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Colimit-dense subcategories

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Dedicated to the memory of Věra Trnková, a great teacher and dear friend

Abstract. Among cocomplete categories, the locally presentable ones can be defined as those with a strong generator consisting of presentable objects. Assuming Vopěnka's Principle, we prove that a cocomplete category is locally presentable if and only if it has a colimit dense subcategory and a generator consisting of presentable objects. We further show that a 3-element set is colimit-dense in $\mathbf{Set}^{\mathrm{op}}$, and spaces of countable dimension are colimit-dense in $\mathbf{Vec}^{\mathrm{op}}$.

Keywords: locally presentable category; colimit-dense subcategory; Vopěnka's Principle

Classification: 18C35, 18A30, 03E55

1. Introduction

Our paper is devoted to the question of existence of *colimit-dense* subcategories of a given category $\mathcal{K}$, i.e., small, full subcategories such that every object of $\mathcal{K}$ is a colimit of a diagram in that subcategory. We show e.g. that a set of 3 elements forms a colimit-dense subcategory of the dual of **Set**. In contrast, finite-dimensional vector spaces are not colimit-dense in the dual of **Vec**—but spaces of countable dimension are.

Recall that a small, full subcategory $\mathcal{G}$ of $\mathcal{K}$ is called *dense* if every object X of $\mathcal{K}$ is the canonical colimit of objects of $\mathcal{G}$. That is, the diagram

$$D_X\colon \mathcal{G}/X \to \mathcal{K}$$

assigning to every object $g\colon G \to X$ its domain has colimit X with the canonical colimit cocone. Whether or not $\mathbf{Set}^{\mathrm{op}}$ has a dense subcategory depends on the following set-theoretical assumption:

(M) There exists a cardinal λ such that every λ-complete ultrafilter is principal.

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Indeed, (M) is equivalent to $\mathbf{Set}^{\mathrm{op}}$ having a dense subcategory, as proved by J. R. Isbell in [7]. We present a short proof in Section 3, and discuss the analogous result for the dual of **Vec**, the category of vector spaces over a given field, in Section 4.

This is closely related to a joint paper of two of the co-authors with Věra Trnková, see [12]. There properties of locally presentable categories depending on the validity of *Vopěnka's Principle* were investigated. This principle states that there is no rigid proper class of graphs, i.e., a class where the only homomorphisms are the identity endomorphisms. This is a famous set-theoretical statement which implies $\neg$(M). And since (M) is consistent with set theory, the negation of Vopěnka's Principle is consistent as well. On the other hand, Vopěnka's Principle is consistent with any set theory in which huge cardinals exist. These and other facts can be found e.g. in Chapter 6 of [2]. In the above joint paper [12] it was proved that if Vopěnka's Principle is assumed, the following hold:

(a) every full subcategory of a locally presentable category is *bounded*, i.e., it has a dense subcategory;
(b) every full subcategory of a locally presentable category closed under limits is reflective and locally presentable;

and

(c) every cocomplete bounded category is locally presentable.

Furthermore, each of the statements (a)–(c) was also proved to imply Vopěnka's Principle.

Unfortunately, one of the statements of [12] turns out to be incorrect, and one of our aims is to repair that statement. According to Theorem 9 of [12] Vopěnka's Principle implies that every category with a colimit-dense subcategory is bounded. This result also appeared as Theorem 6.35 in [2]. However, $\mathbf{Set}^{\mathrm{op}}$ is a counter-example, as mentioned above. We provide a correction by proving a weaker statement. Let us call a generator of a category *presentable* if it consists of presentable objects. The weaker statement proved in Section 2 is the following

Theorem 1.1. *Vopěnka's Principle implies that every cocomplete category having both a colimit-dense subcategory and a presentable generator is bounded.*

Recall from [2] that locally presentable categories are precisely the cocomplete categories with a presentable strong generator. However, a presentable generator is not sufficient: the category **Top** of topological spaces has a presentable generator 1, but it is not locally presentable. And a strong generator (or even a colimit-dense subcategory) is also not sufficient: every complete lattice is colimit-dense in itself, but not every one is algebraic (= locally presentable). The combination as in the above theorem is sufficient under Vopěnka's Principle—and we do not know at present whether that assumption is needed:

Open Problem 1.2. If every category with a presentable generator and a colimit-dense subcategory is bounded, does Vopěnka's Principle follow?

As a byproduct of our study, we present parallel proofs for two classical results: one is that the codensity monad of the embedding of finite sets in **Set** is the ultrafilter monad. This was already proved by J. F. Kennison and D. Gildenhuis in 1971, see [9], and a nice new proof is due to T. Leinster in [10], which we recall in Section 3. The other result is that the codensity monad of the embedding of finite-dimensional vector spaces into **Vec** is the double-dualization monad. This is due to T. Leinster in [10] but the (almost equivalent) fact that the double-dual of a vector space is its profinite completion is well-known, see for example [4] or [3]. We present a new proof, based on Leinster's ideas, in Section 4.

2. Colimit-dense subcategories in general

Recall that an object K of a category $\mathcal{K}$ is *presentable* if its hom-functor preserves λ-filtered colimits for some regular cardinal λ. Recall further that a *generator* is a set of objects whose hom-functors are collectively faithful; it is considered as a full subcategory of $\mathcal{K}$. A generator $\mathcal{G}$ is called

(a) *presentable* if all objects of it are presentable; and
(b) *strong* if every monomorphism $L \rightarrowtail K$ such that all morphisms from $G \in \mathcal{G}$ to K factorize through it is invertible.

Definition 2.1. A small, full subcategory $\mathcal{G}$ of the category $\mathcal{K}$ is called *colimit-dense* if every object K of $\mathcal{K}$ is a colimit of some diagram in $\mathcal{G}$. It is called *consistently colimit-dense* if it contains a generator $\mathcal{G}_0$ of $\mathcal{K}$ for which every object of $\mathcal{K}$ is a colimit of a diagram in $\mathcal{G}$ such that all hom-functors of objects of $\mathcal{G}_0$ preserve this colimit.

It is easy to see that the following implications

$$\text{dense} \quad \Longrightarrow \quad \text{colimit-dense} \quad \Longrightarrow \quad \text{strong generator}$$

hold. None can be reverted: K is clearly colimit-dense in the category **Vec** of vector spaces over K, but it is not dense. (In contrast, $K \times K$ is dense.) All sets of power at most 2 form a strong generator of $\mathbf{Set}^{\mathrm{op}}$, which is demonstrated in Remark 3.10 not to be colimit-dense. And in the category of compact Hausdorff spaces the space 1 forms a strong generator which is not colimit-dense. Finally, not every generator is strong, e.g. the discrete one-element graph is a generator of the category of graphs that is not strong.

Recall from the Introduction that a category is said to be *bounded* if it has a dense generator. And that it is locally presentable if and only if it is cocomplete and has a strong, presentable generator.

Theorem 2.2. *Assume Vopěnka's Principle. Then a category $\mathcal{K}$ with a consistently colimit-dense subcategory is bounded.*

PROOF: The proof of Theorem 9 in [12] is valid with one exception: the calculations in Part (ii) on page 168 are incorrect. (The same is true about item (a) of the proof of Theorem 6.35 of [2].) We correct those calculations as follows:

Let $\mathcal{G}$ be a consistently colimit-dense generator of a category $\mathcal{K}$, and let $\mathcal{G}_0$ be the generator consisting of all members $G \in \mathcal{G}$ such that the hom-functor of G preserves the given colimits $K = \operatorname{colim} B_K$ of diagrams B_K in $\mathcal{G}$ (for all objects K). Take the canonical functor $E\colon \mathcal{K} \to \mathbf{Set}^{\mathcal{G}_0^{\mathrm{op}}}$ assigning to an object K the domain-restriction of $\mathcal{K}(-, K)$ to the dual of $\mathcal{G}_0$. This functor is faithful (because $\mathcal{G}_0$ is a generator) and it preserves colimits of the given diagrams B_K for all objects K. For every object G of $\mathcal{G}_0$ we obtain a presheaf EG on $\mathcal{G}_0$ and define $U_0\colon \mathbf{Set}^{\mathcal{G}_0^{\mathrm{op}}} \to \mathbf{Set}$ as the coproduct of the corresponding representable functors:

$$U_0 = \coprod_{G \in \mathcal{G}_0} \mathbf{Set}^{\mathcal{G}_0^{\mathrm{op}}}(EG, -).$$

Then U_0 is faithful and preserves colimits of the given diagrams B_K postcomposed by E. Therefore, the composite functor

$$U_0 E\colon \mathcal{K} \to \mathbf{Set}$$

is faithful and preserves the colimits of all the diagrams B_K. The above mentioned calculations in (ii) (or in item (a), respectively) are correct when the functor U there is substituted by $U_0 E\colon \mathcal{K} \to \mathbf{Set}$. □

Corollary 2.3. *Assume Vopěnka's Principle. A category is locally presentable if and only if it is cocomplete and has a colimit-dense subcategory and a presentable generator.*

Proof: Let $\mathcal{K}$ be a cocomplete category with a presentable generator $\mathcal{G}_p$ and a colimit-dense one $\mathcal{G}_c$. There is a regular cardinal λ such that $\mathcal{K}(G, -)$ preserves λ-filtered colimits for every $G \in \mathcal{G}_p$. Let $\mathcal{G}$ be the closure of $\mathcal{G}_c$ under λ-small colimits (i.e., colimits of diagrams having less than λ morphisms). For every object K we have the chosen diagram B_K in $\mathcal{G}_c$ and we denote by B'_K its extension to $\mathcal{G}$ obtained by a free completion of the domain of B_K under λ-small colimits. This is a λ-filtered diagram in $\mathcal{G}$, thus hom-functors of objects of $\mathcal{G}_p$ preserve the colimit of B'_K. Therefore, $\mathcal{G} \cup \mathcal{G}_p$ is clearly a consistently colimit-dense subcategory. Thus the result follows from Theorem 2.2. □

However, Vopěnka's Principle does *not* imply that a cocomplete category with a colimit-dense subcategory is locally presentable (that is, Theorem 9 of [12] and Theorems 6.35 and 6.37 of [2] are false). We will see this in the next section. The next example shows that the converse implication holds:

Example 2.4. Assuming the negation of Vopěnka's Principle (which, recall, is consistent with set theory) the following category $\mathcal{K}$ was proved in [1, Example 1] to be cocomplete and non-bounded, although it has a finite colimit-dense subcategory. (Unfortunately, $\mathcal{K}$ does not have a presentable generator.) Let $\mathcal{L}$ be a large rigid class of graphs. Objects of $\mathcal{K}$ are triples (X, Y, α) where X is a set, $Y \subseteq X$ and α is a graph, i.e. a binary relation, on Y. Morphisms

$$f\colon (X, Y, \alpha) \to (X', Y', \alpha')$$

are functions $f\colon X \to X'$ such that f extends a graph homomorphism from (Y,α) to (Y',α') and for every $x \in X \setminus Y$ we have

$$f(x) \in (X' \setminus Y') \cup \bigcup_{L\in\mathcal{L}} \bigcup_{h\colon L\to(Y',\alpha')} h[L].$$

Remark 2.5. The assumptions of Theorem 2.2 can be weakened to the existence of a colimit-dense subcategory $\mathcal{G}$ and a faithful functor $U\colon \mathcal{K} \to \mathbf{Set}$ preserving colimits of the diagrams B_K used in expressing $\mathcal{K}$-objects by $\mathcal{G}$-objects. This follows from our proof: use U in place of U_0E.

3. Colimit-density in $\mathbf{Set}^{\mathrm{op}}$

Recall that an ultrafilter $\mathcal{U}$ on a set X is called λ-*complete* for an infinite cardinal λ, if it is closed under intersections of less than λ members. This can be expressed via λ-*partitions* of X, i.e., partitions $Q = (X_i)_{i\in n}$ of X into $n < \lambda$ nonempty subsets: an ultrafilter is λ-complete if and only if for every λ-partition (X_i) it contains precisely one member which we denote by $\alpha(X_i)$. This gives rise to a function α from the set of all λ-partitions of X to the power-set of X such that $\alpha(Q) = X_j$ for some $j \in n$.

Lemma 3.1 (F. Galvin, A. Horn [6])**.** *For every set X a collection of subsets is a λ-complete ultrafilter if and only if it contains a unique member in every λ-partition of X. That is, the above passage $\mathcal{U} \mapsto \alpha$ is bijective.*

Remark 3.2. The paper [6] also shows that the unique choice corresponding to a λ-complete ultrafilter $\mathcal{U}$ is *coherent*, i.e., if Q_2 is coarser than Q_1 then the chosen member of Q_2 contains that of Q_1.

Let $U_\lambda(X)$ denote the set of all λ-complete ultrafilters on a set X.

For every, possibly finite, λ denote by $\mathbf{Set}_\lambda$ the full subcategory of $\mathbf{Set}$ consisting of sets of cardinality less than λ.

Lemma 3.3. *Let λ be an infinite cardinal. For every set X, $U_\lambda(X)$ is the limit of the canonical diagram formed by all λ-partitions of X.*

Proof: Consider the diagram $D_X\colon X/\mathbf{Set}_\lambda \to \mathbf{Set}$ assigning to every object of $X/\mathbf{Set}_\lambda$ its codomain. Every mapping $f\colon X \to Z$ with $|Z| < \lambda$ induces a λ-partition Q_f of X. Another such mapping $f_2\colon X \to Z_2$ factorizes through $f_1\colon X \to Z_1$ if and only if Q_{f_2} is coarser than Q_{f_1}. Hence, the limit of the diagram D_X consists of coherent choices of elements of λ-partitions Q_f. Thus, our lemma follows from Lemma 3.1 and Remark 3.2. $\square$

Corollary 3.4 (J.R. Isbell [7])**.** *Let λ be an infinite cardinal. Then $\mathbf{Set}_\lambda^{\mathrm{op}}$ is dense in $\mathbf{Set}^{\mathrm{op}}$ if and only if every λ-complete ultrafilter is principal.*

Indeed, the factorizing mapping from X to the limit of the canonical diagram of all λ-partitions sends an element x to the principal ultrafilter generated by x. Thus $\mathbf{Set}_\lambda$ is codense in $\mathbf{Set}$ if and only if every λ-complete ultrafilter is principal.

Corollary 3.5 (J. R. Isbell [7]). *The category* $\mathbf{Set}^{\mathrm{op}}$ *is bounded if and only if (M) holds.*

Indeed, suppose **Set** has a codense subcategory $\mathcal{G}$ and λ is an infinite cardinal larger than $|G|$ for each $G \in \mathcal{G}$. Then $\mathcal{G} \subset \mathbf{Set}_\lambda \subset \mathbf{Set}$. By [7, 1.1], it follows that $\mathbf{Set}_\lambda$ is codense in **Set**, and our result follows from Corollary 3.4.

Remark 3.6. (a) Recall that the *ultrafilter functor* $U\colon \mathbf{Set} \to \mathbf{Set}$ assigns to every set X the set $U(X)$ of all ultrafilters on it and to every mapping $f\colon X \to Y$ the mapping

$$Uf\colon \mathcal{U} \mapsto \{A \subseteq Y\colon f^{-1}(A) \in \mathcal{U}\}.$$

This functor yields a monad $\mathbb{U}$ with the unit $\eta_X\colon X \to U(X)$ given by principal ultrafilters. Indeed, U carries a unique structure of a monad, as proved by R. Börger in [5]. This is based on the fact that U is terminal in the category of all set functors preserving finite coproducts.

The subfunctor U_λ of U of all λ-complete ultrafilters also carries a unique structure of a monad. Börger's proof is easily adapted: U_λ is terminal in the category of all set functors preserving coproducts of size smaller than λ. We thus obtain a submonad $\mathbb{U}_\lambda$ of $\mathbb{U}$.

(b) Recall further the concept of the *codensity monad* of a small, full subcategory $\mathcal{G}$ of a complete category $\mathcal{K}$: this is the monad given by the left Kan extension of the embedding $\mathcal{G} \hookrightarrow \mathcal{K}$ over itself. Explicitly, this is the monad (T, μ, η) where T assigns to an object K the limit of the diagram $D_K\colon K/\mathcal{G} \to \mathcal{K}$ assigning to every object $g\colon K \to G$ its codomain (with a limit cone $\pi_g\colon TK \to G$). To a morphism $k\colon K \to L$ this functor assigns the unique morphism Tk with

$$\pi_g.Tk = \pi_{k.g}$$

for all $g\colon L \to G$ in $L/\mathcal{G}$. The monad unit has components η_K determined by $\pi_g.\eta_K = \mathrm{id}$ for all $g \in K/\mathcal{G}$.

Corollary 3.7. *The monad* $\mathbb{U}_\lambda$ *is the codensity monad of the embedding*

$$\mathbf{Set}_\lambda \hookrightarrow \mathbf{Set}.$$

Indeed, the formula for the codensity monad in (b) above demonstrates that T agrees with U_λ. Thus the corollary follows from (a).

Remark 3.8. The special case $\lambda = \omega$ is the classical result of J. F. Kennison and D. Gildenhuis, see [9], that the ultrafilter monad is the codensity monad of the embedding of finite sets into **Set**. The above proof for this case was presented by T. Leinster in [10].

Surprisingly, $\mathbf{Set}^{\mathrm{op}}$ has a 'very small' colimit-dense subcategory:

Proposition 3.9. *A set of power* 3 *is colimit-dense in* $\mathbf{Set}^{\mathrm{op}}$.

Proof: Every set of power at most 2 can be expressed by using an equalizer of two endomaps of $\{0, 1, 2\}$.

For every set X of power at least 3 we present a diagram D in **Set** whose objects have power 3 and whose limit is X. Choose elements t in X and s outside of X, and for every element $x \neq t$ of X put $K_x = \{t, x, s\}$. Given a subset Y of X and an element $x \in Y$, denote by $f_{Y,x} \colon Y \to K_x$ the function mapping x to itself and the rest to t. For every element $x \in X \setminus \{t\}$ let p_x be an endomap of K_x whose fixed points are x and t but not s.

Objects of D are all the above sets K_x and all three-element subsets $Y = \{t, x, x'\}$ of X. The only connecting morphisms are $f_{Y,x} \colon Y \to K_x$ for all $Y = \{t, x, x'\}$ and all p_x above. We have a cone of D consisting of the functions $f_{X,x} \colon X \to K_x$ and $g_Y \colon X \to Y$, mapping x to x if $x \in Y$, else to t. This is a limit cone.

Indeed, it is sufficient to verify that given a compatible choice of elements of all objects of D:

$$k_x \in K_x \qquad \text{and} \qquad l_Y \in Y,$$

there exists a unique $v \in X$ such that

$$k_v = v \qquad \text{and} \qquad l_Y = v \qquad \text{for all } \; Y = \{t, x, v\},$$

while choosing t in all other cases. Observe that due to the connecting map p_x we have $k_x \neq s$ for every x.

(a) In case that there exists $x \in X \setminus \{t\}$ with $k_x = x$, we put $v = x$. If Y contains v, use the fact that $f_{Y,v}(l_Y) = k_v = v$ to conclude $l_Y = v$. If z is distinct from v, then $k_z = t$: use $f_{Z,z}$ for $Z = \{t, v, z\}$. And we also have $l_Y = t$ if Y does not contain v: choose $z \in Y \setminus \{t\}$ and use $f_{Y,z}$.

(b) In case that $k_x = t$ for all x, put $v = t$. We have $l_Y = t$ for every Y: use $f_{Y,x}(l_Y) = k_x$ for $x \in Y \setminus \{t\}$.

Unicity is clear: if v, v' are distinct in $X \setminus \{t\}$, then l_Y for $Y = \{t, v, v'\}$ demonstrates that they do not both have the above property. □

Remark 3.10. (a) In contrast, sets of power at most 2 (that is, $\mathbf{Set}_3$) do not form a colimit-dense subcategory of $\mathbf{Set}^{\mathrm{op}}$. Indeed, a nonempty limit of any diagram $D \colon \mathcal{D} \to \mathbf{Set}_3$ in **Set** always has cardinality which is a power of 2. To see this, let $\mathcal{D}_1$ be the small category obtained by adjoining to $\mathcal{D}$ the formal inverses of all morphisms u in $\mathcal{D}$ for which Du is an isomorphism. Then the diagram D can be extended to a diagram $D_1 \colon \mathcal{D}_1 \to \mathbf{Set}_3$ in the obvious way, and the limit of D_1 is the same as that of D. Let $\mathcal{D}_0$ be the full subcategory of $\mathcal{D}_1$ on objects d for which there exists no morphism $u \colon d' \to d$ in $\mathcal{D}_1$ such that $D_1(u)$ is a constant mapping. If the limit of D is not empty, then the domain restriction $D_0 \colon \mathcal{D}_0 \to \mathbf{Set}$ of the diagram D_1 has the same limit as D_1 and D. All connecting morphisms of D_0 are isomorphisms between 2-element sets. Thus every connected component of $\mathcal{D}_0$ yields a subdiagram with a limit which is either empty or a two-element set. If $\mathcal{D}_0$ has a connected component with empty limit, then the limit of D is empty. Otherwise, $\mathcal{D}_0$ has k connected components with nonempty limits, thus, the limit of D has cardinality 2^k.

(b) The category $\mathbf{Set}_3$ is colimit-dense in the full subcategory $\mathcal{K}$ of $\mathbf{Set}^{\mathrm{op}}$ on sets that have cardinality 2^k or 0. Assuming $\neg$(M), $\mathcal{K}$ is not bounded. Indeed, it is clear that $\mathbf{Set}$ is the idempotent completion of $\mathcal{K}$: every nonempty set X is a retract of 2^X. And if an idempotent completion $\mathcal{A}$ of a category $\mathcal{K}$ is not bounded, then $\mathcal{K}$ is also not bounded. (Suppose, to the contrary, that $\mathcal{L}$ is a small dense subcategory of $\mathcal{K}$. Then we have a canonical full embedding of $\mathcal{K}$ to the category of presheaves on $\mathcal{L}$. It follows that also $\mathcal{A}$ canonically embeds into that presheaf category, thus $\mathcal{L}$ is dense in $\mathcal{A}$, a contradiction.)

Conclusion 3.11. Assuming $\neg$(M), $\mathbf{Set}^{\mathrm{op}}$ has a colimit-dense subcategory but not a dense one. Assuming (M), Vopěnka's principle does not hold and Example 2.4 yields a cocomplete category having a colimit-dense subcategory but not a dense one.

4. Vector spaces

We now turn to the category $\mathbf{Vec}$ of vector spaces over a given field K. There are numerous analogies to $\mathbf{Set}$, but there are also differences. Let us start with the latter: Whereas, as we have seen, finite sets are colimit-dense in $\mathbf{Set}^{\mathrm{op}}$, finite-dimensional spaces are not colimit-dense in $\mathbf{Vec}^{\mathrm{op}}$. To verify this, recall that the dualization functor $(-)^*\colon \mathbf{Vec} \to (\mathbf{Vec})^{\mathrm{op}}$ given by $X^* = [X, K]$ (the space of all linear forms) is left adjoint to its opposite, and this leads to a monad on $\mathbf{Vec}$,

$$((-)^{**}, \eta, \mu)$$

called the *double-dualization monad.* It assigns to every space X its double-dual $X^{**} = [X^*, K]$ and to a morphism $f\colon X \to Y$ the morphism $f^{**}\colon X^{**} \to Y^{**}$ which takes $x \in X^{**}$ to the linear map

$$f^{**}(x)\colon Y^* \to K, \qquad u \mapsto x(u \cdot f) \qquad \text{for all } \ u\colon Y \to K.$$

The unit has components

$$\eta_X\colon X \to X^{**}, \qquad x \mapsto \mathrm{ev}_x,$$

where ev_x evaluates each $u\colon X \to K$ at x. We thus call the vectors of X^{**} of the form $\eta(x)$ the *evaluation vectors.*

Example 4.1. A vector space A of dimension $\aleph_0$ is not a limit of a diagram of finite-dimensional spaces in $\mathbf{Vec}$. This follows from the fact that A is not isomorphic to the dual of any space Y (since if Y has dimension $n \geq \aleph_0$, then Y^* has dimension $|K|^n$). Given a diagram $D\colon \mathcal{D} \to \mathbf{Vec}$, let $D^{**}\colon \mathcal{D} \to \mathbf{Vec}$ denote the composite of D and $(-)^{**}$. If the objects of D are finite-dimensional, we see that D^{**} is naturally isomorphic to D. Therefore, the limit $\lim D$ in $\mathbf{Vec}$ is the dual space to $\operatorname{colim} D^*$, since $(-)^*$ takes colimits to limits. Thus that limit is not isomorphic to A.

Proposition 4.2. *All vector spaces of countable dimension form a colimit-dense subcategory of* $\mathbf{Vec}^{\mathrm{op}}$.

Proof: For every infinite cardinal n, we construct a diagram D in **Vec** whose objects have dimension 1 or $\aleph_0$ and whose limit is the n-dimensional space V of all functions of finite support from n to K. Our diagram D has as objects

(a) the subspace K_x of V for $x \in n$ of all functions whose support is a subset of $\{x\}$; and
(b) the subspace L_Y of V for every countably infinite subset Y of n of all functions whose support is a subset of Y.

For every Y in (b) and every element $x \in Y$ we have the linear map $f_{Y,x} \colon L_Y \to K_x$ of domain restriction to $\{x\}$. These are precisely all the connecting maps of our diagram. We claim that V is a limit of D with respect to the following cone: $g_x \colon V \to K_x$, domain restrictions to $\{x\}$, and $g_Y \colon V \to L_Y$, domain restrictions to Y. It is easy to see that this is indeed a cone of D.

Let another cone be given by a space T and linear maps $h_x \colon T \to K_x$ and $h_Y \colon T \to L_Y$. Let $h \colon T \to K^n$ have components h_x, $x \in n$, (using the obvious isomorphism of K_x and K). Choose an arbittrary object L_Y. Due to compatibility, we know for every element $x \in Y$ that

$$h_x = f_{Y,x} \cdot h_Y.$$

Therefore, we get a commutative square as follows

$$\begin{tikzcd} T \arrow[r, "h"] \arrow[d, "h_Y"'] & K^n \arrow[d, "p_Y"] \\ L_Y \arrow[r, "e_Y"'] & K^Y \end{tikzcd}$$

where p_Y is the projection and e_Y the canonical embedding. (Indeed, by post-composing this square with the projection of K^Y corresponding to $x \in Y$, we get the equality above.) This implies that for every element t of T the restriction of the function $h(t) \colon n \to K$ to Y has finite support. Since this holds for all countable subsets Y of n, we conclude that $h(t)$ has finite support for every $t \in T$. In other words, h has a codomain restriction $h' \colon T \to V$. This is the desired factorization of the given cone. Indeed the equality

$$h_Y = g_Y \cdot h'$$

follows from the above square. Combined with $h_x = f_{Y,x} \cdot h_Y$ above, this implies

$$h_x = g_x \cdot h'.$$

The unicity of h' is obvious.

$\square$

Theorem 4.3. *The category* $\mathbf{Vec}^{\mathrm{op}}$ *is bounded if and only if (M) holds.*

It seems curious that this result has not been published before: the sufficiency was already proved by J. R. Isbell in 1964: [8] Theorem 8.1. For the necessity, see [11] Theorem 1.6.

By a *finite-dimensional linear partition* of a vector space X we mean a surjective linear map $a\colon X \twoheadrightarrow K^n$, where $n \in \mathbb{N}$. Every vector $t \in X^{**}$ yields a choice of a member of the partition a (or, equivalently, a choice of a vector $\alpha(a)$ of K^n): if $n = 1$ we have $a \in X^*$ and the choice is simply $t(a) \in K$. In general, a has components $a_1, \dots, a_n \in X^*$ and we put

$$\alpha(a) = (t(a_1), \dots, t(a_n)) \in K^n.$$

This choice is *coherent* in the expected sense: for every commutative triangle in $\mathbf{Vec}$

$$\begin{tikzcd} & X \arrow[dl, two heads, "b"'] \arrow[dr, two heads, "a"] & \\ K^m \arrow[rr, "u"'] & & K^n \end{tikzcd}$$

we have

$$\alpha(a) = u(\alpha(b)).$$

Indeed, u is a matrix (u_{ij}) and a has components $a_i = \sum\limits_j u_{ij} b_j$. Since t is a linear map, $\alpha(a)$ has components

$$t(a_i) = \sum u_{ij} t(b_j).$$

This is precisely the ith component of $u(t(b_1), \dots, t(b_m))$.

Lemma 4.4. *For every space X the vectors of X^{**} are precisely the coherent choices of a member of every finite-dimensional linear partition of X. That is, the above passage $t \mapsto \alpha$ is bijective.*

Proof: Let α be a coherent choice. We prove that there exists a unique $t \in X^{**}$ with $\alpha(a) = (t(a_1), \dots, t(a_n))$ for every $a = \langle a_1, \dots, a_n\rangle\colon X \twoheadrightarrow K^n$.

Given $a \in X^*$, then either $a = 0$ or $a\colon X \twoheadrightarrow K$ is surjective. We define $t \in X^{**}$ by

$$t(0) = 0 \qquad \text{and} \qquad t(a) = \alpha(a) \qquad \text{for} \quad a \neq 0.$$

(1) Then t is linear. Indeed, to prove $t(ka) = kt(a)$ for every $k \in K$, we can restrict ourselves to $k \neq 0$. Then ka is surjective whenever a is. And we have

a commutative triangle in **Vec** as follows

$$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "ka"] & \\ K \arrow[rr, "k\cdot(-)"'] & & K \end{tikzcd}$$

Since α is coherent, this yields $t(ka) = kt(a)$.

To prove $t(a_1 + a_2) = t(a_1) + t(a_2)$, we can clearly assume $a_i \neq 0$ and also $a_1 + a_2 \neq 0$ (for the case $a_1 = -a_2$ use $k = -1$ above). Let $\langle a_1, a_2\rangle \colon X \to K^2$ have the image factorization $j \cdot e$ where $e\colon X \to A$ for $A = K$ or K^2 is surjective and j is injective. The following triangles

$$\begin{tikzcd} & & X \arrow[dll, "a_1"'] \arrow[d, "e"] \arrow[drr, "a_2"] & & \\ K & K^2 \arrow[l, "\pi_1"] & A \arrow[l, "j"] \arrow[r, "j"'] & K^2 \arrow[r, "\pi_2"'] & K \end{tikzcd}$$

imply $\alpha(a_i) = \pi_i \cdot j \cdot \alpha(e)$, therefore

$$\alpha(a_1) + \alpha(a_2) = \oplus \cdot j \cdot \alpha(e)$$

for the addition $\oplus\colon K^2 \to K$. And the following triangle

$$\begin{tikzcd} & X \arrow[dl, "e"'] \arrow[dr, "a_1+a_2"] & \\ A \arrow[r, "j"'] & K^2 \arrow[r, "\oplus"'] & K \end{tikzcd}$$

yields $\alpha(a_1) + \alpha(a_2) = \alpha(a_1 + a_2)$. This proves that t is linear.

(2) We have that t satisfies $\alpha(a) = (t(a_1), \dots, t(a_n))$. This is clear for $n = 1$. For general n, given $a_i \neq 0$ use the coherence of α on the following triangles

$$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "a_i"] & \\ K^n \arrow[rr, "\pi_i"'] & & K \end{tikzcd}$$

(3) Finally, t is unique – this is obvious. □

For every infinite cardinal λ denote by

$$\mathbf{Vec}_\lambda$$

the full subcategory of **Vec** on spaces of dimension less that λ. Recall from the Introduction Leinster's result that the full embedding

$$\mathbf{Vec}_\omega \hookrightarrow \mathbf{Vec}$$

of finite-dimensional spaces has the double-dualization monad as the codensity monad. Our lemma allows for a direct proof. Börger's result mentioned above about the unique monad structure on the ultrafilter functor $\mathbb{U}$ is based on the fact, proved in [5], that U is the terminal object of the category of set functors preserving finite coproducts. We now prove an analogous fact about $(-)^{**}$. A functor $F\colon \mathbf{Vec} \to \mathbf{Vec}$ is called *linear* if the induced maps $\mathbf{Vec}(X,Y) \to \mathbf{Vec}(FX,FY)$ are linear for all spaces X, Y.

Lemma 4.5. *For every linear endofunctor F of* **Vec** *with $FK \cong K$ there exists a natural transformation $\alpha\colon F \to (-)^{**}$. It is unique up to a scalar multiple, i.e., every other such transformation has the form $k\alpha$ for some $k \in K$.*

Proof: Without loss of generality assume $FK = K$.

(1) Existence. For every space X define a function

$$\alpha_X \colon FX \to X^{**}$$

as follows: given $x \in FX$, then $\alpha_X(x)\colon X^* \to K$ is defined by

$$\alpha_X(x)(t) = Ft(x) \qquad \text{for all } \; t\colon X \to K.$$

Since F is a linear functor for every vector x of X the map $\alpha_X(x)(-)$ is linear. And since each $Ft(-)$ is a linear map, α_X is linear. Let us verify the naturality squares

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & X^{**} \arrow[d, "f^{**}"] \\ FY \arrow[r, "\alpha_Y"'] & Y^{**} \end{tikzcd}$$

The upper passage applied to $x \in FX$ yields $\alpha_X(x) \cdot f^*$. To every $s\colon Y \to K$ this function assigns

$$f^{**}(\alpha_X(x))(s) = \alpha_X(x)(s \cdot f) = F(s \cdot f)(x).$$

The lower passage assigns to s the value

$$\alpha_Y(Ff(x))(s) = Fs(Ff(x)),$$

which is the same one.

(2) Uniqueness. Let $\beta\colon F \to (-)^{**}$ be a natural transformation. The component

$$\beta_K\colon K \to K^{**} \quad (\cong K)$$

is a linear map, hence, it is given by a scalar $k \in K$ in the sense that

$$\beta_K(l)(u) = u(kl) \qquad \text{for all } \ l \in K,\ u\colon K \to K.$$

We are going to prove that $\beta = k\alpha$, i.e., for all $x \in FX$ and $t\colon X \to K$ we have

$$\beta_X(x)(t) = k \cdot Ft(x).$$

This follows from the naturality square

$$\begin{tikzcd} FX \arrow[r, "\beta_X"] \arrow[d, "Ft"'] & X^{**} \arrow[d, "t^{**}"] \\ K \arrow[r, "\beta_K"'] & K^{**} \end{tikzcd}$$

We apply it to $x \in FX$ and get $t^{**}(\beta_X(x))$ which to $\mathrm{id}_K \in K^*$ assigns the value $\beta_X(x)(t)$. The lower passage applied to x assigns to id_K the value $\beta_K(Ft(x)) = k \cdot Ft(x)$. $\square$

Corollary 4.6. *Let $((-)^{**}, \eta, \mu)$ be the double-dualization monad. Then every monad structure on the endofunctor $(-)^{**}$ has for some scalar $k \in K \setminus \{0\}$ the form $((-)^{**}, k\eta, k^{-1}\mu)$.*

Indeed, the above lemma implies that the unit is $k\eta$ and the multiplication is $l\mu$ for k, $l \in K$. From the unit law $(l\mu) \cdot [(-)^{**}(k\eta)] = \mathrm{id}$, i.e., $lk\mu \cdot [(-)^{**}\eta] = \mathrm{id}$, we deduce $lk = 1$.

Corollary 4.7. *The codensity monad of the embedding*

$$\mathbf{Vec}_\omega \hookrightarrow \mathbf{Vec}$$

is the double-dualization monad.

This is analogous to Corollary 3.7 for $\mathbb{U}$: we first verify that if (T, η^T, μ^T) is the codensity monad, then the endofunctor T can be chosen to be $(-)^{**}$. This follows from Remark 3.10 just as for $\mathbb{U}$ above: recall that limits in **Vec** are created by the forgetful functor to **Set**. Thus T assigns to X a limit of the diagram D_X of all $a\colon X \to K^n$, $n \in \mathbb{N}$, which consists of all compatible choices of elements of K^n for all a's. And Lemma 4.4 allows us to put $TX = X^{**}$ with limit projections

$$\pi_a\colon\ X^{**} \xrightarrow{a^{**}} (K^n)^{**} \cong K^n$$

for all a. Given a morphism $f\colon X \to Y$, then Tf is, following Remark 3.6 (b), defined by $\pi_a \cdot Tf = \pi_{a\cdot f}$, and it is easy to see that f^{**} satisfies these equalities, thus $Tf = f^{**}$.

Next recall that the monad unit $\eta^T\colon \mathrm{Id} \to T$ has, according to the above remark, components $\eta_X\colon X \to TX$ determined by the commutative triangles below

$$\begin{tikzcd} X \arrow[rr, "\eta^T_X"] \arrow[dr, "a"'] & & TX \arrow[dl, "\pi_a"] \\ & K^n & \end{tikzcd} \qquad \text{for all } \ a\colon X \to K^n$$

Due to the above choice of the limit cone π_a, we see that the unit of $(-)^{**}$ satisfies $\pi_a \cdot \eta_X = a$ for all a, thus, $\eta = \eta^T$. This means that for (η^T, μ^T) the scalar in the preceding lemma is $k = 1$.

Definition 4.8. Let λ be an infinite cardinal.

By a *linear λ-partition* of a space X we mean a surjective linear map onto a space of dimension less that λ.

A vector x of X^{**} is called *λ-complete* if for every linear λ-partition $a\colon X \to A$ we have: $a^{**}(x)$ is an evaluation vector (i.e., it lies in the image of η_A).

Lemma 4.9. *Let λ be an infinite cardinal. For every space X the λ-complete vectors of X^{**} are precisely the coherent choices of members of linear λ-partitions of X.*

Proof: First recall that all linear forms of a given space are collectively monic. Indeed, given two distinct vectors p and q, we can assume p nonzero and choose a basis containing p. Moreover, in case q is not a scalar multiple of p we include q in that basis. The linear form assigning to every vector its p-coordinate separates p and q: it sends p to 1 and q to 0. In case $p = kq$ for some scalar k, we know that $k \neq 1$, and the same linear form separates p and q again.

For every coherent choice α as above it is our task to show that there exists a vector x of X^{**} with $\alpha(a) = a^{**}(x)$ for every linear map $a\colon X \to A$ with $\dim A < \lambda$. Since α yields, in particular, a choice for all finite-dimensional linear partitions, we know from the above lemma that an x exists such that $\alpha(a) = a^{**}(x)$ holds for all linear forms a. Given a linear λ-partition, we prove $\alpha(a) = a^{**}(x)$ by verifying that every linear form $u\colon A \to K$ merges both sides. By coherence,

$$u(\alpha(a)) = \alpha(u \cdot a) = (u \cdot a)(x).$$

And this is precisely the value of u at $a^{**}(x)$. $\square$

Notation 4.10. For every infinite cardinal λ denote by $(-)^{**}_\lambda$ the subfunctor of the double-dualization monad assigning to every space all λ-complete vectors of its double dual.

Corollary 4.11. *The codensity monad of the emebedding*

$$\mathbf{Vec}_\lambda \hookrightarrow \mathbf{Vec}$$

is the submonad of the double-dualization monad carried by $(-)^{**}_\lambda$.

The proof is completely analogous to that of Corollary 4.7. The following is the analogue of Corollary 3.4.

Theorem 4.12. *The category $\mathbf{Vec}_\lambda^{\mathrm{op}}$ is dense in $\mathbf{Vec}^{\mathrm{op}}$ if and only if every λ-complete vector of X^{**} is an evaluation vector (for all spaces X).*

Proof: For every space X let $D_X\colon X\big/\mathbf{Vec}_\lambda \to \mathbf{Vec}$ be the diagram assigning to each $a\colon X \to A$ the codomain. The limit of D_X is the subspace of X^{**} formed by all λ-complete vectors. To see this, recall that limits are created by the forgetful functor to **Set**. Thus, $\lim D_X$ can be described as the space of all compatible choices $(\hat{-})$ of elements $\hat{a} \in A$ for all $a\colon X \to A$ with $\dim A < \lambda$. By the preceding lemma, we conclude that if λ has the property in our theorem, then $\mathbf{Vec}_\lambda$ is codense in $\mathbf{Vec}$. Indeed, for every space X the canonical limit of D_X is $\eta_X[X] \cong X$. The converse implication is a particular case of the next proposition. $\square$

Proposition 4.13. *If $\mathcal{B}$ is a small codense subcategory in* $\mathbf{Vec}$ *and λ is an infinite cardinal larger than the dimensions of all spaces of $\mathcal{B}$, then for all spaces X, every λ-complete vector of X^{**} is an evaluation vector.*

Proof: Clearly, $\mathcal{B} \subset \mathbf{Vec}_\lambda \subset \mathbf{Vec}$. By [7, 1.1], it follows that $\mathbf{Vec}_\lambda$ is codense in $\mathbf{Vec}$. Denote by $D_X\colon X/\,\mathbf{Vec}_\lambda \to \mathbf{Vec}$ the canonical diagram. Then, for every space X the limit of D_X, consisting of all λ-complete vectors of X^{**}, yields $X \cong \eta[X]$. Therefore, λ has the property of our theorem and proposition. $\square$

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