On commutative rings whose maximal ideals are idempotent

Farid Kourki, Rachid Tribak

Abstract. We prove that for a commutative ring R, every noetherian (artinian) R-module is quasi-injective if and only if every noetherian (artinian) R-module is quasi-projective if and only if the class of noetherian (artinian) R-modules is socle-fine if and only if the class of noetherian (artinian) R-modules is radical-fine if and only if every maximal ideal of R is idempotent.

Keywords: artinian module; modules of finite length; noetherian module; quasi-injective module; quasi-projective module; radical-fine class of modules; socle-fine class of modules

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1. Introduction

Rings will be associative and commutative with identity and modules will be unitary. A module is called *semiartinian* if each nonzero factor module has a simple submodule. A module M is called *S-primary* for a simple module S if each nonzero factor module of M has a simple submodule isomorphic to S. S. E. Dickson in [6] calls a ring R a *T-ring* if each semiartinian R-module decomposes into a direct sum of its primary components. Examples of a T-ring include any noetherian ring and any semilocal ring (see [6, Corollary 2.7]). T. J. Cheatham and J. R. Smith in [5] used T-rings to characterize rings for which certain modules are semisimple. In fact, they showed in [5, Theorem 5] that a ring R is a T-ring such that any maximal ideal of R is idempotent if and only if the class of semiartinian R-modules coincides with the class of semisimple R-modules. So it is of a natural interest to consider the following question: For which rings R, is every artinian R-module (noetherian R-module, R-module of finite length) semisimple? Recently, we have provided an answer to this question. In fact, we proved that for a ring R, every artinian R-module (noetherian R-module, R-module of finite length) is semisimple if and only if every maximal ideal of R is idempotent (Lemma 2.1). It turns out that this result has many interesting applications. A class $\mathcal{C}$ of modules is said to be *socle-fine* (*radical-fine*) if for every pair M, N in $\mathcal{C}$: $\mathrm{Soc}(M) \cong \mathrm{Soc}(N) \Leftrightarrow M \cong N$ ($M/\mathrm{Rad}(M) \cong N/\mathrm{Rad}(N) \Leftrightarrow M \cong N$). These two notions were introduced by A. Idelhadj and A. Kaidi (see [11] and [13]). One of the interesting problems that can be studied is to characterize rings by some of

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their classes of modules which are socle-fine (radical-fine). In this way D. M. Baquero, A. Idelhadj, C. M. González, A. Kaidi and A. Yahya have proved important results that can be found in [11], [12], [13], [14] and [15]. Nevertheless, most of the classes considered have homological properties (as projectivity and injectivity) and the problem of characterizing rings over which a class of modules satisfying a chain condition (as the ascending or the descending chain condition or having a composition series) is socle-fine (radical-fine) seems to be never considered before. In this article, we show that the class of artinian R-modules (noetherian R-modules, R-modules of finite length) is socle-fine if and only if this class is radical-fine if and only if every maximal ideal of R is idempotent (Theorems 3.3 and 3.5). A module M is said to be a C3-*module* if whenever N and L are direct summands of M such that $N \cap L = 0$, then $N \oplus L$ is a direct summand of M. D3-*modules* can be defined dually. In [1] and [23], I. Amin, Y. Ibrahim and M. F. Yousif established new characterizations of several well known classes of rings in terms of C3-modules and D3-modules. However, the study of the question of characterizing rings over which any module satisfying some kind of chain condition (as ACC, DCC, having a composition series) is a C3-module (D3-module) does not appear anywhere. We show that every noetherian R-module (artinian R-module, R-module of finite length) is a C3-module if and only if every noetherian R-module (artinian R-module, R-module of finite length) is a D3-module if and only if every maximal ideal of R is idempotent (Theorems 4.2 and 4.3).

Throughout this article, R is a commutative ring with unity. Let M be an R-module. We denote by $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$ and $E(M)$ the Jacobson radical, the socle and the injective hull of M, respectively. We use the notations $N \subseteq M$ and $N \leq M$ to denote that N is a subset of M and N is a submodule of M, respectively. By $\mathbb{Z}$ we denote the ring of integer numbers and $\mathbb{Z}_n$ denotes $\mathbb{Z}/n\mathbb{Z}$.

2. When are noetherian (artinian) R-modules V-modules?

We begin with a result taken from [16, Theorem 2.15]. It is the main motivation of this work.

Lemma 2.1. *The following conditions are equivalent for a ring R:*

(i) *Any R-module of finite length is semisimple.*
(ii) *Any artinian R-module is semisimple.*
(iii) *Any noetherian R-module is semisimple.*
(iv) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of R.*

Let M be a module. Following [9], the Krull dimension (denoted K-dim) is defined as follows: $K\text{-dim}(M) = -1$ when $M = 0$. Given an ordinal α, and assuming that the concept $K\text{-dim}(M) < \alpha$ is already defined, then $K\text{-dim}(M)$ is defined to be α if $K\text{-dim}(M) \not< \alpha$ and there exists no descending sequence $M = M_0 \geq M_1 \geq \cdots$ of submodules of M with $K\text{-dim}(M_{n-1}/M_n) \not< \alpha$ for all $n \geq 1$. It is easily seen that $K\text{-dim}(M) = 0$ if and only if M is a nonzero artinian module. Note that every noetherian module has Krull dimension (see [9, Proposition 1.3]). Recall that a module M is called *tall* if it contains a submodule N such that both M/N and N are non-noetherian. A ring R is called *tall* if every non-noetherian R-module is tall (for example, max rings are tall by [18, Corollary 1.2]). In 1976, B. Sarath showed that the class of rings R for which every module having Krull dimension is noetherian is exactly that of tall rings (see [19, Theorem 2.7]). Next, we characterize the class of rings R for which every R-module with Krull dimension is semisimple.

Proposition 2.2. *The following conditions are equivalent for a ring R:*

(i) *Any R-module with Krull dimension is semisimple.*
(ii) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of R.*

Proof: (i) $\Rightarrow$ (ii) Let M be a noetherian R-module. By [9, Proposition 1.3], M has Krull dimension. So M is semisimple. Hence (ii) follows from Lemma 2.1.

(ii) $\Rightarrow$ (i) Let $\mathfrak{m}$ be a maximal ideal of R. By hypothesis, we have $\mathfrak{m}^n = \mathfrak{m}$ for every integer $n \geq 1$. Therefore $R/\bigcap_{n\geq 1}\mathfrak{m}^n = R/\mathfrak{m}$ is a tall ring. Applying [18, Corollary 2.7], it follows that R is a tall ring. Now let M be an R-module with Krull dimension. By [19, Theorem 2.7], M is a noetherian module. So M is semisimple by Lemma 2.1. This completes the proof. $\square$

An R-module M is called a *V-module* (or a *cosemisimple module*) if every proper submodule of M is an intersection of maximal submodules. If the R-module R is a V-module, then the ring R is called a *V-ring*. It is well known that the class of V-modules is closed under isomorphic images, submodules, factor modules and direct sums (see, for example, [2, page 216, Exercise 23] or [10, Proposition 3.3]). It follows that an R-module M is a V-module if and only if every cyclic submodule Rx of M is a V-module. Note that for every $0 \neq x \in M$, $Rx \cong R/\operatorname{Ann}(x)$ (as R-modules) and the commutative V-rings are exactly the commutative von Neumann regular rings. Then a nonzero module M is a V-module if and only if for every $0 \neq x \in M$, $R/\operatorname{Ann}(x)$ is a von Neumann regular ring (see also [3, Theorem 2.3]).

Lemma 2.3. *Let M be a V-module. If M is also noetherian (artinian or of finite length), then M is semisimple.*

Proof: Assume that M is a noetherian (an artinian) module. Let $0 \neq x \in M$. Since M is a V-module, $R/\operatorname{Ann}(x)$ is a von Neumann regular ring. By hypothesis, $Rx \cong R/\operatorname{Ann}(x)$ is a noetherian (an artinian) R-module. So $R/\operatorname{Ann}(x)$ is a noetherian (an artinian) ring. This clearly forces that $R/\operatorname{Ann}(x)$ is a semisimple ring. Therefore Rx is a semisimple R-module. Consequently, M is a semisimple module. $\square$

T. J. Cheatham and J. R. Smith showed in [5, Theorem 6] that a ring R has all its maximal ideals idempotent if and only if each semiartinian R-module is a V-module. In the next theorem we show that the result of T. J. Cheatham and J. R. Smith remains valid if we replace the semiartinian condition by some chain conditions.

Theorem 2.4. *The following conditions are equivalent for a ring R:*

(i) *Any R-module of finite length is a V-module.*
(ii) *Any artinian R-module is a V-module.*
(iii) *Any noetherian R-module is a V-module.*
(iv) *Any R-module with Krull dimension is a V-module.*
(v) *Any semiartinian R-module is a V-module.*
(vi) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of* R.

Proof: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (vi) These follow from Lemmas 2.1 and 2.3.

(v) $\Leftrightarrow$ (vi) By [5, Theorem 6].

(iv) $\Rightarrow$ (vi) Let M be an R-module with Krull dimension. By assumption, M is also a V-module. By [22, Theorem 1], M is noetherian. Hence M is semisimple by Lemma 2.3. Using Proposition 2.2, we conclude that $\mathfrak{m}^2 = \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R.

(vi) $\Rightarrow$ (iv) This follows from Proposition 2.2 and the fact that every semisimple module is a V-module. $\square$

3. Socle-fine and radical-fine classes of modules

A class $\mathcal{C}$ of modules is said to be socle-fine if whenever $M, N \in \mathcal{C}$ with $\mathrm{Soc}(M) \cong \mathrm{Soc}(N)$ then $M \cong N$.

Lemma 3.1. *Let $\mathcal{C}$ be a class of modules which is closed under submodules. Then the following statements are equivalent:*

(i) *The class $\mathcal{C}$ is socle-fine.*
(ii) *Any module belonging to $\mathcal{C}$ is semisimple.*

Proof: (i) $\Rightarrow$ (ii) Let $M \in \mathcal{C}$. As the class $\mathcal{C}$ is closed under submodules, $\mathrm{Soc}(M) \in \mathcal{C}$. Since $\mathrm{Soc}(M) \cong \mathrm{Soc}(\mathrm{Soc}(M))$ and $\mathcal{C}$ is a socle-fine class, we have $M \cong \mathrm{Soc}(M)$. Hence M is a semisimple module.

(ii) $\Rightarrow$ (i) It is clear that any class of semisimple modules is socle-fine. $\square$

Lemma 3.2. (i) *The class of modules that are noetherian (artinian or of finite length) is closed under submodules, factor modules and finite direct sums.*

(ii) *The class of modules with Krull dimension is closed under submodules, factor modules and finite direct sums.*

Proof: (i) is well known and (ii) follows from [9, Lemma 1.1]. $\square$

Combining Lemmas 2.1, 3.1 and 3.2 and Proposition 2.2, we obtain the following result.

Theorem 3.3. *Let R be a ring. The following conditions are equivalent:*

(i) *The class of R-modules of finite length is socle-fine.*
(ii) *The class of artinian R-modules is socle-fine.*
(iii) *The class of noetherian R-modules is socle-fine.*
(iv) *The class of R-modules having Krull dimension is socle-fine.*
(v) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of* R.

Recall that a class $\mathcal{C}$ of modules is said to be radical-fine if whenever $M, N \in \mathcal{C}$ with $M/\operatorname{Rad}(M) \cong N/\operatorname{Rad}(N)$ then $M \cong N$.

Lemma 3.4. *Let $\mathcal{C}$ be a class of modules which is closed under factor modules. Then the following statements are equivalent:*

(i) *The class $\mathcal{C}$ is radical-fine.*
(ii) *Any module belonging to $\mathcal{C}$ is a V-module.*

Proof: (i) $\Rightarrow$ (ii) Let $M \in \mathcal{C}$. Since the class $\mathcal{C}$ is closed under factor modules, $M/\operatorname{Rad}(M) \in \mathcal{C}$. Note that $M/\operatorname{Rad}(M) \cong (M/\operatorname{Rad}(M))/\operatorname{Rad}(M/\operatorname{Rad}(M))$. Since the class $\mathcal{C}$ is radical fine, we have $M \cong M/\operatorname{Rad}(M)$. Hence $\operatorname{Rad}(M) = 0$. Using again the fact that $\mathcal{C}$ is closed under factor modules, we see that $\operatorname{Rad}(M/N) = 0$ for every proper submodule N of M. Therefore M is a V-module.

(ii) $\Rightarrow$ (i) This follows from the fact that any V-module has zero Jacobson radical. □

Combining Theorem 2.4 and Lemma 3.4, we get the following result.

Theorem 3.5. *Let R be a ring. The following conditions are equivalent:*

(i) *The class of R-modules of finite length is radical-fine.*
(ii) *The class of artinian R-modules is radical-fine.*
(iii) *The class of noetherian R-modules is radical-fine.*
(iv) *The class of R-modules having Krull dimension is radical-fine.*
(v) *The class of semiartinian R-modules is radical-fine.*
(vi) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of R.*

Recall that a ring R is called a *T-ring* if each semiartinian R-module decomposes into a direct sum of its primary components (see [6]). From Theorems 3.3 and 3.5 arises the following question: For which rings R is the class of semiartinian R-modules socle-fine? The following result gives an answer.

Proposition 3.6. *Let R be a ring. The following conditions are equivalent:*

(i) *The class of semiartinian R-modules is socle-fine.*
(ii) *Any semiartinian R-module is semisimple.*
(iii) *The ring R is a T-ring and* $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of R.*

Proof: (i) $\Leftrightarrow$ (ii) Note that the class of semiartinian modules is closed under submodules (see, for example, [20, Proposition 2.3]). The equivalence follows from Lemma 3.1.

(ii) $\Leftrightarrow$ (iii) By [5, Theorem 5]. □

It is clear that the class of V-modules is radical-fine. Next, we characterize the class of rings R over which the class of V-modules is socle-fine.

Proposition 3.7. *The following conditions are equivalent for a ring R:*

(i) *The class of V-R-modules is socle-fine.*
(ii) *Any V-R-module is semisimple.*
(iii) *The ring R is a T-ring and each V-R-module is semiartinian.*

Proof: (i) $\Leftrightarrow$ (ii) This follows from Lemma 3.1 and the fact that the class of V-modules is closed under submodules.

(ii) $\Leftrightarrow$ (iii) By [5, Theorem 4]. $\square$

4. Quasi-injective modules, Ci-modules and their duals

Consider the following conditions on a module M:

(C2): If a submodule N of M is isomorphic to a direct summand of M, then N is a direct summand of M.

(C3): If N and L are direct summands of M such that $N \cap L = 0$, then $N \oplus L$ is a direct summand of M.

(C4): If $M = N \oplus L$ with $N, L \leq M$ and $f\colon N \to L$ is a monomorphism, then $\operatorname{Im} f$ is a direct summand of L.

A module M is said to be a Ci-*module* if it satisfies the condition (Ci), $i = 2, 3, 4$.

We have the following hierarchy (see [17, page 18] and [7]):

$$\text{injective} \Rightarrow \text{quasi-injective} \Rightarrow (\mathrm{C2}) \Rightarrow (\mathrm{C3}) \Rightarrow (\mathrm{C4}).$$

Dually, consider the following conditions on a module M:

(D2): If N is a submodule of M such that M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

(D3): If N and L are direct summands of M such that $N + L = M$, then $N \cap L$ is a direct summand of M.

(D4): If $M = N \oplus L$ with $N, L \leq M$ and $f\colon N \to L$ is an epimorphism, then $\operatorname{Ker} f$ is a direct summand of N.

A module M is called a Di-*module* if it satisfies the condition (Di), $i = 2, 3, 4$.

From [17, Lemma 4.6 and Proposition 4.38] and [8, Theorem 2.2], it follows that the following implications hold:

$$\text{projective} \Rightarrow \text{quasi-projective} \Rightarrow (\mathrm{D2}) \Rightarrow (\mathrm{D3}) \Rightarrow (\mathrm{D4}).$$

Lemma 4.1. *Let $\mathcal{C}$ be a class of R-modules which is closed under submodules, factor modules and finite direct sums. For each $i = 2, 3, 4$, the following conditions are equivalent:*

(i) *Any R-module belonging to $\mathcal{C}$ is a* Ci*-module.*
(ii) *Any R-module belonging to $\mathcal{C}$ is a quasi-injective module.*
(iii) *Any R-module belonging to $\mathcal{C}$ is a* Di*-module.*
(iv) *Any R-module belonging to $\mathcal{C}$ is a quasi-projective module.*
(v) *Any R-module belonging to $\mathcal{C}$ is semisimple.*

Proof: Note that any semisimple module is quasi-injective and quasi-projective. So any semisimple module is a Ci-module and a Di-module for all $i \in \{2, 3, 4\}$. The proof is completed by showing the implications (i) $\Rightarrow$ (v) and (iii) $\Rightarrow$ (v) for $i = 4$.

(i) $\Rightarrow$ (v) Suppose that any module in $\mathcal{C}$ is a C4-module. Let $M \in \mathcal{C}$ and let N be a submodule of M. Since the class $\mathcal{C}$ is closed under submodules and finite direct sums, $N \oplus M \in \mathcal{C}$. So $N \oplus M$ is a C4-module. By considering the inclusion map $i\colon N \to M$, we see that N is a direct summand of M. Consequently, M is a semisimple module.

(iii) $\Rightarrow$ (v) Assume that any module in $\mathcal{C}$ is a D4-module. Let $M \in \mathcal{C}$ and let N be a submodule of M. Since the class $\mathcal{C}$ is closed under factor modules and finite direct sums, $M \oplus M/N \in \mathcal{C}$. Hence $M \oplus M/N$ is a D4-module. By considering the natural epimorphism $p\colon M \to M/N$, we see that N is a direct summand of M. Consequently, M is semisimple. $\square$

Applying Proposition 2.2 and Lemmas 2.1, 3.2 and 4.1, we obtain the following two theorems.

Theorem 4.2. *Let R be a ring. For each $i = 2, 3, 4$, the following conditions are equivalent:*

(i) *Any R-module of finite length is a Ci-module.*
(ii) *Any artinian R-module is a Ci-module.*
(iii) *Any noetherian R-module is a Ci-module.*
(iv) *Any R-module having Krull dimension is a Ci-module.*
(v) *Any R-module of finite length is quasi-injective.*
(vi) *Any artinian R-module is quasi-injective.*
(vii) *Any noetherian R-module is quasi-injective.*
(viii) *Any R-module having Krull dimension is quasi-injective.*
(ix) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of R.*

Theorem 4.3. *Let R be a ring. For each $i = 2, 3, 4$, the following conditions are equivalent:*

(i) *Any R-module of finite length is a Di-module.*
(ii) *Any artinian R-module is a Di-module.*
(iii) *Any noetherian R-module is a Di-module.*
(iv) *Any R-module having Krull dimension is a Di-module.*
(v) *Any R-module of finite length is quasi-projective.*
(vi) *Any artinian R-module is quasi-projective.*
(vii) *Any noetherian R-module is quasi-projective.*
(viii) *Any R-module having Krull dimension is quasi-projective.*
(ix) $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of R.*

Next, we characterize a subclass of the class of rings whose maximal ideals are idempotent in terms of semiartinian modules.

Proposition 4.4. *Let R be a ring. For $i = 2, 3, 4$, the following conditions are equivalent:*

(i) *Any semiartinian R-module is a Ci-module.*
(ii) *Any semiartinian R-module is a Di-module.*
(iii) *Any semiartinian R-module is quasi-injective.*
(iv) *Any semiartinian R-module is quasi-projective.*
(v) *Any semiartinian R-module is semisimple.*
(vi) *The ring R is a T-ring and* $\mathfrak{m}^2 = \mathfrak{m}$ *for any maximal ideal* $\mathfrak{m}$ *of R*

Proof: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) follow by using Lemma 4.1 and the fact that the class of semiartinian modules is closed under submodules, factor modules and finite direct sums (see [20, Proposition 2.3]).

(v) $\Leftrightarrow$ (vi) By [5, Theorem 5]. □

Replacing "semiartinian" with "V-module" in Proposition 4.4, we get the following result.

Proposition 4.5. *Let R be a ring. For each $i = 2, 3, 4$, the following conditions are equivalent:*

(i) *Any V-R-module is a Ci-module.*
(ii) *Any V-R-module is a Di-module.*
(iii) *Any V-R-module is quasi-injective.*
(iv) *Any V-R-module is quasi-projective.*
(v) *Any V-R-module is semisimple.*
(vi) *The ring R is a T-ring and every V-R-module is semiartinian.*

Proof: The equivalence of (i), (ii), (iii), (iv) and (v) follows by using Lemma 4.1 and the fact that the class of V-modules is closed under submodules, factor modules and direct sums.

(v) $\Leftrightarrow$ (vi) By [5, Theorem 4]. □

Now, by replacing "quasi-injective" with "injective" in both Propositions 4.4 and 4.5, we obtain the next proposition.

Proposition 4.6. *The following conditions are equivalent for a ring R:*

(i) *Any semiartinian R-module is injective.*
(ii) *Any V-R-module is injective.*
(iii) *The ring R is a semisimple ring.*

Proof: The equivalence of these conditions comes from [4, page 236, Corollary] and the fact that semisimple modules are V-modules and semiartinian modules. □

The next proposition is an extension of [16, Proposition 2.7].

Proposition 4.7. *The following conditions are equivalent for a ring R:*

(i) *Any R-module of finite length is injective.*
(ii) *Any artinian R-module is injective.*
(iii) *Any noetherian R-module is injective.*
(iv) *Any R-module having Krull dimension is injective.*
(v) *The ring R is a von Neumann regular ring.*

Proof: The equivalence of (ii), (iii) and (v) is shown in [16, Proposition 2.7].

(iv) $\Rightarrow$ (iii) Let M be a noetherian R-module. By [9, Proposition 1.3] M has Krull dimension. So M is injective by (iv).

(iii) $\Rightarrow$ (i) This is obvious.

(i) $\Rightarrow$ (v) Note that R is a commutative ring. The implication comes from the fact that any simple R-module is of finite length.

(v) $\Rightarrow$ (iv) Let M be an R-module with Krull dimension. Since R is von Neumann regular, $\mathfrak{m}^2 = \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of R. By Proposition 2.2 M is semisimple. Moreover, M has finite uniform dimension by [9, Proposition 1.4]. Therefore M is a finite direct sum of simple R-modules each of them is injective. It follows that M is an injective R-module. □

Examples 4.8. (i) Let R be a von Neumann regular ring which is not a T-ring. For a particular example, we can take the ring $S = \prod_{n\in\mathbb{N}} F_n$, where $F_n = \mathbb{Z}_2$, $n \geq 1$. Then we consider the subring R of S generated by $\bigoplus_{n\geq 1} F_n$ and 1_S. Of course, we have

$$R = \{(a_1, \ldots, a_n, b, b, b, \ldots) \colon a_i \in \mathbb{Z}_2,\ b \in \mathbb{Z}_2,\ n \geq 1\}.$$

It is well known that R is a von Neumann regular ring. But R is not a T-ring by [6, page 355, Examples].

(1) Noetherian R-modules, artinian R-modules and modules having Krull dimension are semisimple injective by Lemma 2.1 and Propositions 2.2 and 4.7.

(2) There exists a V-R-module M which is not quasi-injective by Proposition 4.5 and there exists a semiartinian R-module which is not quasi-injective by Proposition 4.4.

(3) By Theorem 3.5 and Proposition 3.6, the class of semiartinian R-modules is radical-fine but it is not socle-fine.

(4) The class of V-R-modules is radical-fine (see Lemma 3.4), but it is not socle-fine by Proposition 3.7.

(ii) Let R be a semilocal ring which is not semisimple such that $\mathfrak{m}^2 = \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R. Then R is a T-ring by [6, Corollary 2.7]. From Propositions 4.4 and 4.6, it follows that every semiartinian R-module is quasi-injective and quasi-projective but there exists a semiartinian R-module which is not injective.

To construct an example of a ring R satisfying the above conditions, let F be a field and let $S = F[X_1, X_2, \ldots]$ be the polynomial ring with countably many commuting indeterminates X_i, $i \geq 1$. Consider the ring $R = S/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal of S generated by the set $\{X_1^2, X_n^2 - X_{n-1} \colon n \geq 2\}$. Let $\mathfrak{m}$ be the ideal of R generated by all $\overline{X_i} = X_i + \mathfrak{a}$, $i \geq 1$. By [21, page 635], $\mathfrak{m}$ is the unique maximal ideal of R and $\mathfrak{m}^2 = \mathfrak{m}$.

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F. Kourki:
Centre Régional des Métiers de l'Education et de la Formation (CRMEF)-Tanger, Annexe de Larache, B. P. 4063, Larache, Morocco

E-mail: kourkifarid@hotmail.com

R. Tribak:
Centre Régional des Métiers de l'Education et de la Formation (CRMEF)-Tanger, Avenue My Abdelaziz, Souani, B. P. 3117, Tangier, Morocco

E-mail: tribak12@yahoo.com

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