2 + BT\delta\varrho + BKLT\varrho \\
\leq & \, T(\delta + KL)(T\alpha^2 + T\alpha + B\varrho),
\end{aligned}$$

by virtue of (3.7), we obtain

$$|(\mathcal{N}_1\varphi)(t) + (\mathcal{N}_2\psi)(t)| \leq T(\delta + KL)(T\alpha^2 + T\alpha + B\varrho) + BTL\sum_{k=0}^{n} L_{a_k} \leq L.$$

The proof is complete. □

The same steps of the previous proof can prove the following lemma.

**Lemma 14.** *For any $\varphi, \psi \in P_T(L, M)$, $a_k \in P_T(L_{a_k}, M_{a_k})$, $k = \overline{0, n}$ and if (3.7) is satisfied then*

$$|(\mathcal{N}_3\varphi)(t) + (\mathcal{N}_4\psi)(t)| \leq L.$$

**Lemma 15.** *If $t_1, t_2 \in [0, T]$ then*

$$|H_{1,1}(t_1,s) - H_{1,1}(t_2,s)| \leq |t_2 - t_1|(T\alpha^2 + \varrho B), \tag{3.8}$$

*and*

$$|H_{1,2}(t_1,s)-H_{1,2}(t_2,s)|\le |t_2-t_1|(T\alpha^2+\varrho B). \tag{3.9}$$

Proof: By a direct calculus, we have

$$\frac{\partial}{\partial t}H_{1,1}(t,s)=\alpha^2(t-s)\,\mathrm{e}^{\varrho(s-t)}+\alpha^2(s-t+T)\,\mathrm{e}^{\varrho(s-t-T)}-\varrho H_{1,1}(t,s),$$

and

$$\frac{\partial}{\partial t}H_{1,2}(t,s)=\alpha^2(t-s+T)\,\mathrm{e}^{-\varrho(t-s+T)}+\alpha^2(s-t)\,\mathrm{e}^{-T\varrho}\,\mathrm{e}^{-\varrho(t-s+T)}-\varrho H_{1,2}(t,s).$$

So,

$$\begin{aligned}\left|\frac{\partial}{\partial t}H_{1,1}(t,s)\right| &\le |\alpha^2(t-s)\,\mathrm{e}^{\varrho(s-t)}+\alpha^2(s-t+T)\,\mathrm{e}^{\varrho(s-t-T)}-\varrho H_{1,1}(t,s)|\\ &\le |\alpha^2(t-s)\,\mathrm{e}^{\varrho(s-t)}+\alpha^2(s-t+T)\,\mathrm{e}^{\varrho(s-t-T)}|+|\varrho H_{1,1}(t,s)|\\ &\le T\alpha^2+\varrho B,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial t}H_{1,2}(t,s)\right| &\le |\alpha^2(t-s+T)\,\mathrm{e}^{-\varrho(t-s+T)}+\alpha^2(s-t)\,\mathrm{e}^{-T\varrho}\,\mathrm{e}^{-\varrho(t-s+T)}\\ &\qquad -\varrho H_{1,2}(t,s)|\\ &\le |\alpha^2(t-s+T)\,\mathrm{e}^{-\varrho(t-s+T)}+\alpha^2(s-t)\,\mathrm{e}^{-T\varrho}\,\mathrm{e}^{-\varrho(t-s+T)}|\\ &\qquad +|\varrho H_{1,2}(t,s)|\\ &\le T\alpha^2+\varrho B.\end{aligned}$$

Let $a=(t_1,s)$ and $b=(t_2,s)$, $s\in[0,T]$. From the finite increments theorem applied in $a$ and $b$, there exist $c_1=(\alpha_1,\beta_1)$ and $c_2=(\alpha_2,\beta_2)$ in the open segment $]a,b[$ such that

$$\begin{aligned}|H_{1,1}(t_1,s)-H_{1,1}(t_2,s)| &\le |t_2-t_1|\left|\frac{\partial}{\partial t}H_{1,1}(\alpha_1,\beta_1)\right|+|s-s|\left|\frac{\partial}{\partial s}H_{1,1}(\alpha_1,\beta_1)\right|\\ &\le |t_2-t_1|(T\alpha^2+\varrho B),\end{aligned}$$

and

$$\begin{aligned}|H_{1,2}(t_1,s)-H_{1,2}(t_2,s)| &\le |t_2-t_1|\left|\frac{\partial}{\partial t}H_{1,2}(\alpha_2,\beta_2)\right|+|s-s|\left|\frac{\partial}{\partial s}H_{1,2}(\alpha_2,\beta_2)\right|\\ &\le |t_2-t_1|(T\alpha^2+\varrho B).\end{aligned}$$

This completes the proof. □

The following lemma can be proved by using the same techniques as in the preceding proof.

**Lemma 16.** *If* $t_1, t_2 \in [0,T]$ *then*

$$\frac{\partial}{\partial t} H_{1,1}(t,s) = -\alpha^2 (t-s)\,\mathrm{e}^{-\varrho(s-t)} - \alpha^2 (s-t+T)\,\mathrm{e}^{-\varrho(s-t-T)} + \varrho H_{1,2}(t,s),$$

*and*

$$\frac{\partial}{\partial t} H_{2,2}(t,s) = -\alpha^2\,\mathrm{e}^{\varrho(T-s+t)}(T-s+t) - \alpha^2\,\mathrm{e}^{T\varrho}\,\mathrm{e}^{\varrho(T-s+t)}(s-t) + \varrho H_{2,2}(t,s).$$

*Furthermore*

$$|H_{2,1}(t_1,s) - H_{2,1}(t_2,s)| \le |t_2 - t_1|(T\alpha^2 + \varrho B),$$

*and*

$$|H_{2,2}(t_1,s) - H_{2,2}(t_2,s)| \le |t_2 - t_1|(T\alpha^2 + \varrho B).$$

**Lemma 17.** *If* $t_1, t_2 \in [0,T]$ *with* $t_2 > t_1$ *and if*

$$\begin{aligned} &(T^2\alpha\varrho(3\alpha+2) + 2B\varrho(T\varrho+1) + 4T\alpha)(\delta + KL) \\ &\qquad + 2(B + T^2\alpha^2 + BT\varrho)\sum_{k=0}^{n} L_{a_k} \le M, \end{aligned} \tag{3.10}$$

*then*

$$|((\mathcal{N}_1\varphi)(t_2) + (\mathcal{N}_2\psi)(t_2)) - ((\mathcal{N}_1\varphi)(t_1) + (\mathcal{N}_2\psi)(t_1))| \le M|t_2 - t_1|, \quad \forall\, \varphi, \psi \in P_T(L,M).$$

Proof: Let

$$g_1(t,s) = \mathrm{e}^{-\varrho(t+T-s)}, \qquad g_2(t,s) = (t-s)\,\mathrm{e}^{-\varrho(t-s)},$$
$$g_3(t,s) = (s-t)\,\mathrm{e}^{-\varrho(t+T-s)}.$$

We have

$$g_1(t,s) \le 1, \qquad g_2(t,s) \le T, \qquad g_3(t,s) \le T. \tag{3.11}$$

By virtue of the finite increments theorem applied in $(t_1,s)$ and $(t_2,s)$, $s \in [0,T]$, there exist $c_3 = (\alpha_3, \beta_3)$, $c_4 = (\alpha_4, \beta_4)$ and $c_5 = (\alpha_5, \beta_5)$ in the open segment $]a,b[$ such that

$$\begin{aligned} |g_1(t_2,s) - g_1(t_1,s)| &= |t_2 - t_1| \Big| \frac{\partial}{\partial t} g_1(\alpha_3, \beta_3) \Big| \\ &= |t_2 - t_1| |\varrho\,\mathrm{e}^{-\varrho(T-\beta_3+\alpha_3)}| \le \varrho |t_2 - t_1|, \end{aligned} \tag{3.12}$$

$$\begin{aligned} |g_2(t_2,s)-g_2(t_1,s)| &= |t_2-t_1|\left|\frac{\partial}{\partial t} g_2(\alpha_4,\beta_4)\right| \\ &= |t_2-t_1|\,\mathrm{e}^{\varrho(\beta_4-\alpha_4)}|\beta_4\varrho-\alpha_4\varrho+1| \\ &\le |t_2-t_1|(T\varrho+1), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} |g_3(t_2,s)-g_3(t_1,s)| &= |t_2-t_1|\left|\frac{\partial}{\partial t} g_3(\alpha_5,\beta_5)\right| \\ &= |t_2-t_1|\,\mathrm{e}^{-\varrho(T-\beta_5+\alpha_5)}|(\beta_5\varrho-\alpha_5\varrho+1)| \\ &\le |t_2-t_1|(T\varrho+1). \end{aligned} \tag{3.14}$$

Now, for $t_1, t_2 \in [0,T]$ and $\varphi, \psi \in P_T(L,M)$, we get

$$\begin{aligned} &|((\mathcal{N}_1\varphi)(t_2)+(\mathcal{N}_2\psi)(t_2))-((\mathcal{N}_1\varphi)(t_1)+(\mathcal{N}_2\psi)(t_1))| \\ &\qquad \le |(\mathcal{N}_1\varphi)(t_2)-(\mathcal{N}_1\varphi)(t_1)|+|(\mathcal{N}_2\psi)(t_2)-(\mathcal{N}_2\psi)(t_1)|. \end{aligned}$$

From Lemma 3, (3.8) and (3.9)

$$\begin{aligned} &|(\mathcal{N}_1\varphi)(t_2)-(\mathcal{N}_1\varphi)(t_1)| \\ &\quad \le \left|\int_0^{t_2} H_{1,1}(t_2,s)\sum_{k=0}^n a_k(s)\varphi^{[k]}(s)\,\mathrm{d}s-\int_0^{t_1} H_{1,1}(t_1,s)\sum_{k=0}^n a_k(s)\varphi^{[k]}(s)\,\mathrm{d}s\right| \\ &\qquad + \left|\int_{t_2}^{T} H_{1,2}(t_2,s)\sum_{k=0}^n a_k(s)\varphi^{[k]}(s)\,\mathrm{d}s-\int_{t_1}^{T} H_{1,2}(t_1,s)\sum_{k=0}^n a_k(s)\varphi^{[k]}(s)\,\mathrm{d}s\right| \\ &\quad \le \sum_{k=0}^n\int_{t_1}^{t_2} |H_{1,1}(t_2,s)||a_k(s)\varphi^{[k]}(s)|\,\mathrm{d}s \\ &\qquad + \sum_{k=0}^n\int_0^{t_1} |H_{1,1}(t_2,s)-H_{1,1}(t_1,s)||a_k(s)\varphi^{[k]}(s)|\,\mathrm{d}s \\ &\qquad + \sum_{k=0}^n\int_{t_1}^{t_2} |H_{1,2}(t_1,s)||a_k(s)\varphi^{[k]}(s)|\,\mathrm{d}s \\ &\qquad + \sum_{k=0}^n\int_{t_2}^{T} |H_{1,2}(t_2,s)-H_{1,2}(t_1,s)||a_k(s)\varphi^{[k]}(s)|\,\mathrm{d}s \\ &\quad \le \left(2(B+T^2\alpha^2+BT\varrho)\sum_{k=0}^n L_{a_k}\right)|t_2-t_1|. \end{aligned}$$

By using Lemma 3, (3.8), (3.9) and (3.11)–(3.14)

$$
\begin{aligned}
&|(\mathcal{N}_2\psi)(t_2)-(\mathcal{N}_2\psi)(t_1)| \\
&\qquad\le T\alpha^2(KL+\delta)\int_0^T |g_1(t_2,s)-g_1(t_1,s)|\,\mathrm{d}s \\
&\qquad\quad+\alpha(KL+\delta)\int_{t_1}^{t_2} g_2(t_2,s)\,\mathrm{d}s \\
&\qquad\quad+\alpha(KL+\delta)\int_0^{t_1} |g_2(t_2,s)-g_2(t_1,s)|\,\mathrm{d}s \\
&\qquad\quad+\alpha(KL+\delta)\int_{t_1}^{t_2} g_3(t_1,s)\,\mathrm{d}s \\
&\qquad\quad+\alpha(KL+\delta)\int_{t_2}^{T} |g_3(t_2,s)-g_2(t_3,s)|\,\mathrm{d}s \\
&\qquad\quad+\varrho(KL+\delta)\int_{t_1}^{t_2} H_{1,1}(t_2,s)\,\mathrm{d}s \\
&\qquad\quad+\varrho(KL+\delta)\int_0^{t_1} |H_{1,1}(t_2,s)-H_{1,1}(t_1,s)|\,\mathrm{d}s \\
&\qquad\quad+\varrho(KL+\delta)\int_{t_1}^{t_2} H_{1,2}(t_1,s)\,\mathrm{d}s \\
&\qquad\quad+\varrho(KL+\delta)\int_{t_2}^{T} |H_{1,2}(t_2,s)-H_{1,2}(t_1,s)|\,\mathrm{d}s \\
&\qquad\le T^2\alpha^2(KL+\delta)\varrho|t_2-t_1|+2\alpha T(KL+\delta)|t_2-t_1| \\
&\qquad\quad+2\alpha T(KL+\delta)(T\varrho+1)|t_2-t_1|+2\varrho(KL+\delta)B|t_2-t_1| \\
&\qquad\quad+2\varrho T(KL+\delta)(T\alpha^2+\varrho B)|t_2-t_1| \\
&\qquad= (T^2\alpha\varrho(3\alpha+2)+2B\varrho(T\varrho+1)+4T\alpha)(\delta+KL)|t_2-t_1|.
\end{aligned}
$$

So

$$
\begin{aligned}
&|(\mathcal{N}_1\varphi)(t_2)+(\mathcal{N}_2\psi)(t_2)-((\mathcal{N}_1\varphi)(t_1)+(\mathcal{N}_2\psi)(t_1))| \\
&\qquad\le \Bigg[\Bigg(2(B+T^2\alpha^2+BT\varrho)\sum_{k=0}^{n} L_{a_k}\Bigg)+(T^2\alpha\varrho(3\alpha+2) \\
&\qquad\quad+2B\varrho(T\varrho+1)+4T\alpha)(\delta+KL)\Bigg]|t_2-t_1| \\
&\qquad\le M|t_2-t_1|.
\end{aligned}
$$

The proof is complete. $\square$

Similarly to the proof of Lemma 17, we can show the following lemma.

**Lemma 18.** *If $t_1, t_2 \in [0,T]$ with $t_2 < t_1$ and if (3.10) is satisfied then*

$$|(\mathcal{N}_3\varphi)(t_2)+(\mathcal{N}_4\psi)(t_2)-((\mathcal{N}_3\varphi)(t_1)+(\mathcal{N}_4\psi)(t_1))| \leq M|t_2-t_1|, \ \forall\, \varphi,\psi \in P_T(L,M).$$

**Theorem 2.** *Suppose that $a_k \in P_T(L_{a_k}, M_{a_k})$, $k=\overline{0,n}$ and conditions (2.1), (3.6), (3.7) and (3.10) hold. Then (1.1) has at least one solution $x \in P_T(L,M)$.*

Proof: From Lemma 7, we see that the fixed points of $\mathcal{N}_1+\mathcal{N}_2$ are solutions of (1.1) and vice versa. From Lemmas 9 and 11, conditions (ii) and (iii) of the Krasnoselskii's fixed point theorem are satisfied. According to Lemmas 13 and 17 combined with Lemma 5, the condition (i) is also satisfied. Consequently the operator $\mathcal{N}_1+\mathcal{N}_2$ has at least one fixed point in $P_T(L,M)$ which prove that equation (1.1) has at least one solution in $P_T(L,M)$. □

Similar as in the proof of Theorem 2 we can prove the following theorem.

**Theorem 3.** *Suppose that $a_k \in P_T(L_{a_k}, M_{a_k})$, $k=\overline{0,n}$ and conditions (2.1), (3.6), (3.7) and (3.10) hold. Then (1.2) has a solution $x \in P_T(L,M)$.*

### 3.2 Uniqueness of periodic solutions.

**Theorem 4.** *Suppose that $a_k \in P_T(L_{a_k}, M_{a_k})$, $k=\overline{0,n}$ and conditions (2.1), (3.7) and (3.10) hold. If*

$$KT(B\varrho+T\alpha+T\alpha^2)+BT\sum_{k=1}^{n}\sum_{j=0}^{k-1}L_{a_k}M^j<1, \tag{3.15}$$

*then (1.1) has a unique solution $x \in P_T(L,M)$.*

Proof: For $\varphi,\psi \in P_T(L,M)$ and $a_k \in P_T(L_{a_k}, M_{a_k})$, $k=\overline{0,n}$ we have

$$\begin{aligned}
&|(\mathcal{N}_1+\mathcal{N}_2)(\varphi)(t)-(\mathcal{N}_1+\mathcal{N}_2)(\psi)(t)|\\
&\qquad\leq |(\mathcal{N}_1\varphi)(t)-(\mathcal{N}_1\psi)(t)|+|(\mathcal{N}_2\varphi)(t)-(\mathcal{N}_2\psi)(t)|\\
&\qquad\leq \left(KT(B\varrho+T\alpha+T\alpha^2)+BT\sum_{k=1}^{n}\sum_{j=0}^{k-1}L_{a_k}M^j\right)\|\varphi-\psi\|.
\end{aligned}$$

From (3.15) and the contraction mapping principle, $\mathcal{N}_1+\mathcal{N}_2$ has a fixed point in $P_T(L,M)$ and by Lemma 7 this fixed point is a solution of (1.1). These complete the proof. □

The same steps of the preceding proof prove the following theorem.

**Theorem 5.** *Suppose that $a_k \in P_T(L_{a_k}, M_{a_k})$, $k=\overline{0,n}$ and conditions (2.1), (3.7), (3.10) and (3.15) hold. Then (1.2) has a unique solution $x \in P_T(L,M)$.*

### 3.3 Stability of the periodic solution.

**Theorem 6.** *The solution obtained in Theorem 4 depends continuously on the functions $a_k$, $k = \overline{0,n}$ and $f$.*

PROOF: Under the assumptions of Theorem 4, there are two unique corresponding functions $\varphi, \psi \in P_T(L, M)$ such that

$$\begin{aligned}
\varphi(t) = {} & \sum_{k=0}^{n} \int_0^t H_{1,1}(t,s) e_k(s) \varphi^{[k]}(s)\,\mathrm{d}s + \sum_{k=0}^{n} \int_t^T H_{1,2}(t,s) a_k(s) \varphi^{[k]}(s)\,\mathrm{d}s \\
& + T\alpha^2 \int_0^T f_1\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t+T-s)}\,\mathrm{d}s \\
& - \alpha \int_0^t (s-t) f_1\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s \\
& - \alpha\, \mathrm{e}^{-\varrho T} \int_t^T (s-t) f_1\big(s, \varphi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s \\
& - \varrho \int_0^t f_1(s, \varphi\big(s-\tau(s))\big) H_{1,1}(t,s)\,\mathrm{d}s \\
& - \varrho \int_t^T f_1\big(s, \varphi(s-\tau(s))\big) H_{1,2}(t,s)\,\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
\psi(t) = {} & \sum_{k=0}^{n} \int_0^t H_{1,1}(t,s) b_k(s) \psi^{[k]}(s)\,\mathrm{d}s + \sum_{k=0}^{n} \int_t^T H_{1,2}(t,s) b_k(s) \psi^{[k]}(s)\,\mathrm{d}s \\
& + T\alpha^2 \int_0^T f_2\big(s, \psi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t+T-s)}\,\mathrm{d}s \\
& - \alpha \int_0^t (s-t) f_2\big(s, \psi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s \\
& - \alpha\, \mathrm{e}^{-\varrho T} \int_t^T (s-t) f_2\big(s, \psi(s-\tau(s))\big)\, \mathrm{e}^{-\varrho(t-s)}\,\mathrm{d}s \\
& - \varrho \int_0^t f_2\big(s, \psi(s-\tau(s))\big) H_{1,1}(t,s)\,\mathrm{d}s \\
& - \varrho \int_t^T f_2\big(s, \psi(s-\tau(s))\big) H_{1,2}(t,s)\,\mathrm{d}s,
\end{aligned}$$

where $a_k, b_k \in P_T(L_{a_k}, M_{a_k})$. We have

$$\begin{aligned}
|\varphi(t)-\psi(t)| \leq{}& \sum_{k=0}^{n}\int_0^t |H_{1,1}(t,s)||a_k(s)-b_k(s)||\varphi^{[k]}(s)|\,\mathrm{d}s \\
&+\sum_{k=0}^{n}\int_t^T |H_{1,2}(t,s)||a_k(s)-b_k(s)||\varphi^{[k]}(s)|\,\mathrm{d}s \\
&+\sum_{k=0}^{n}\int_0^t |H_{1,1}(t,s)||\varphi^{[k]}(s)-\psi^{[k]}(s)||b_k(s)|\,\mathrm{d}s \\
&+\sum_{k=0}^{n}\int_t^T |H_{1,2}(t,s)||\varphi^{[k]}(s)-\psi^{[k]}(s)||b_k(s)|\,\mathrm{d}s \\
&+T\alpha^2\int_0^T \Big(\big|f_1\big(s,\varphi(s-\tau(s))\big)-f_1\big(s,\psi(s-\tau(s))\big)\big| \\
&\qquad+\big|(f_1-f_2)\big(s,\psi(s-\tau(s))\big)\big|\Big)\,\mathrm{d}s \\
&+\alpha\int_0^t (t-s)\Big(\big|f_1\big(s,\varphi(s-\tau(s))\big)-f_1\big(s,\psi(s-\tau(s))\big)\big| \\
&\qquad+\big|(f_1-f_2)\big(s,\psi(s-\tau(s))\big)\big|\Big)\,\mathrm{d}s \\
&+\alpha\int_t^T (s-t)\Big(\big|f_1\big(s,\varphi(s-\tau(s))\big)-f_1\big(s,\psi(s-\tau(s))\big)\big| \\
&\qquad+\big|(f_1-f_2)\big(s,\psi(s-\tau(s))\big)\big|\Big)\,\mathrm{d}s \\
&+\varrho\int_0^t |H_{1,1}(t,s)|\Big(\big|f_1\big(s,\varphi\big(s-\tau(s)\big)\big)-f_1\big(s,\psi\big(s-\tau(s)\big)\big)\big| \\
&\qquad+\big|(f_1-f_2)\big(s,\psi(s-\tau(s))\big)\big|\Big)\,\mathrm{d}s \\
&+\varrho\int_t^T |H_{1,2}(t,s)|\Big(\big|f_1\big(s,\varphi(s-\tau(s))\big)-f_1\big(s,\psi(s-\tau(s))\big)\big| \\
&\qquad+\big|(f_1-f_2)\big(s,\psi(s-\tau(s))\big)\big|\Big)\,\mathrm{d}s.
\end{aligned}$$

So,

$$\begin{aligned}
\|\varphi-\psi\| \leq{}& BLT\sum_{k=0}^{n}\|a_k-b_k\| + BT\sum_{k=1}^{n}\sum_{j=0}^{k-1} L_{a_k}M^j\|\varphi-\psi\| \\
&+KT(B\varrho+T\alpha+T\alpha^2)\|\varphi-\psi\| + T(B\varrho+T\alpha+T\alpha^2)\|f_1-f_2\| \\
={}& \left(KT(B\varrho+T\alpha+T\alpha^2)+BT\sum_{k=1}^{n}\sum_{j=0}^{k-1}L_{a_k}M^j\right)\|\varphi-\psi\| \\
&+T(B\varrho+T\alpha+T\alpha^2)\|f_1-f_2\|.
\end{aligned}$$

Using (3.15)

$$KT(B\varrho + T\alpha + T\alpha^2) + BT\sum_{k=1}^{n}\sum_{j=0}^{k-1} L_{a_k} M^j < 1,$$

gives

$$\|\varphi - \psi\| \le \frac{T(B\varrho + T\alpha + T\alpha^2)\|f_1 - f_2\| + BLT\sum_{k=0}^{n}\|a_k - b_k\|}{1 - \left(KT(B\varrho + T\alpha + T\alpha^2) + BT\sum_{k=1}^{n}\sum_{j=0}^{k-1} L_{a_k} M^j\right)}.$$

This completes the proof. □

The same steps of the preceding proof prove the following theorem.

**Theorem 7.** *The solution obtained in Theorem 5 depends continuously on the functions $a_k$, $k = \overline{0,n}$ and $f$.*

**3.4 An example.** We consider the following equation

$$\begin{aligned} x'''(t) &+ 12x''(t) + 48x'(t) + 64x(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\cos 4\pi x(t) - \frac{1}{18}x(t - \cos 4\pi x)\cos^2 4\pi x\Big) \\ &\quad - \frac{1}{6}(\sin 3\pi t)x(t) + \frac{1}{8}(\sin 4\pi t)x^2(t), \end{aligned} \tag{3.16}$$

where

$$\varrho = 4, \quad f(t,x) = \cos 4\pi x - \frac{1}{18}x\cos^2 4\pi t, \quad \tau(t) = \cos 4\pi x, \quad T = \frac{1}{2}, \quad K = \frac{1}{18},$$
$$a_1(t) = a_2(t) = \frac{1}{8}\sin 4\pi t, \qquad a_1,\, a_2 \in P_{1/2}\Big(\frac{1}{8}, \frac{1}{8}\Big).$$

By taking

$$L = \frac{2\pi}{5} \qquad \text{and} \qquad M = 6\pi,$$

we find

$$\alpha = 1.156\,5, \qquad B = 0.21953, \qquad \delta = 1.$$

So,

$$KT(B\varrho + T\alpha + T\alpha^2) = 5.903\,2 \times 10^{-2} < 1,$$

$$BTL\sum_{k=0}^{n} L_{a_k} + T(\delta + KL)(T\alpha^2 + T\alpha + B\varrho) = 1.171\,2 \le \frac{2\pi}{5},$$

$$\begin{aligned} (T^2\alpha\varrho(3\alpha + 2) + 2B\varrho(T\varrho + 1) + 4T\alpha)(\delta + KL) + 2(B + T^2\alpha^2 + BT\varrho)\sum_{k=0}^{n} L_{a_k} \\ = 18.468 \le 6\pi, \end{aligned}$$

$$KT(B\varrho + T\alpha + T\alpha^2) + BT\sum_{k=1}^{n}\sum_{j=0}^{k-1} L_{a_k} M^j = 0.345\,1 < 1.$$

All conditions of Theorem 4 are satisfied and consequently (3.16) has a unique solution in $P_T(2\pi/5, 6\pi)$.

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