Continuous images of Lindelöf p-groups, σ-compact groups, and related results

Alexander V. Arhangel'skii

Dedicated to the Memory of Věra Trnková, a sunny person and a symbol of Prague topology

Abstract. It is shown that there exists a σ-compact topological group which cannot be represented as a continuous image of a Lindelöf p-group, see Example 2.8. This result is based on an inequality for the cardinality of continuous images of Lindelöf p-groups (Theorem 2.1). A closely related result is Corollary 4.4: if a space Y is a continuous image of a Lindelöf p-group, then there exists a covering γ of Y by dyadic compacta such that $|\gamma| \leq 2^\omega$. We also show that if a homogeneous compact space Y is a continuous image of a *cdc*-group G, then Y is a dyadic compactum (Corollary 3.11).

Keywords: Lindelöf p-group; homogeneous space; Lindelöf Σ-space; dyadic compactum; countable tightness; σ-compact; *cdc*-group; p-space

Classification: 54A25, 54B05

1. Introduction and preliminaries

All spaces considered below are assumed to be Tychonoff. We use, with a few exceptions, the terminology and notation from [7], [9], and [4], [5]. If X is a set, then $|X|$ is the cardinality of this set. Let $N = \{1, 2, \dots\}$ be the set of positive integers with the discrete topology. For a space X and a cardinal number τ, we say that the Lindelöf degree $l(X)$ of X does not exceed τ, if every open covering μ of X contains a subcovering μ_0 such that $|\mu_0| \leq \tau$. A space X is *a paracompact p-space* if there exists a perfect mapping of X onto some metrizable space, see [1]. Lindelöf p-spaces are preimages of separable metrizable spaces under perfect mappings. It is well-known that every Lindelöf Σ-space is a continuous image of a Lindelöf p-space—in fact, this can be accepted as a definition of Lindelöf Σ-spaces, see [10]. But for topological groups the situation is different. A Lindelöf Σ-group is a topological group which is a Lindelöf Σ-space. Similarly, a Lindelöf p-group is a topological group which is a Lindelöf p-space. It has been shown in [3] that there exists a Lindelöf Σ-group which cannot be represented as a continuous image of a Lindelöf p-group. However, the group presented in [3] was not σ-compact. Notice that σ-compact spaces are Lindelöf Σ-spaces. In this paper,

DOI 10.14712/1213-7243.2019.027

we investigate further what are the reasons for Lindelöf Σ-groups not to be continuous images of Lindelöf p-groups. We establish some cardinal inequalities for mappings, which bring forward these reasons, see Sections 2, 3, 4, Theorem 2.1, for example, and we discover many σ-compact topological groups of this kind.

An important role in this article belongs to coverings of spaces by dyadic compacta. In the last section, we define a cardinal function in these terms and apply it to formulate some quite general theorems on mappings. Topological groups are somehow involved in all of them, and this assumption cannot be dropped. These theorems describe certain situations in which there is no continuous surjection of one space onto another. Theorems 4.6, 4.9, Corollary 4.7 are among the main results in this direction.

2. Some theorems on continuous images of small paracompact p-groups

Theorem 2.1. *Suppose that f is a continuous mapping of a paracompact p-group G onto a space Y which is covered by a countable family η of closed subspaces with countable tightness. Suppose also that the Lindelöf degree $l(G)$ of G does not exceed 2^ω. Then $|Y| \leq 2^\omega$. If, in addition, G is Lindelöf, then Y is separable.*

This theorem will be proved with the help of several results below, where the first statement is a suitable combination of some well-known facts.

Proposition 2.2. *For every paracompact p-group G such that the Lindelöf degree $l(G)$ of G does not exceed 2^ω, there exists a disjoint covering γ of G such that the following conditions are satisfied:*

(1) $|\gamma| \leq 2^\omega$;

(2) *every $F \in \gamma$ is a dyadic compactum.*

If, in addition, G is Lindelöf, then we can select the covering γ so that conditions (1) and (2) are satisfied, and there exists a countable subfamily μ of γ such that $\bigcup \mu$ is dense in G.

Proof: We can fix a perfect and open mapping f of the space G onto a metrizable space M such that $f^{-1}(p)$ is homeomorphic to a compact subgroup of the group G for each $p \in M$, see [7, Theorems 4.3.20 and 4.3.35]. Put $\gamma = \{f^{-1}(p) \colon p \in M\}$. Clearly, $l(M) \leq l(X) \leq 2^\omega$. Since M is metrizable, it follows that $|M| \leq 2^\omega$ and that $|\gamma| \leq 2^\omega$. Obviously, γ is a covering of G. Thus, condition (1) holds. Condition (2) also holds. Indeed, every $F \in \gamma$ is homeomorphic to a compact group. Therefore, every $F \in \gamma$ is a dyadic compactum, by Ivanovskij–Kuz'minov theorem, see [7, Theorem 4.1.7]. □

We also need the next two statements:

Proposition 2.3. *Suppose that f is a continuous mapping of a space X onto a space Y, where X is covered by a family γ of dyadic compact subspaces of X such that $|\gamma| \leq 2^\omega$, and Y is covered by a countable family η of closed subspaces* *of Y such that the tightness of P is countable, for every $P \in \eta$. Then the cardinality of Y does not exceed 2^ω.*

Proof: We can assume that $\eta = \{P_i : i \in \omega\}$. Fix $F \in \gamma$, and put $\eta_F = \{F_i : i \in \omega\}$, where $F_i = f(F) \cap P_i$. Since η covers Y, we have $f(F) = \bigcup\{F_i : i \in \omega\}$. Clearly, every F_i is a compact space with countable tightness. Since $f(F)$ is also compact, it follows from Ranchin's theorem in [11] that the tightness of $f(F)$ is countable. But $f(F)$ is a dyadic compactum, since F is a dyadic compactum. Therefore, $f(F)$ is metrizable, by a theorem of A. V. Arhangel'skii and V. I. Ponomarev in [6] on metrizability of every dyadic compactum with countable tightness. Hence, $f(F)$ is separable, and $|f(F)| \leq 2^\omega$. Since this is true for every $F \in \gamma$, and $|\gamma| \leq 2^\omega$, we conclude that the cardinality of Y does not exceed 2^ω. □

The proof of the next statement is similar to the proof of the preceding statement, so we omit it.

Proposition 2.4. *Suppose that f is a continuous mapping of a space X onto a space Y, and that Z is a dense subspace of X such that $Z = \bigcup \gamma$, where each $F \in \gamma$ is a dyadic compact subspace of X, $|\gamma| \leq \omega$, and Y is covered by a countable family of closed subspaces of Y with countable tightness. Then the density of Y is countable.*

Proof of Theorem 2.1: Proposition 2.2 implies that the mapping f satisfies all the restrictions imposed on f in Proposition 2.3. Applying Proposition 2.3, we conclude that the cardinality of Y does not exceed 2^ω. Similarly, Proposition 2.2 implies that f satisfies the restrictions imposed on f in Proposition 2.4. Applying Proposition 2.4, we conclude that Y is separable. Theorem 2.1 is proved. □

The proof of Theorem 2.1 shows that the following version of this theorem also holds.

Theorem 2.5. *Suppose that f is a continuous mapping of a Lindelöf p-group G onto a space Y such that the tightness of every compact subspace of Y is countable. Then $|Y| \leq 2^\omega$, and Y is separable.*

It is also worth mentioning the next fact which obviously follows from Theorem 2.1, Proposition 2.2, and from the classical Ivanovskij–Kuz'minov theorem, which says that every compact topological group is a dyadic compactum, see [7, Theorem 4.1.7]:

Theorem 2.6. *Suppose that f is a continuous mapping of a paracompact p-group G such that the Lindelöf degree $l(G)$ of G does not exceed 2^ω, onto a space Y. Then there exists a family γ of dyadic compact subspaces of Y such that γ covers Y, and $|\gamma| \leq 2^\omega$.*

Now we present another result of the same type as Theorem 2.1, but with a slightly different proof.

Let Y be a topological space, and let τ be an infinite cardinal number. We will say that Y is *τ-transparent* with respect to the pseudocharacter if there exists a family $\{Y_\alpha \colon \alpha < \tau\}$ of closed subspaces of Y such that, for each $\alpha < \tau$, each $y \in Y_\alpha$ is a G_δ-point in Y_α, and $Y = \bigcup\{Y_\alpha \colon \alpha < \tau\}$.

Theorem 2.7. *Suppose that f is a continuous mapping of a paracompact p-space X onto a space Y which is covered by a family $\eta = \{Y_\alpha \colon \alpha < 2^\omega\}$ of closed subspaces with countable pseudocharacter. Suppose also that the Lindelöf degree $l(X)$ of X does not exceed 2^ω. Then $|Y| \leq 2^\omega$.*

Proof: Since X is a paracompact p-space, we can fix a perfect mapping g of the space X onto a metrizable space M, see [1]. Put $\gamma = \{g^{-1}(p) \colon p \in M\}$. Clearly, $l(M) \leq l(X) \leq 2^\omega$. Since M is metrizable, it follows that $|M| \leq 2^\omega$ and that $|\gamma| \leq 2^\omega$. Put $\xi = \{f(F) \colon F \in \gamma\}$. Clearly, ξ is a covering of Y by compact subspaces, and $|\xi| \leq 2^\omega$. Put $\mu = \{S \cap P \colon S \in \xi,\ P \in \eta\}$. We have $|\mu| \leq 2^\omega$, since the same is true for ξ and for η. It is also clear that, for each $H \in \mu$, H is compact and each $y \in H$ is a G_δ-point in H. Therefore, every $H \in \mu$ is a first-countable compactum, which implies that $|H| \leq 2^\omega$, see [2]. Since $|\mu| \leq 2^\omega$ and $Y = \bigcup \mu$, we conclude that $|Y| \leq 2^\omega$. □

Finally, let us show that there exists a σ-compact topological group which cannot be obtained as a continuous image of a Lindelöf p-group.

Example 2.8. Let $c = 2^\omega$ and $\tau = 2^c$. Fix a discrete space A such that $|A| = \tau$, and let $b(A)$ be the one point compactification of A. Then $b(A)$ is a compact sequential space. In fact, $b(A)$ is an Eberlein compactum, since $b(A)$ is homeomorphic to a compact subspace of the space $C_p(b(A))$ of continuous real-valued functions on $b(A)$ in the topology of pointwise convergence, see Proposition 3.3.2 in [4]. Now, let G be the free topological group of the space $b(A)$. Then G is σ-compact and sequential [7, Corollary 7.4.9]. In particular, the tightness of G is countable. On the other hand, $|G| > |A| > 2^\omega$. Therefore, Theorem 2.1 implies that G cannot be obtained as a continuous image of a paracompact p-group with the Lindelöf degree less than or equal to 2^ω. Hence, G is not a continuous image of a Lindelöf p-group.

Every σ-compact space Y can be represented as a continuous image of a σ-compact locally compact space X. This space X is a Lindelöf p-space.

The requirement in Theorem 2.1 that G be a topological group cannot be dropped.

Example 2.9. Let $b(A)$ be the one point compactification of a discrete space A with the cardinality 2^c, where $c = 2^\omega$. Then $b(A)$ is a compact space with countable tightness, see Example 2.8. In particular, $b(A)$ is a paracompact p-space, and $b(A)$ can be mapped onto a one point space by a perfect and open mapping. We put $Y = b(A)$, and define f as the identity mapping of $b(A)$ onto itself. Then all conditions in Theorem 2.1 are satisfied, except one: $b(A)$ is not a topological group. Notice, in connection with Theorem 2.1, that $|Y| = 2^c > 2^\omega$.

3. Topological groups with countable disjoint coverings by compact subspaces

An interesting special case of σ-compact groups constitutes topological groups which can be covered by a disjoint countable family of compact subspaces. Such topological groups are called below *cdc-groups*. If a space X is an image of a topological group G under a perfect mapping f, and X can be covered by a disjoint countable family of compact subspaces, then we say that X is a *cdc-space*. Some simple examples of such objects are easily available. The product $G \times N$, where G is a compact group and N is the discrete group of integers, is a *cdc*-group. The group of rational numbers is also a *cdc*-group. In fact, all countable topological groups are *cdc*-groups. It is also clear that the product of any finite family of *cdc*-groups is a *cdc*-group. Any closed subgroup of a *cdc*-group is a *cdc*-group.

Example 3.1. Suppose that G is a topological group with a compact subgroup H such that the quotient set G/H is countable. Then G is a *cdc*-group.

The example is not so special, as the next lemma shows.

Lemma 3.2. *Suppose that η is a countable disjoint covering of a topological group G by compact subspaces. Then every member F of η is a dyadic compactum.*

Proof: Clearly, F is a G_δ-subset of G. Since G is a topological group and F is compact, it follows from a theorem of M. M. Choban in [8] that F is a dyadic compactum. □

Proposition 3.3. *If a space X is an image of a topological group G under a perfect mapping f, and γ is a disjoint countable covering of X by compact subspaces, then each member F of γ is a dyadic compactum.*

Proof: Put $\eta = \{f^{-1}(F) \colon F \in \gamma\}$. Every member of η is compact, since f is perfect. It is also clear that η is a countable disjoint covering of G. Since G is a topological group, it follows from Lemma 3.2 that each member $f^{-1}(F)$ of η is a dyadic compactum. Therefore, each $F \in \gamma$ is also a dyadic compactum. □

The next statement will help us to identify a σ-compact topological group which is not a *cdc*-group.

Theorem 3.4. *Suppose that f is a continuous mapping of a cdc-space X onto a Tychonoff space Y such that the tightness of every compact subspace of Y is countable. Then Y has a countable network.*

Proof: We can fix a countable disjoint family γ of compact subspaces of X such that $X = \bigcup \gamma$. Put $\eta = \{f(F) \colon F \in \gamma\}$. By Proposition 3.3, each $F \in \gamma$ is a dyadic compactum. Therefore, every member of η is a dyadic compactum. Clearly, the tightness of every $P \in \eta$ is countable. Since every dyadic compactum with countable tightness is metrizable, by a theorem in [6], we conclude that each $P \in \eta$ is a separable metrizable space. Since η is countable, it follows that Y has a countable network. □

We have seen in the preceding section (Example 2.8) that there exists a σ-compact group G with countable tightness such that $|G| > 2^\omega$. On the other hand, the next statement obviously follows from Theorem 3.4:

Corollary 3.5. *Every cdc-space X with countable tightness has a countable network, and hence, the cardinality of it does not exceed 2^ω.*

Corollary 3.6. *There exists a σ-compact sequential zero-dimensional topological group which is not a cdc-group.*

PROOF: Indeed, take the free topological group G of an uncountable convergent sequence $b(A)$, see Example 2.8. Then G is not a *cdc*-group, since $b(A)$ does not have a countable network. □

Recall that a linearly ordered topological space is a space the topology of which is generated by a linear ordering. It is well-known that every linearly ordered topological space is hereditarily normal, see [9].

Theorem 3.7. *Suppose that f is a continuous mapping of a cdc-space X onto a linearly ordered topological space Y. Then Y has a countable network.*

PROOF: The argument is similar to the proof of Theorem 3.4. The only new step is the reference to the following fact: every linearly ordered dyadic compactum is metrizable. □

The last statement could have been derived from the next one:

Theorem 3.8. *Suppose that f is a continuous mapping of a cdc-space G onto a hereditarily normal space Y. Then Y has a countable network.*

PROOF: The argument is again similar to the proof of Theorem 3.4. The only new step is the reference to this fact: every hereditarily normal dyadic compactum is metrizable. This is so, since there exists a non-normal zero-dimensional space with the weight ω_1. □

As an application of the above theorem, we give the following result which obviously follows from it:

Corollary 3.9. *Suppose that a cdc-group G acts continuously and transitively on a hereditarily normal space Y. Then Y has a countable network.*

Every compact topological group is a *cdc*-group. Every compactum, which is a continuous image of a compact topological group, is a dyadic compactum. Now the next question naturally comes to mind: is every compact space, which is a continuous image of a *cdc*-group, dyadic? The answer to this question is in the negative. An example was given in [3, Example 5.33]. But, we give below some positive partial results in this direction.

Proposition 3.10. *Suppose that a compact space Y is a continuous image of a cdc-group G. Then Y has a nonempty open subspace W which is homeomorphic to an open subspace of some dyadic compactum (that is, Y is locally dyadic at some point).*

Proof: It follows from Lemma 3.2 that there exists a countable covering γ of Y by dyadic compacta. Clearly, some $F \in \gamma$ contains a nonempty open subset V of Y. □

We also have the following obvious corollary of the last statement:

Corollary 3.11. *Suppose that a homogeneous compact space Y is a continuous image of a cdc-group G. Then Y is a dyadic compactum.*

4. The dyadic covering number

An important technical role in this article belongs to coverings of spaces by dyadic compacta. Below we define a cardinal function in these terms and apply it to formulate some quite general theorems on mappings. A special feature of these theorems is that topological groups are somehow involved in all of them and that this assumption cannot be dropped. Roughly, the theorems describe situations in which there is no continuous surjection of one space onto another space. Theorems 4.6, 4.9, and Corollary 4.7 are among the most general results in this article in this direction.

Let X be a topological space. Then the *dyadic covering number* $\mathrm{dcn}(X)$ of X is the smallest cardinal number τ such that there exists a covering γ of X by dyadic compact subspaces of X with $|\gamma| = \tau$. Clearly, the next statement follows from Proposition 3.3:

Proposition 4.1. *If a space X is a continuous image of a cdc-space under a continuous mapping, then* $\mathrm{dcn}(X) \leq \omega$.

Obviously, the converse to the above statement also holds. We can give to Proposition 4.1 a more general form as follows:

Proposition 4.2. *If a space Y is a continuous image of a space X under a continuous mapping, then* $\mathrm{dcn}(Y) \leq \mathrm{dcn}(X)$.

The next statement follows immediately from Theorem 2.6:

Corollary 4.3. *Suppose that f is a continuous mapping of a paracompact p-group G such that the Lindelöf degree $l(G)$ of G does not exceed 2^ω, onto a space Y. Then* $\mathrm{dcn}(Y) \leq 2^\omega$.

In particular, we have:

Corollary 4.4. *If a space Y is a continuous image of a Lindelöf p-group, then* $\mathrm{dcn}(Y) \leq 2^\omega$.

On the other hand, we have the following fact, see Example 2.8, which is now obvious:

Proposition 4.5. *For every cardinal number τ, there exists a σ-compact topological group G such that* $\mathrm{dcn}(G) > 2^\tau$.

Theorem 4.6. *Suppose that G is a topological group which admits a continuous mapping h onto a space Z with compact preimages $h^{-1}(z)$ of points in Z. Suppose also that each point in Z is a G_δ-point, and that $|Z| \leq 2^\omega$. Then, for every continuous mapping f of G into a space Y, which is covered by a countable family η of closed subspaces with countable tightness, we have: $|f(G)| \leq 2^\omega$.*

PROOF: Put $\gamma = \{h^{-1}(z) \colon z \in Z\}$. Clearly, every member of γ is a compact G_δ-subset of G. Hence, each member of γ is a dyadic compactum, by a theorem of M. M. Choban in [8]. We also see that $|\gamma| \leq |Z| \leq 2^\omega$. Hence, by Proposition 2.3, we have $|f(G)| \leq 2^\omega$. $\square$

Corollary 4.7. *Suppose that G is a topological group with a compact subgroup H such that the quotient space G/H has a countable network. Then, for every continuous mapping f of G into a space Y which is covered by a countable family η of closed subspaces with countable tightness, we have $|f(G)| \leq 2^\omega$.*

PROOF: The natural quotient mapping q of G onto G/H is perfect, since H is compact. Every point in $Z = G/H$ is a G_δ-point, since G/H has a countable network. For the same reason, $|G/H| \leq 2^\omega$. Thus, all conditions in Theorem 4.6 are satisfied. Therefore, by this theorem, $|f(G)| \leq 2^\omega$. $\square$

Corollary 4.8. *Suppose that G is a topological subgroup of the topological group $C_p(X)$, where X is a compact space, and that h is a continuous mapping of G into a space Z with compact preimages of points in Z. Suppose also that each point in Z is a G_δ-point, and that $|Z| \leq 2^\omega$. Then $|G| \leq 2^\omega$.*

PROOF: The tightness of $C_p(X)$ is countable, since X is compact [4, Theorem 2.1.1]. Hence, the tightness of G is also countable. Therefore, the statement above follows from Theorem 4.6. $\square$

Theorem 4.9. *Suppose that X is the product $H \times Z$ of a compact topological group H with a space Z such that $|Z| \leq 2^\omega$. Then, for every continuous mapping f of X into a space Y which is covered by a countable family η of closed subspaces with countable tightness, we have $|f(X)| \leq 2^\omega$.*

PROOF: Clearly, $\mathrm{dcn}(X) \leq 2^\omega$. Hence, by Proposition 2.3, we have $|f(G)| \leq 2^\omega$. $\square$

References

[1] Arhangel'skiĭ A. V., *A class of spaces which contains all metric and all locally compact spaces*, Mat. Sb. (N.S.) **67(109)** (1965), 55–88 (Russian); English translation in Amer. Math. Soc. Transl. **92** (1970), 1–39.

[2] Arhangel'skii A. V., *The power of bicompacta with first axiom of countability*, Dokl. Akad. Nauk SSSR **187** (1969), 967–970 (Russian); English translation in Soviet Math. Dokl. **10** (1969), 951–955.

[3] Arhangel'skiĭ A. V., *Topological homogeneity. Topological groups and their continuous images*, Uspekhi Mat. Nauk **42** (1987), no. 2(254), 69–105 (Russian); English translation in Russian Math. Surveys **42** (1987), 83–131.

[4] Arhangel'skiĭ A. V., *Topological Function Spaces*, Mathematics and Its Applications (Soviet Series), 78, Kluwer Academic Publishers Group, Dordrecht, 1992.
[5] Arhangel'skii A. V., van Mill J., *Topological homogeneity*, Recent Progress in General Topology, III, Atlantis Press, Paris, 2014, 1–68.
[6] Arhangel'skiĭ A. V., Ponomarev V. I., *Dyadic bicompacta*, Dokl. Akad. Nauk SSSR **182** (1968), 993–996 (Russian); English translation in Soviet Math. Dokl. **9** (1968), 1220–1224.
[7] Arhangel'skii A., Tkachenko M., *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, 1, Atlantis Press, Paris, World Scientific Publishing, Hackensack, 2008.
[8] Čoban M. M., *Topological structure of subsets of topological groups and their quotient spaces*, Topological Structures and Algebraic Systems, Mat. Issled. Vyp. **44** (1977), 117–163, 181 (Russian).
[9] Engelking R., *General Topology*, Mathematical Monographs, 60, PWN—Polish Scientific Publishers, Warsaw, 1977.
[10] Nagami K., Σ-*spaces*, Fund. Math. **65** (1969), 169–192.
[11] Rančin D. V., *Tightness, sequentiality, and closed coverings*, Dokl. Akad. Nauk SSSR **232** (1977), no. 5, 1015–1018 (Russian); English translation in Soviet Math. Dokl. **18** (1977), no. 1, 196–200.

A. V. Arhangel'skii:
Moscow State Pedagogical University Malaya Pirogovskaya Ulitsa, 1/1, Moskva 119435, Rusko

E-mail: arhangel.alex@gmail.com

(*Received* November 24, 2018, *revised* December 18, 2018)