Contractible simplicial objects

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Dedicated to the memory of Věra Trnková

Abstract. We raise the question of when a simplicial object in a catetgory is deemed contractible. The literature offers three definitions. One is the existence of an "extra degeneracy", indexed by -1, which does not quite live up to the name. This can be strengthened to a "strong extra degeneracy". Another possibility is that it be homotopic to a constant simplicial object. Despite claims in the literature to the contrary, we show that all three are distinct concepts with strong extra degeneracy implies extra degeneracy implies homotopic to a constant and give explicit examples to show the converses fail.

Keywords: contractible; homotopic to a constant; reduced homotopy; partial simplicial object

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1. Introduction

The notion of simplicial homotopy is well known, see Section 2 below. But it raises several interesting questions that we have not found answered in the literature, including one for which the literature is wrong.

First we raise the question of when a simplicial object in some category is contractible. We begin by showing that, assuming the category is idempotent complete, it does not matter whether we are dealing with a simplicial object that is augmented or not. Looking through the literature, we find essentially three definitions of contractibility. The first two are in terms of what are called "extra degeneracies" although there is some question what that means, and the third is that a simplicial object is contractible if it is homotopic to a constant.

There are at least three places in the literature that claim that being homotopic to a constant is equivalent to (one of) the extra degeneracy definitions. Regrettably, one of the three is in [1, Theorem 3.3]. Unfortunately, what is proved there is only that the extra degeneracy implies homotopic to a constant (see Theorem 4.3, below); the converse is ignored. In fact, the converse is false, as we will show below, based on Subsection 4.5. The second, [4, Proposition 1.2.12] repeats and cites the claim from [1]. The third appears in [9, Lemma 4.5.1], which cites [8, Theorem 6.4], which seems to be the required result but includes one equation (the one labeled $(C_i)^n$, in our notation $d^i h^0 = h^0 d^{i-1}$, $1 \le i \le n+1$), which is

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not generally satisfied by a homotopy when $i = 1$. Thus although Meyer's result as stated is correct, the extra hypothesis means that it does not support the implication claimed by E. Riehl. See the discussion just before Subsection 4.5 for a further explanation.

It turns out that there are two versions of extra degeneracies, which we will call *extra degeneracies* and *strong extra degeneracies*. We will show that

$$\text{strong extra degeneracies} \Longrightarrow \text{extra degeneracies} \Longrightarrow \text{homotopic to a constant}$$

and give examples to show that both implications are strict.

As the names suggest, having strong extra degeneracies immediately implies having extra degeneracies. Theorem 4.3, as noted above, shows that having extra degeneracies implies homotopic to a constant.

The proofs that these implications are strict are done using examples that begin with a truncated simplicial set, Subsections 4.4 and 4.5. and are completed using a construction called the coskeleton. This is usually described using a Kan extension, but we have not found an explicit description of the coskeleton and so we include one; see Section 5.1. One problem with coskeletons is that they do not get along well with simplicial homotopies. We use a notion we call *reduced homotopy* which is fully equivalent to the usual, yet does lift to coskeletons. See Section 3 for details.

We say that a simplicial object is *contractible*, respectively *strongly contractible*, if it has extra degeneracies, respectively strong extra degeneracies. One question that started us looking at these things was wondering whether a retract of a contractible simplicial object is contractible. It turns out that for simplicial objects in an idempotent-complete category, the properties of being homotopic to a constant and of being contractible are closed under the formation of retracts, but the property of being strongly contractible is not. See Section 6 for details.

Section 7 gives an explicit equational proof that if a topological space is topologically contractible to a point, then its singular simplicial set is contractible, but not necessarily strongly contractible. It is this example that leads us to take one of the senses of contractibility as definitive. We have not been able to find this explicit construction in the literature.

Since many of the computations involving simplicial objects are long and complicated, we have relegated several of them to appendices.

2. Simplicial objects and partial simplicial objects

To make this self-contained, we briefly describe simplicial objects in a category. A *simplicial object* in a category $\mathcal{X}$ consists of a countable sequence of objects $\{X_n : n \geq 0\}$; arrows $d^i_n = d^i : X_n \longrightarrow X_{n-1}$ for $n > 0$ and $0 \leq i \leq n$, called *face operators*; and arrows $s^i_n = s^i : X_n \longrightarrow X_{n+1}$ for $n \geq 0$ and $0 \leq i \leq n$, called *degeneracies*. These are subject to the following equations. Note that, as already indicated, we usually omit the lower indices.

∘ $d^i d^j = d^{j-1} d^i$ for $i < j$;
∘ $s^i s^j = s^j s^{i-1}$ for $j < i$;
∘ $d^i s^j = \begin{cases} s^{j-1} d^i & \text{if } \ i < j, \\ \mathrm{id} & \text{if } \ i = j \ \text{ or } \ i = j+1, \\ s^j d^{i-1} & \text{if } \ i > j+1. \end{cases}$

An *augmented simplicial object* $X \longrightarrow X_{-1}$ consists of a simplicial object X and a map $d_0^0 = d^0 \colon X_0 \longrightarrow X_{-1}$ such that $d^0 d^0 = d^0 d^1 \colon X_1 \longrightarrow X_{-1}$.

If X and Y are simplicial objects, a *simplicial map* $f \colon X \longrightarrow Y$ consists of morphisms $f_n \colon X_n \longrightarrow Y_n$ that commute with the faces and degeneracies in the obvious way. If $f, g \colon X \longrightarrow Y$ is a pair of simplicial maps a *homotopy*, written $h \colon f \rightsquigarrow g$, consists of morphisms $h_n^i = h^i \colon X_n \longrightarrow Y_{n+1}$ such that

∘ $d^0 h^0 = f_n$;
∘ $d^{n+1} h^n = g_n$;
∘ $d^i h^j = \begin{cases} h^{j-1} d^i & \text{if } \ i < j, \\ d^i h^{i-1} & \text{if } \ i = j, \\ h^j d^{i-1} & \text{if } \ i > j+1; \end{cases}$
∘ $s^i h^j = \begin{cases} h^j s^{i-1} & \text{if } \ i > j, \\ h^{j+1} s^i & \text{if } \ i \le j. \end{cases}$

Note that the relation $\rightsquigarrow$ is neither symmetric nor transitive. It is reflexive. We leave it as an exercise to show that $h \colon f \rightsquigarrow f$ if we define $h^i = s^i f$.

We will also have occasion to deal with *partial simplicial objects*, also known as *truncated simplicial objects*. An m-partial simplicial object X is a finite sequence $X_0, X_1, \ldots, X_m$, face maps $d^i = d_n^i \colon X_n \longrightarrow X_{n-1}$ for $0 < n \le m$ and $0 \le i \le n$, and degeneracies $s^i = s_n^i \colon X_n \longrightarrow X_{n+1}$ for $0 \le n < m$ and $0 \le i \le n$ satisfying the same identities as a simplicial object insofar as they are defined. We will show that every partial simplicial object is the truncation of a full simplicial object, Section 5.1.

3. Reduced homotopy[1]

By a *reduced homotopy* between $f, g \colon X \longrightarrow Y$ we mean a family $r^i = r_n^i \colon X_n \longrightarrow Y_n$ for all n and $0 \le i \le n+1$ such that

RH-1. $r^0 = f_n$;
RH-2. $r^{n+1} = g_n$;
RH-3. $d^i r^j = \begin{cases} r^{j-1} d^i & \text{for } \ i < j, \\ r^j d^i & \text{for } \ i \ge j; \end{cases}$

[1]As far as we are aware, the definition of reduced homotopy is new. There are hints in the literature, but we have not found a precise definition nor a theorem such as Proposition 3.1. We needed this because the usual definition of homotopy does not work well with coskeleton Subsection 5.1. The difficulty lies in describing the value of $d_{n+1}^{i+1} h_n^i = d_{n+1}^i h_n^i$, in terms of X_n.

RH-4. $s^i r^j = \begin{cases} r^{j+1} s^i & \text{for } i < j, \\ r^j s^i & \text{for } i \geq j. \end{cases}$

Some special cases of this are worth mentioning. When $j = 0$, it follows that $d^i r^0 = r^0 d^i$ and $s^i r^0 = r^0 s^i$, which are just the conditions that f is simplicial. When $j = n+1$, it follows that $d^i r^{n+1} = r^n d^i$, while $s^i r^{n+1} = r^{n+2} s^i$, which just express that g is simplicial.

Proposition 3.1. *There is a bijection between homotopies and reduced homotopies between pairs of arrows* $X \rightrightarrows Y$.

Proof: Let $f, g\colon X \longrightarrow Y$ and $h\colon f \rightsquigarrow g$. Define $r^0 = f$, $r^{n+1} = g$ and $r^i = d^i h^i = d^i h^{i-1}$ for $1 \leq i \leq n$. Then the first two equations of RH are satisfied. For RH-3, we consider cases:

$i < j < n+1$:	$d^i r^j = d^i d^j h^j = d^{j-1} d^i h^j = d^{j-1} h^{j-1} d^i = r^{j-1} d^i$;
$i < j = n+1$:	$d^i r^{n+1} = d^i g = g d^i = r^n d^i$;
$i > j$:	$d^i r^j = d^i d^j h^j = d^j d^{i+1} h^j = d^j h^j d^i = r^j d^i$;
$i = j > 0$:	$d^i r^i = d^i d^i h^i = d^i d^i h^{i-1} = d^i d^{i+1} h^{i-1} = d^i h^{i-1} d^i = r^i d^i$;
$i = j = 0$:	$d^0 r^0 = d^0 f = f d^0 = r^0 d^0$.

To verify RH-4, we calculate $s^i r^j$. When $i < j$, we have

$$s^i r^j = s^i d^j h^j = d^{j+1} s^i h^j = d^{j+1} h^{j+1} s^i = r^{j+1} s^i.$$

When $i \geq j$, we have

$$s^i r^j = s^i d^j h^j = d^j s^{i+1} h^j = d^j h^j s^i = r^j s^i.$$

In the other direction, given a reduced homotopy r, we let $h^i = r^{i+1} s^i$. To see that h is a homotopy, first we calculate

$$d^0 h^0 = d^0 r^1 s^0 = r^0 d^0 s^0 = f$$

$$d^{n+1} h^n = d^{n+1} r^{n+1} s^n = r^{n+1} d^{n+1} s^n = g.$$

Next we see that

$$d^i h^i = d^i r^{i+1} s^i = r^i d^i s^i = r^i$$

while

$$d^i h^{i-1} = d^i r^i s^{i-1} = r^i d^i s^{i-1} = r^i.$$

For $i < j$ we have

$$d^i h^j = d^i r^{j+1} s^j = r^j d^i s^j = r^j s^{j-1} d^i = h^{j-1} d^i.$$

For $i > j+1$ we have

$$d^i h^j = d^i r^{j+1} s^j = r^{j+1} d^i s^j = r^{j+1} s^j d^{i-1} = h^j d^{i-1}.$$

Next we calculate $s^i h^j$.

$i \leq j$:	$s^i h^j = s^i r^{j+1} s^j = r^{j+2} s^i s^j = r^{j+2} s^{j+1} s^i = h^{j+1} s^i$;
$i > j$:	$s^i h^j = s^i r^{j+1} s^j = r^{j+1} s^i s^j = r^{j+1} s^j s^{i-1} = h^j s^{i-1}$.

Now we must show that these constructions are inverse to each other. If we begin with h and define $r^i = d^i h^i$, then $r^{i+1}s^i = d^{i+1}h^{i+1}s^i = d^{i+1}s^i h^i = h^i$. The other way around, if we start with r and let $h^i = r^{i+1}s^i$, then $d^i h^i = d^i r^{i+1} s^i = r^i d^i s^i = r^i$. □

4. Contractible simplicial objects

What does it mean for a simplicial object to be contractible? We know what it means for a topological set to be contractible. A space S is *topologically contractible* if there exists a continuous map $H\colon S \times I \longrightarrow S$, where I is the unit interval, such that $H(s,0) = s$ for all $s \in S$ and $H(s,1)$ is constantly equal to some $s_0 \in S$.

This definition makes special use of a one point space. The most important feature, at least from our point of view, of a one point space is that it is discrete, which implies that its singular simplicial complex (see Section 7) is constant in the following sense.

4.1 Constant simplicial objects. A *constant simplicial object* C is one for which every term is the same, say A and every face and degeneracy is the identity. We will say that X is homotopic to C if there are maps $f\colon C \longrightarrow X$ and $g\colon X \longrightarrow C$ such that $gf = \mathrm{id}_C$ and $\mathrm{id}_X \rightsquigarrow fg$. On the one hand, this is (apparently) too weak, as will be discussed later. On the other hand, we ought to be content with $gf \rightsquigarrow 1$ and both instances of $\rightsquigarrow$ should be replaced by the equivalence relation it generates. If we did this for fg, the problem would become intractible. As for gf, the only map homotopic to id_C is itself (easy exercise) so that point resolves itself. We stick to the above definition.

4.2 Extra degeneracies. But there is another way to look at a contraction. Given a space S we will show in Section 7 that the singular simplicial complex (see Section 7) has an "extra degeneracy". This is a sequence of maps $t_n\colon X_n \longrightarrow X_{n+1}$ such that

- ∘ $d^0 t = \mathrm{id}$;
- ∘ $d^i t = t d^{i-1}$ for $i > 0$;
- ∘ $s^i t = t s^{i-1}$ for $i > 0$.

This almost satisfies the same equations as a degeneracy labeled s^{-1}. But such a degeneracy would also satisfy $s^0 s^{-1} = s^{-1} s^{-1}$ or, in the notation we are using, $s^0 t = tt$. We will call t a *contraction*, and say that X is *contractible* or has *extra degeneracies* if t satisfies the three equations. We will call t a *strong contraction*, and say that X is *strongly contractible* or has *strong extra degeneracies* if t satisfies the three equations above and, in addition, satisfies $s^0 t = tt$.

Example 4.4 below shows that contractibility does not imply strong contractibility. What we do have is Theorem 6.1 which, for simplicial sets, is [3, Lemma III.5.1]. Incidentally, that reference (page 200) defines an extra degeneracy the way we do. But first we need an interlude to discuss augmentations.

4.3 Augmentation. We show that if the ambient category has split idempotents, then it does not matter whether we are dealing with an augmented or non-augmented simplicial object. For the augmentation can be added in essentially one way. If X is a simplicial set and the category has coequalizers, it is natural to augment it by letting $d_0^0 \colon X_0 \longrightarrow X_{-1}$ be the coequalizer of $d_1^0, d_1^1 \colon X_1 \longrightarrow X_0$. It is slightly surprising that if X is a contractible simplicial object in a category $\mathcal{X}$ that is idempotent complete, you do not need coequalizers.

Assuming a contraction t, we begin with $d^1td^1t = d^1d^2tt = d^1d^1tt = d^1td^0t = d^1t$. We then split the idempotent d^1t as $X_0 \xrightarrow{d^0} X_{-1} \xrightarrow{t} X_0$ so that $d^1t = td^0$ as required. We also have

$$d^0d^0 = d^0td^0d^0 = d^0d^1td^0 = d^0d^1d^1t = d^0d^1d^2t = d^0d^1td^1 = d^0td^0d^1 = d^0d^1$$

so that we have an augmented simplicial object. It is immediate that

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & X_0 \arrow[r, "d^0"] & X_{-1} \end{tikzcd}$$

is a coequalizer. These considerations are closely related to Beck's precise tripleableness theorem which is not quite stated in [2, Theorem 1] but is clearly explained in [7, Section 1].

4.4 A contractible simplicial set that is not strongly contractible. We begin with a contractible truncated augmented simplicial set X, defined only in dimensions $-1, 0, 1, 2$. The elements of the X_n are as shown in this table:

	X_{-1}	X_0	X_1		X_2			
	α	β	γ	δ	ε	ζ	η	θ
d^0		α	β	β	γ	δ	γ	δ
d^1			β	β	γ	γ	γ	δ
d^2					γ	γ	δ	δ
s^0		δ	η	θ				
s^1			ζ	θ				
t	β	γ	ε	ζ				

Proposition 4.1. *These equations define a short simplicial object with t as a contraction.*

Proof: There are many computations; see Appendix A for details. □

Proposition 4.2. *Neither t nor any other map is a strong contraction.*

Proof: First we observe that if $\tau \colon X_0 \longrightarrow X_1$ satisfies $\tau\tau = s^0\tau$ as well as all the other identities it has to satisfy to be a strong contraction, then so does

$\tau\colon X_{-1} \longrightarrow X_0$. In fact

$$\tau\tau = \tau\tau d^0\tau = d^2\tau\tau\tau = d^2 s^0\tau\tau = s^0\tau d^0\tau = s^0\tau.$$

Now suppose we had a candidate τ for a strong contraction. Since $\tau(\alpha) = \beta$, we must have $\tau(\beta) = \tau\tau(\alpha) = s^0\tau(\alpha) = s^0(\beta) = \delta$. In a similar way, $\tau(\delta) = \theta$. As for $\tau(\gamma)$, the first constraint is that $d^0\tau(\gamma) = \gamma$, which forces $\tau(\gamma)$ to be either ε or η. But we must also satisfy $d^1\tau(\gamma) = \tau d^0(\gamma) = \tau(\beta) = \delta$, which forces $\tau(\gamma) = \theta$, a contradiction. □

This is now completed to an example using the coskeleton, Section 5.1.

Theorem 4.3. *A contractible simplicial object in an idempotent complete category is homotopic to a constant simplicial object.*

PROOF: Suppose $X \xrightarrow{d^0} A$ has a contraction $t = \{t_n\colon X_n \longrightarrow X_{n+1} : n \geq 0\}$. Let $A \xrightarrow{t_{-1}} X_0 \xrightarrow{d^0} A$ split the idempotent $d^1 t_0$. We define $f_n = (s^0)^n t\colon A \longrightarrow X_n$ and $g_n = (d^0)^{n+1}\colon X_n \longrightarrow A$. We begin by showing that these are simplicial maps between X and the constant simplicial object A. We must show that the diagram

$$\begin{tikzcd}
A \arrow[d, "(s^0)^{n+1}t"'] & A \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d, "(s^0)^n t"'] & A \arrow[d, "(s^0)^{n-1}t"'] \\
X_{n+1} \arrow[d, "(d^0)^{n+2}"'] & X_n \arrow[l, "s^i"'] \arrow[r, "d^i"] \arrow[d, "(d^0)^{n+1}"'] & X_{n-1} \arrow[d, "(d^0)^n"'] \\
A & A \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] & A
\end{tikzcd}$$

(4.3.1)

commutes. We have $s^i(s^0)^n t = (s^0)^{n+1}t$, $d^i(s^0)^n t = (s^0)^{n-1}t$, $(d^0)^{n+2}s^i = (d^0)^{n+1}$, and $(d^0)^n d^i = (d^0)^{n+1}$. Clearly $gf = \mathrm{id}$, and we wish to show that $\mathrm{id} \rightsquigarrow fg$. We define $h^i = (s^0)^i t(d^0)^i\colon X_n \longrightarrow X_{n+1}$. A number of equations have to be satisfied.

(1) $d^0h^0 = d^0t = \mathrm{id}$.
(2) $d^{n+1}h^n = d^{n+1}(s^0)^n t(d^0)^n = (s^0)^n d^1 t(d^0)^n = (s^0)^n t(d^0)^{n+1} = f_n g_n$.
(3) $d^ih^i = d^i(s^0)^i t(d^0)^i = (s^0)^{i-1}d^1 s^0 t(d^0)^i = (s^0)^{i-1}t(d^0)^i$, while $d^ih^{i-1} = d^i(s^0)^{i-1}t(d^0)^{i-1} = (s^0)^{i-1}t(d^0)^i$.
(4) If $i > j+1$, $d^ih^j = d^i(s^0)^j t(d^0)^j = (s^0)^j d^{i-j}t(d^0)^j = (s^0)^j t d^{i-j-1}(d^0)^j = (s^0)^j t(d^0)^j d^{i-1} = h^j d^{i-1}$.
(5) If $i < j$, $d^ih^j = d^i(s^0)^j t(d^0)^j = (s^0)^{j-1}t(d^0)^j$, while $h^{j-1}d^i = (s^0)^{j-1}t(d^0)^{j-1}d^i = (s^0)^{j-1}t(d^0)^j$.

(6) If $i > j$, $s^i h^j = s^i (s^0)^j t(d^0)^j = (s^0)^j s^{i-j} t(d^0)^j = (s^0)^j t s^{i-j-1}(d^0)^j = (s^0)^j t(d^0)^j s^{i-1} = h^j s^{i-1}$.

(7) If $i \le j$, $s^i h^j = s^i (s^0)^j t(d^0)^j = (s^0)^{j+1} t(d^0)^j$, while $h^{j+1} s^i = (s^0)^{j+1} t(d^0)^{j+1} s^i = (s^0)^{j+1} t(d^0)^j$.

Thus X is homotopic to a constant simplicial object. □

Note that in the third equation above, we could continue to get

$$d^i h^{i-1} = (s^0)^{i-1} t(d^0)^i = (s^0)^{i-1} t(d^0)^{i-1} d^{i-1} = h^{i-1} d^{i-1}$$

which is not necessarily satisfied by a homotopy that makes a simplicial object homotopic to a constant simplicial object. This is the equation that [8] added without comment that allowed him to conclude that homotopic to a constant implied extra degeneracy.

Suppose C is a constant simplicial object in which every term is A every face and degeneracy is the identity and that $f\colon C \longrightarrow X$ and $g\colon X \longrightarrow C$ are such that $gf = \mathrm{id}_C$ and $h\colon \mathrm{id}_X \rightsquigarrow fg$. Meyer defined $t_n = h_n^0$ in each dimension. But one of the equations that has to be satisfied by a contraction is $d^1 t = t d^0$ or $d^1 h^0 = h^0 d^0$. This equation is not satisfied by homotopies in general, but it is exactly Meyer's additional equation.

4.5 A partial simplicial object homotopic to a constant, but not contractible. We let C denote the 2-partial simplicial set which is constantly equal to $\{*\}$, the one-point set whose only element is $*$. We embed C as a subsimplicial set of $Y = \{Y_0, Y_1, Y_2\}$. We let $Y_0 = \{*\}$ and let Y_1 be the set which is generated by $*$, an element α, and by action of the map $r_1^1\colon Y_1 \longrightarrow Y_1$, which we will denote by u. We assume that $u(*) = *$ and we let $Y_1 = \{*, \alpha, u\alpha, \ldots, u^n\alpha, \ldots\}$.

In this chart, n, k are non-negative but l has to be strictly positive.

	Y_0	Y_1			Y_2								
	$*$	$*$	α	$u^n\alpha$	$*$	β	γ	$v^n\beta$	$v^n\gamma$	$w^k\beta$	$w^l\gamma$	$v^nw^k\beta$	$v^nw^l\gamma$
d^0		$*$	$*$	$*$	$*$	α	$*$	α	$*$	$u^k\alpha$	$*$	$u^k\alpha$	$*$
d^1		$*$	$*$	$*$	$*$	α	α	$u^n\alpha$	$u^n\alpha$	$u^k\alpha$	$u^l\alpha$	$u^{n+k}\alpha$	$u^{n+l}\alpha$
d^2					$*$	$*$	α	$*$	$u^n\alpha$	$*$	$*$	$*$	$*$
s^0	$*$	$*$	β	$w^n\beta$									
s^1	$*$	$*$	γ	$v^n\gamma$									
r^0	$*$	$*$	α	$u^n\alpha$	$*$	β	γ	$v^n\beta$	$v^n\gamma$	$w^k\beta$	$w^l\gamma$	$v^nw^k\beta$	$v^nw^l\gamma$
r^1	$*$	$*$	$u\alpha$	$u^{n+1}\alpha$	$*$	$v\beta$	$v\gamma$	$v^{n+1}\beta$	$v^{n+1}\gamma$	$vw^k\beta$	$vw^l\gamma$	$v^{n+1}w^k\beta$	$v^{n+1}w^l\gamma$
r^2		$*$	$*$	$*$	$*$	$w\beta$	$w\gamma$	$v^nw\beta$	$v^nw\gamma$	$w^{k+1}\beta$	$w^{l+1}\gamma$	$v^nw^{k+1}\beta$	$v^nw^{l+1}\gamma$
r^3					$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

To construct Y_2, we let $\beta = s^0(\alpha)$ and $\gamma = s^1(\alpha)$. We also let $v = r_2^1\colon Y_2 \longrightarrow Y_2$ and $w = r_2^2\colon Y_2 \longrightarrow Y_2$. For convenience we look for an example for which $vw = wv$

and such that Y_2 is the set generated by $*, \beta, \gamma$, and the action of v, w. We let Y_2 have the elements $*, \beta, \gamma$, and all elements of the form $v^n(\beta)$, $w^n(\beta)$, $v^n\gamma$, $w^n(\gamma)$, $v^k w^l(\beta)$, $v^k w^l(\gamma)$. The definitions of the faces and degeneracies and of the reduced homotopy are then given by the chart above.

There is some redundancy in the chart as, for example, the values of $s^i(\alpha)$, $d^j(\alpha)$, $r^k(\alpha)$ can be determined by looking at the column headed by α or at the next column, headed by $u^n(\alpha)$ and letting $n = 0$. Note that we set u^0, v^0, w^0 equal to the identity function. For further redundancy see Appendix B. Also note that we cannot set $l = 0$. For example, the column for $w^l(\gamma)$ would contradict the column for γ if we set l equal to zero.

We claim that this defines a truncated simplicial homotopy between the constant simplicial object C and the the truncated simplicial set Y. See Appendix B for details.

We claim that this is not contractible. If there were a contraction t, we focus on $t\alpha$. From $d^0 t\alpha = \alpha$, we infer from the chart that $t\alpha$ can only be $v^n\beta$ for some $n \geq 0$. From $t* = td^0\alpha = d^1 t\alpha = d^1 v^n \beta = u^n\alpha$, we see that $t* = u^n\alpha$. From $t* = td^1\alpha = d^2 t\alpha = d^2 v^n\beta = *$ we derive a contradiction.

Finally, we note that by using the coskeleton, see below, we can construct a full simplicial set which is homotopic to a constant but not contractible.

5. The coskeleton of a partial simplicial object

5.1 Coskeleton. Suppose that $\mathcal{X}$ has finite limits. Here we show how to extend a truncated (augmented) simplicial object, such as the ones described in Subsections 4.4 and 4.5, to a simplicial object. This construction is well known; we have included it to make this note self contained. It can be described as the right Kan extension from the inclusion of the truncation of the standard simplex into the full standard simplex.

We will show that if two maps between $(n-1)$-partial simplicial objects are reduced homotopic, so are the induced maps between the coskeletons. We will also show that if the partial object is (strongly) contractible, the full object will be also. By an obvious induction, it will suffice to begin with a partial (augmented) simplicial object $X = \{X_{-1}, X_0, X_1, \ldots, X_{n-1}\}$, together with the relevant faces and degeneracies.

In the argument below, we pretend that the category is set-based and that the limits can be defined by elements. This can of course always be replaced by actual limits. Then we let $\mathrm{Cosk}_n(X) = \{X_{-1}, X_0, \ldots, X_{n-1}, X_n\}$ where X_n is the set of all $(n+1)$-tuples $\mathbf{x} = (x^0, x^1, \ldots, x^n) \in (X_{n-1})^{n+1}$ for which $d^i x^j = d^{j-1} x^i$ when $i < j$. We define $d^i\mathbf{x} = x^i$, from which it is immediate that $d^i d^j \mathbf{x} = d^{j-1} d^i \mathbf{x}$ when $i < j$.

Moreover, given $x \in X_{n-1}$, the definition of $s^i(x) \in X_n$ is forced because the simplicial identities determine the values of $d^k s^i$. So we define $s^i \colon X_{n-1} \longrightarrow X_n$

by:

$$s^i x = \mathbf{x} = (s^{i-1}d^0x, s^{i-1}d^1x, \ldots, s^{i-1}d^{i-1}x, x, x, s^i d^{i+1}x, \ldots, s^i d^{n-1}x).$$

More precisely, $s^i x = (x^0, \ldots, x^n)$, where

$$x^k = \begin{cases} s^{i-1}d^k x & \text{if } k < i, \\ x & \text{if } k = i,\ i+1, \\ s^i d^{k-1}x & \text{if } k > i+1. \end{cases}$$

Defining $s^i\colon X_{n-1} \longrightarrow X_n$ as above and $d^i\colon X_n \longrightarrow X_{n-1}$ by $d^i(x^0, x^1, \ldots, x^n) = x^i$, we claim that we have extended X to $\mathrm{Cosk}_n(X)$, a partial simplicial object of degree n. Moreover, the extended object will preserve the contractibility (and strong contractibility) when the original $(n-1)$-partial simplicial object X has these properties. See Appendix C for details. The first step is to prove $s^i(x) = (x^0, x^1, \ldots, x^n) \in \mathrm{Cosk}_n(X)$ for all $x \in X_{n-1}$, by showing $d^j x^k = d^{j-1}x^j$ whenever $j < k$.

Proposition 5.1. *Suppose X and Y are $(n-1)$-partial simplicial objects, f, g: $X \longrightarrow Y$ are $(n-1)$-partial simplicial maps and $r\colon f \rightsquigarrow g$ is an $(n-1)$-partial reduced homotopy. Then f, g, and r extend to the nth coskeletons.*

Proof: That f and g extend is obvious. To extend r, let $(x^0, x^1, \ldots, x^n)$ be an element of the $\mathrm{Cosk}_n(X)$. This means that for $0 \le i < j \le n$, $d^i x^j = d^{j-1}x^i$. For $0 \le k \le n+1$, let $r^k(x^0, \ldots, x^n) = (r^{k-1}x^0, \ldots, r^{k-1}x^{k-1}, r^k x^k, \ldots, r^k x^n)$. In particular

$$r^0(x^0, \ldots, x^n) = (r^0x^0, \ldots, r^0x^n) = (fx^0, \ldots, fx^n) = f(x^0, \ldots, x^n)$$

and

$$r^{n+1}(x^0, \ldots, x^n) = (r^nx^0, \ldots, r^nx^n) = (gx^0, \ldots, gx^n) = g(x^0, \ldots, x^n).$$

We must show that $r^k(x^0, \ldots, x^n) \in \mathrm{Cosk}_n(Y)$. We have to consider cases.

$$\begin{array}{ll} i < j < k\colon & d^i r^{k-1} x^j = r^{k-2} d^i x^j = r^{k-2} d^{j-1} x^i = d^{j-1} r^{k-1} x^i; \\ i < k \le j\colon & d^i r^k x^j = r^{k-1} d^i x^j = r^{k-1} d^{j-1} x^i = d^{j-1} r^{k-1} x^i; \\ k \le i < j\colon & d^i r^k x^j = r^k d^i x^j = r^k d^{j-1} x^i = d^{j-1} r^k x^i. \end{array}$$

It is clear from the definition that for $i < j$, $d^i r^j = r^{j-1} d^i$ and for $i \ge j$, $d^i r^j = r^j d^i$. These equations are where the formula came from. We must show that the commutation equations of the r with the degeneracies is satisfied. (Recall that for $x \in X_{n-1}$, $s^i x = (s^{i-1}d^0x, \ldots, s^{i-1}d^{i-1}x, x, x, s^i d^{i+1}x, \ldots, s^i d^{n-1}x)$.) We must show that $j < k$ implies that $s^j r^k = r^{k-1} s^j$. Since the face operators in $\mathrm{Cosk}_n(Y)$ are collectively monic, it suffices to show that $d^i s^j r^k = d^i r^{k-1} s^j$ for $i = 0, \ldots, n$. Again we consider cases.

$i < j < k$:

$$d^i s^j r^k = s^{j-1} d^i r^k = s^{j-1} r^{k-1} d^i = r^{k-2} s^{j-1} d^i = r^{k-2} d^i s^j = d^i r^{k-1} s^j;$$

$j < k$ and $i = j, j+1$: $\qquad d^i s^i r^k = r^k = r^k d^i s^i = d^i r^{k+1} s^i;$

$j < i+1 \le k$:
$$d^i s^j r^k = s^j d^{i-1} r^k = s^j r^{k-1} d^{i-1} = r^k s^j d^{i-1} = r^k d^i s^j = d^i r^{k+1} s^j;$$
$j < k < i+1$:
$$d^i s^j r^k = s^j d^{i-1} r^k = s^j r^k d^{i-1} = r^{k+1} s^j d^{i-1} = r^{k+1} d^i s^j = d^i r^{k+1} s^j.$$
□

In order to apply this to Subsection 4.5 we need the following.

Proposition 5.2. *If C is a constant partial simplicial object, then its coskeleton is also constant.*

Proof: Suppose C is defined up to degree $n-1$. As in the definition of coskeleton, we will pretend we are in sets. Then $C_n = \{(x^0, \dots, x^n) \in C_n^{n+1} \colon d^i x^j = d^{j-1} x^i \text{ for } i < j\}$. But all d^i are identities, so this says that all x^i are equal so that $C_n = C_{n-1}$ and it is easy to see that all faces and degeneracies are the identity. □

6. Retracts

One of our original motivations for this paper was to discover whether a retract of a contractible simplicial object is contractible.

Theorem 6.1. *A retract of a contractible simplicial object is contractible; every contractible simplicial object is a retract of a strongly contractible simplicial object.*

Proof: Suppose $Y \xrightarrow{f} X \xrightarrow{g} Y$ are simplicial maps such that $gf = \mathrm{id}$ and that X has a contraction t^X. We define $t^Y = g t^X f$. The proof that t^Y is a contraction is trivial.

To go the other way, suppose t is a contraction on X. The cone CX is defined by $(CX)_n = X_{n+1}$, $(Cd)^i = d^{i+1}$ and $(Cs)^i = s^{i+1}$. We claim that the s^0 constitute a strong contracting homotopy on CX. In fact, $(Cd)^0 s^0 = d^1 s^0 = \mathrm{id}$. For $i > 0$, $(Cd)^i s^0 = d^{i+1} s^0 = s^0 d^i = s^0 (Cd)^{i-1}$ and $(Cs)^i s^0 = s^{i+1} s^0 = s^0 s^i = s^0 (Cs)^{i-1}$. In addition, $(Cs)^0 s^0 = s^1 s^0 = s^0 s^0$ so that s^0 is a strong contraction on CX. Now we wish to show that the existence of a contraction on X gives X as a retract of CX. In fact, $t_n \colon X_n \longrightarrow (CX)_n = X_{n+1}$ and $d^0 \colon (CX)_n = X_{n+1} \longrightarrow X_n$ exhibit X_n as a retract of $(CX)_n$ so it suffices to show that these are simplicial maps. We have $(Cd)^i t = d^{i+1} t = t d^i$ and similarly $(Cs)^i t = t s^i$. Finally, for $i \ge 0$, $(Cd)^i d^0 = d^{i+1} d^0 = d^0 d^i$ and similarly, $(Cs)^i d^0 = d^0 s^i$. □

Corollary 6.2. *A retract of a strongly contractible simplicial set need not be strongly contractible.*

Proof: This is immediate from Example 4.4. □

The story of simplicial objects that are homotopic to a constant is a bit more complicated. We begin with three lemmas of which the first is standard and left to the reader. The third says that the horizontal composite of homotopies is a homotopy. This is doubtless known although the use of reduced homotopy makes it trivial. It remains the case that homotopies do not generally compose vertically.

Lemma 6.3. *Suppose that $h\colon g \rightsquigarrow k$ in the diagram $X \xrightarrow{f} Y \underset{k}{\overset{g}{\rightrightarrows}} Z \xrightarrow{l} W$ of simplicial objects, then $lhf\colon lgf \rightsquigarrow lkf$.*

Lemma 6.4. *Suppose C is a constant simplicial object. Then any two homotopic maps $X \longrightarrow C$ are equal.*

Proof: Assume that $f, g\colon X \longrightarrow C$ and $h\colon f \rightsquigarrow g$. Then from $f_0 = d^0h^0 = h^0$ and $g_0 = d^1h^0 = h^0$ in degree 0, we see that $f_0 = g_0$. If we suppose that $f_{n-1} = g_{n-1}$, then we have that $f_n = d^0 f_n = f_{n-1}d^0 = g_{n-1}d^0 = d^0 g_n = g_n$. □

Lemma 6.5. *Suppose that in the diagram $X \underset{g}{\overset{f}{\rightrightarrows}} Y \underset{l}{\overset{k}{\rightrightarrows}} Z$, we have $f \rightsquigarrow g$ and $k \rightsquigarrow l$, then $kf \rightsquigarrow lg$.*

Proof: Although it must be possible to prove this using ordinary homotopies, the use of reduced homotopies renders it easy. Assuming that $r\colon f \rightsquigarrow g$ and $q\colon k \rightsquigarrow l$ are reduced homotopies, let $p_n^i = q_n^i r_n^i$ for $0 \le i \le n+1$. It is now a trivial computation to see that the p_n^i define a reduced homotopy $kf \rightsquigarrow lg$. □

Theorem 6.6. *If idempotents split in $\mathcal{X}$, a retract of a simplicial object in $\mathcal{X}$ that is homotopic to a constant is also homotopic to a constant.*

Proof: Suppose that we have a diagram

$$
\begin{tikzcd}
C \arrow[r, "f"] \arrow[dr, "kf"'] & X \arrow[r, "g"] \arrow[d, shift right, "k"'] & C \\
& Y \arrow[u, shift right, "l"'] \arrow[ur, "gl"'] &
\end{tikzcd}
$$

in which C is constant, $gf = \mathrm{id}$, $\mathrm{id} \rightsquigarrow fg$, and $kl = \mathrm{id}$. We claim that $glkf\colon C \longrightarrow C$ is idempotent. Observe that $glkfglkf \rightsquigarrow glklkf$ by Lemma 6.3 as $fg \rightsquigarrow \mathrm{id}$. By Lemma 6.4 this implies $glkfglkf = glklkf$ which equals $glkf$ as $kl = \mathrm{id}$. We then split this idempotent getting maps $B \xrightarrow{u} C \xrightarrow{v} B$ such that $vu = \mathrm{id}$ and $uv = glkf$. Then $vglkfu = vuvu = \mathrm{id}$. For the other composite $kfuvgl = kfglkfgl$, we have $\mathrm{id} = klkl \rightsquigarrow kfglkfgl$ from Lemma 6.5. □

7. Singular simplicial complexes

As usual, Δ_n denotes the set of all points $(a_0, \dots, a_n) \in \mathbb{R}^n$ such that all $a_i \geq 0$ and $a_0 + \cdots + a_n = 1$. We map $\delta^i \colon \Delta_{n-1} \longrightarrow \Delta_n$ by

$$\delta^i(a_0, \dots, a_{n-1}) = (a_0, \dots, a_{i-1}, 0, a_i, \dots, a_{n-1})$$

and $\sigma^i \colon \Delta_{n+1} \longrightarrow \Delta_n$ by

$$\sigma^i(a_0, \dots, a_{n+1}) = (a_0, \dots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \dots, a_n).$$

If X is a topological space, we let $\mathrm{SS}(X)$ denote the simplicial set whose nth term is $\mathrm{Hom}(\Delta_n, X)$ with d^i given by $d^i = \mathrm{Hom}(\delta^i, X)$ and $s^i = \mathrm{Hom}(\sigma^i, X)$. Then $\mathrm{SS}(X)$ becomes a simplicial set, as is well-known and readily verified. We also note that if $f \colon X \longrightarrow Y$ is continuous, then $\mathrm{SS}(f) \colon \mathrm{SS}(X) \longrightarrow \mathrm{SS}(Y)$, defined so that $f(u) = fu$, is easily seen to be a simplicial map.

Theorem 7.1. *Suppose X and Y are topological spaces, $f, g \colon X \longrightarrow Y$ maps and $H \colon X \times I \longrightarrow Y$ a map such that $H(x, 0) = fx$ and $H(x, 1) = gx$. Then H induces a simplicial homotopy* $\mathrm{SS}(f) \rightsquigarrow \mathrm{SS}(g)$.

PROOF: Define $r^i \colon \mathrm{SS}_n(X) \longrightarrow \mathrm{SS}_n(Y)$ by letting $u \colon \Delta_n \longrightarrow X$ and defining $r^i u(a_0, \dots, a_n) = H(u(a_0, \dots, a_i, a_{i+1}, \dots, a_n), a_0 + \cdots + a_{i-1})$. We give the details in Appendix D. □

It is standard that a topological space X is *contractible* to the point $* \in X$ if there is a map $H \colon X \times I \longrightarrow X$ such that $H(x, 1) = x$ and $H(x, 0) = *$ for all $x \in X$. It will be convenient in this section to denote $H(x, a)$ by ax for $x \in X$ and $a \in I$, treating it as the set I acting on X. Should it happen that it is the *monoid* I that acts, meaning that $(ab)x = a(bx)$ for all $x \in X$ and $a, b \in I$, we will say that we have a *regular contraction* and that X is *regularly contractible.*

Any convex set in $\mathbb{R}^n$ is regularly contractible. By translating, we can assume that it contains the origin and then we can let $* = 0$ and ax have its standard value. In fact, it would suffice that there be a single element $*$ in the set such that the line segment between each other point and $*$ lie in the set. Such a set is called *star-shaped.*

Theorem 7.2. *Let X be a contractible topological space. Then the singular simplicial set over X is contractible; if X is regularly contractible, then its singular simplicial set is strongly contractible.*

PROOF: Suppose $X \times I \longrightarrow X$ is contraction, denoted $(x, a) \mapsto ax$. We define a contraction $t \colon \mathrm{Hom}(\Delta_n, X) \longrightarrow \mathrm{Hom}(\Delta_{n+1}, X)$ by

$$tu(a_0, a_1, \dots, a_{n+1}) = \begin{cases} (1 - a_0) u\left(\frac{a_1}{1-a_0}, \dots, \frac{a_{n+1}}{1-a_0}\right) & \text{if } a_0 \neq 1, \\ * & \text{otherwise.} \end{cases}$$

The details, including the proof that tu is well-defined and continuous, are found in Appendix D. Note that this equation is not like the one used in the preceding theorem.

7.1 A contractible space that is not regularly contractible. Unfortunately, the space is not Hausdorff, nor even T_1. It is equivalent to a subspace of the Khalimsky topology on the integers, see [6], [5].

We let E be the space with five elements we will denote v, w, x, y, z and whose basic open sets are $\{v\}$, $\{v, w, x\}$, $\{z\}$, and $\{x, y, z\}$. We define a topological contraction H of E to the single point v, as follows, where $H(u, r) = ru$ for $u \in E$ and $r \in [0, 1]$:

(1) $rv = v$ for all $0 \le r \le 1$;

(2) $rw = \begin{cases} v & \text{for } 0 \le r < \frac{1}{5}, \\ w & \text{for } \frac{1}{5} \le r \le 1; \end{cases}$

(3) $rx = \begin{cases} v & \text{for } 0 \le r < \frac{1}{5}, \\ w & \text{for } \frac{1}{5} \le r \le \frac{2}{5}, \\ x & \text{for } \frac{2}{5} < r \le 1; \end{cases}$

(4) $ry = \begin{cases} v & \text{for } 0 \le r < \frac{1}{5}, \\ w & \text{for } \frac{1}{5} \le r \le \frac{2}{5}, \\ x & \text{for } \frac{2}{5} < r < \frac{3}{5}, \\ y & \text{for } \frac{3}{5} \le r \le 1; \end{cases}$

(5) $rz = \begin{cases} v & \text{for } 0 \le r < \frac{1}{5}, \\ w & \text{for } \frac{1}{5} \le r \le \frac{2}{5}, \\ x & \text{for } \frac{2}{5} < r < \frac{3}{5}, \\ y & \text{for } \frac{3}{5} \le r \le \frac{4}{5}, \\ z & \text{for } \frac{4}{5} < r \le 1. \end{cases}$

Continuity follows from the calculation

(1) $H^{-1}(\{v\}) = (\{v\} \times [0, 1]) \cup (E \times [0, 1/5))$;
(2) $H^{-1}(\{v, w, x\}) = (\{v, w, x\} \times [0, 1]) \cup (E \times [0, 3/5))$;
(3) $H^{-1}\{z\}) = (\{z\} \times (4/5, 1]$;
(4) $H^{-1}\{x, y, z\} \times (2/5, 1]$.

Finally, we will show that neither this action nor any other is regular. Suppose "$r \cdot$" is a regular action with base point $u \in E$. We can suppose without loss of generality that $u \in \{x, y, z\}$ since there is a symmetry on E that exchanges $\{x, y, z\}$ with $\{v, w, x\}$. Since $1 \cdot v = v$ and $\{v\}$ is open, there is some $r < 1$ such that $r \cdot v = v$. But then $r^n \cdot v = v$ for all n. Since $\lim r^n = 0$, it follows that $\lim r^n v = 0 \cdot v = u$. But $\{x, y, z\}$ is a neighbourhood of u that excludes every $r^n \cdot v$, so that is impossible. $\square$

Appendix A. Details for example in Subsection 4.4

These are the detailed computations required in Subsection 4.4:

(1) For $i < j$, $d^i d^j = d^{j-1} d^i$. But all composites of faces end in a one element set, so this is automatic.

(2) For $i < j$, $s^j s^i = s^i s^{j-1}$: $\quad s^1 s^0(\beta) = s^1(\delta) = \theta = s^0(\delta) = s^0 s^0(\beta)$.

(3) $d^0 s^0 = d^1 s^0 = d^1 s^1 = d^2 s^1 = \mathrm{id}$:

$$d^0 s^0(\beta) = d^0(\delta) = \beta; \qquad d^0 s^0(\gamma) = d^0(\eta) = \gamma; \qquad d^0 s^0(\delta) = d^0(\theta) = \delta;$$
$$d^1 s^0(\beta) = d^1(\delta) = \beta; \qquad d^1 s^0(\gamma) = d^1(\eta) = \gamma; \qquad d^1 s^0(\delta) = d^1(\theta) = \delta;$$
$$d^1 s^1(\gamma) = d^1(\zeta) = \gamma; \qquad d^1 s^1(\delta) = d^1(\theta) = \delta;$$
$$d^2 s^1(\gamma) = d^2(\zeta) = \gamma; \qquad d^2 s^1(\delta) = d^2(\theta) = \delta.$$

(4) $d^0 s^1 = s^0 d^0$:

$$d^0 s^1(\gamma) = d^0(\zeta) = \delta = s^0(\beta) = s^0 d^0(\gamma); \quad d^0 s^1(\delta) = d^0(\theta) = \delta = s^0(\beta) = s^0 d^0(\delta).$$

(5) $d^2 s^0 = s^0 d^1$:

$$d^2 s^0(\gamma) = d^2(\eta) = \delta = s^0(\beta) = s^0 d^1(\gamma); \quad d^2 s^0(\delta) = d^2(\theta) = \delta = s^0(\beta) = s^0 d^1(\delta).$$

Proposition. *The map t is a contraction.*

Proof: Again there are a number of computations.

(1) $d^0 t = \mathrm{id}$:

$$d^0 t(\alpha) = d^0(\beta) = \alpha; \qquad d^0 t(\beta) = d^0(\gamma) = \beta;$$
$$d^0 t(\gamma) = d^0(\varepsilon) = \gamma; \qquad d^0 t(\delta) = d^0(\zeta) = \delta.$$

(2) $d^1 t = t d^0$:

$$d^1 t(\beta) = d^1(\gamma) = \beta = t(\alpha) = t d^0(\beta); \qquad d^1 t(\gamma) = d^1(\varepsilon) = \gamma = t(\beta) = t d^0(\gamma);$$
$$d^1 t(\delta) = d^1(\zeta) = \gamma = t(\beta) = t d^0(\delta).$$

(3) $d^2 t = t d^1$:

$$d^2 t(\gamma) = d^2(\varepsilon) = \gamma = t(\beta) = t d^1(\gamma); \qquad d^2 t(\delta) = d^2(\zeta) = \gamma = t(\beta) = t d^1(\delta).$$

□

Appendix B. Details for Subsection 4.5

We will verify that Y, as given in Subsection 4.5, is homotopically equivalent to the constant partial simplicial set C. In what follows, steps 1–5 show that Y is a partial simplicial set, while steps 6–11 verify that the maps r^0, r^1, r^2, r^3 define a reduced homotopy with the required properties.

It is not necessary to prove that two maps into Y_0, from the same domain, are equal as this is immediate, because Y_0 has only one element. Similarly, two maps from Y_0 into the same codomain are, for the maps we are using, always equal because these maps always preserve the element labeled $*$.

We often prove that two maps, say p and q, are equal by showing that $p(x) = q(x)$ for all x in their common domain. We can omit the case of $x = *$ because the maps we are using always preserve $*$.

If the common domain of p and q is Y_1, then to prove $p = q$, we only need to verify that $p(x) = q(x)$ for $x = u^n\alpha$ because the case $x = \alpha$ follows when $n = 0$.

If the common domain of p and q is Y_2, then to prove $p = q$, we only need to verify that $p(x) = q(x)$ for $x = v^n\gamma$, $v^k w^k\beta$ and $v^k w^l\gamma$ because the case $x = *$ is immediate and the other cases follow by setting n or k equal to 0.

We find it convenient to use notation such as "$r^2 \equiv *$ on Y_1" to indicate that r^2 maps every element of Y_1 to $*$.

(1) Proof that $d^i d^j = d^{j-1} d^i$ for $i < j$.
This follows because $d^i d^j$ and $d^{j-1} d^i$ both map to Y_0 (see above).

(2) Proof that $s^i s^j = s^j s^{i-1}$ for $i > j$.
This follows because $s^i s^j$ and $s^j s^{i-1}$ both map from Y_0 (see above).

(3) Proof that $d^i s^j = s^{j-1} d^i$ when $i < j$.
Since the only degeneracies in this example of a partial simplicial set are s^0, s^1 and since $i < j$, we only have to show that $d^0 s^1 = s^0 d^0$.
$d^0 s^1 = s^0 d^0$: $\quad d^0 s^1(u^n\alpha) = d^0(v^n\gamma) = * = s^0(*) = s^0 d^0(u^n\alpha)$.

(4) Proof that $d^i s^j = \mathrm{id}$ for $i = j$ or $i = j + 1$.
$d^0 s^0 = \mathrm{id}$: $\quad d^0 s^0(u^n\alpha) = d^0(w^n\beta) = u^n\alpha$;
$d^1 s^0 = \mathrm{id}$: $\quad d^1 s^0(u^n\alpha) = d^1(w^n\beta) = u^n\alpha$;
$d^1 s^1 = \mathrm{id}$: $\quad d^1 s^1(u^n\alpha) = d^1(v^n\gamma) = u^n\alpha$;
$d^2 s^1 = \mathrm{id}$: $\quad d^2 s^1(u^n\alpha) = d^2(v^n\gamma) = u^n\alpha$.

(5) Proof that $d^i s^j = s^j d^{i-1}$ for $i > j + 1$.
The only case that meets this condition is:
$d^2 s^0 = s^0 d^1$: $\quad d^2 s^0(u^n\alpha) = d^2(w^n\beta) = * = s^0(*) = s^0 d^1(u^n\alpha)$.

(6) Proof of RH-1, that r^0 is the identity.
A glance at the chart makes it clear that $r^0(x) = x$ for all $x \in Y_0 \cup Y_1 \cup Y_2$.

(7) Proof of RH-2, that $r^{n+1} \equiv *$ on Y_n.
Three glances at the chart make it clear that $r^1 \equiv *$ on Y_0; that $r^2 \equiv *$ on Y_1 and that $r^3 \equiv *$ on Y_2.

(8) Proof that $d^i r^j = r^{j-1} d^i$ for $i < j$ (first half of RH-3).
Cases of the form $d^i r^3 = r^2 d^i$ are immediate because $d^i r^3 \equiv *$ as $r^3 \equiv *$ on Y_2 and $r^2 d^i \equiv *$ as $r^2 \equiv *$ on Y_1. Aside from maps to Y_0, the following cases remain:

$d^1r^2 = r^1d^1$:

$$d^1r^2(v^n\gamma) = d^1(v^nw\gamma) = u^{n+1}\alpha = r^1(u^n\alpha) = r^1d^1(v^n\gamma),$$
$$d^1r^2(v^nw^k\beta) = d^1(v^nw^{k+1}\beta) = u^{n+k+1}\alpha = r^1(u^{n+k}\alpha) = r^1d^1(v^nw^k\beta),$$
$$d^1r^2(v^nw^l\gamma) = d^1(v^nw^{l+1}\gamma) = u^{n+l+1}\alpha = r^1(u^{n+l}\alpha) = r^1d^1(v^nw^l\gamma);$$

$d^0r^2 = r^1d^0$:

$$d^0r^2(v^n\gamma) = d^0(v^nw\gamma) = * = r^1(*) = r^1d^0(v^n\gamma),$$
$$d^0r^2(v^nw^k\beta) = d^0(v^nw^{k+1}\beta) = u^{k+1}\alpha = r^1(u^k\alpha) = r^1d^0(v^nw^k\beta),$$
$$d^0r^2(v^nw^l\gamma) = d^0(v^nw^{l+1}\gamma) = * = r^1(*) = r^1d^0(v^nw^l\gamma);$$

$d^0r^1 = r^0d^0$:

$$d^0r^1(v^n\gamma) = d^0(v^{n+1}\gamma) = * = r^0(*) = r^0d^0(v^n\gamma),$$
$$d^0r^1(v^nw^k\beta) = d^0(v^{n+1}w^k\beta) = u^k\alpha = r^0(u^k\alpha) = r^0d^0(v^nw^k\beta),$$
$$d^0r^1(v^nw^l\gamma) = d^0(v^{n+1}w^l\gamma) = * = r^0(*) = r^0d^0(v^nw^l\gamma).$$

(9) Proof that $d^ir^j = r^jd^i$ for $i \geq j$. (Second half of RH-3.)
Note that if $j = 0$ then $d^ir^j = r^jd^i$ is immediate as r^0 is the identity
$d^2r^2 = r^2d^2$:

$$d^2r^2(v^n\gamma) = d^2(v^nw\gamma) = * = r^2d^2(v^n\gamma),$$
$$d^2r^2(v^nw^k\beta) = d^2(v^nw^{k+1}\beta) = * = r^2d^2(v^nw^k\beta),$$
$$d^2r^2(v^nw^l\gamma) = d^2(v^nw^{l+1}\gamma) = * = r^2d^2(v^nw^l\gamma);$$

$d^2r^1 = r^1d^2$:

$$d^2r^1(v^n\gamma) = d^2(v^{n+1}\gamma) = u^{n+1}\alpha = r^1(u^n\alpha) = r^1d^2(v^n\gamma),$$
$$d^2r^1(v^nw^k\beta) = d^2(v^{n+1}w^k\beta) = * = r^1(*) = r^1d^2(v^nw^k\beta),$$
$$d^2r^1(v^nw^l\gamma) = d^2(v^{n+1}w^l\gamma) = * = r^1(*) = r^1d^2(v^nw^l\gamma);$$

$d^1r^1 = r^1d^1$:

$$d^1r^1(v^n\gamma) = d^1(v^{n+1}\gamma) = u^{n+1}\alpha = r^1(u^n\alpha) = r^1d^1(v^n\gamma),$$
$$d^1r^1(v^nw^k\beta) = d^1(v^{n+1}w^k\beta) = u^{n+1+k}\alpha = r^1(u^{n+k}\alpha) = r^1d^1(v^nw^k\beta),$$
$$d^1r^1(v^nw^l\gamma) = d^1(v^{n+1}w^l\gamma) = u^{n+l+1}\alpha = r^1(u^{n+l}\alpha) = r^1d^1(v^nw^l\gamma).$$

(10) Proof that $s^ir^j = r^{j+1}s^i$ for $i < j$. (First half of RH-4.)
The proof that $s^ir^2 = r^3s^i$ for $i < j$ is immediate because we have $s^1r^2 \equiv *$ as $r^2 \equiv *$ on Y_1 and $r^3s^i \equiv *$ because $r^3 \equiv *$ on Y_2. The only remaining

case is:
$s^0r^1 = r^2s^0$: $s^0r^1(u^n\alpha) = s^0(u^{n+1}\alpha) = w^{n+1}\beta = r^2(w^n\beta) = r^2s^0(u^n\alpha)$.

(11) Proof that $s^ir^j = r^js^i$ for $i \geq j$. (Second half of RH-4, so this will complete the proof that Example 4.5 has the indicated properties.)

Note that i can only be 0 or 1 as there is no s^2 in the partial simplicial set of Subsection 4.5. Also, if $j = 0$, then the result is immediate as r^0 is the identity. And if $i = 0$ then we must have $j = 0$ as $i \geq j$. The maps s^id^j with domain Y_0 are trivially equal to d^js^i when $i \geq j$ because these maps agree on $*$, the only element of Y_0. It follows that it only remains to show that $s^1r^1 = r^1s^1$:
$s^1r^1 = r^1s^1$: $s^1r^1(u^n\alpha) = s^1(u^{n+1}\alpha) = v^{n+1}\gamma = r^1(v^n\gamma) = r^1s^1(u^n\alpha)$.

Appendix C. The coskeleton equations

We start by completing the proof that the function s^i actually maps X_{n-1} to X_n. It clearly suffices to show that $d^jd^ks^ix = d^{k-1}d^js^ix$ for $x \in X_{n-1}$ and $j < k$. Recall that

$$s^ix = (s^{i-1}d^0x, s^{i-1}d^1x, \ldots, s^{i-1}d^{i-1}x, x, x, s^id^{i+1}x, \ldots, s^id^{n-1}x).$$

There are a number of cases to consider:

(1) $j < k < i$:

$$d^jd^ks^ix = d^js^{i-1}d^kx = s^{i-2}d^jd^kx = s^{i-2}d^{k-1}d^jx,$$
$$d^{k-1}d^js^ix = d^{k-1}s^{i-1}d^jx = s^{i-2}d^{k-1}d^jx;$$

(2) $j < k = i, i+1$:

$$d^jd^ks^ix = d^jx,$$
$$d^{k-1}d^jx = d^{k-1}s^{i-1}d^jx = d^jx;$$

(3) $j < i < k-1$:

$$d^jd^ks^ix = d^js^id^{k-1}x = s^{i-1}d^jd^{k-1}x = s^{i-1}d^{k-2}d^jx,$$
$$d^{k-1}d^js^ix = d^{k-1}s^{i-1}d^jx = s^{i-1}d^{k-2}d^jx;$$

(4) $j = i, i+1$, $k > i+1$:

$$d^jd^ks^ix = d^js^id^{k-1}x = d^{k-1}x,$$
$$d^{k-1}d^js^ix = d^{k-1}x;$$

(5) $i+1 < j < k$:

$$d^jd^ks^ix = d^js^id^{k-1}x = s^id^{j-1}d^{k-1}x = s^id^{k-2}d^{j-1}x,$$
$$d^{k-1}d^js^ix = d^{k-1}s^id^{j-1}x = s^id^{k-2}d^{j-1}x.$$

The simplicial identities for $d^i d^j$ and $d^i s^j$ are immediate consequences of the definitions. Next we show that for $i > j$, $s^i s^j = s^j s^{i-1}$. The $\{d^k\}$, being the projections on the limit, are collectively monic, so it suffices to show that $d^k s^i s^j = d^k s^j s^{i-1}$, which allows an inductive argument. We must consider a number of cases depending on where k falls with respect to i and j. We always suppose $i > j$.

(1) $k < j$:

$$d^k s^i s^j = s^{i-1} d^k s^j = s^{i-1} s^{j-1} d^k = s^{j-1} s^{i-2} d^k,$$
$$d^k s^j s^{i-1} = s^{j-1} d^k s^{i-1} = s^{j-1} s^{i-2} d^k;$$

(2) $k = j, j+1$:

$$d^k s^i s^j = s^{i-1} d^k s^j = s^{i-1},$$
$$d^k s^j s^{i-1} = s^{i-1};$$

(3) $i > k > j+1$:

$$d^k s^i s^j = s^{i-1} d^k s^j = s^{i-1} s^j d^{k-1} = s^j s^{i-2} d^{k-1},$$
$$d^k s^j s^{i-1} = s^j d^{k-1} s^{i-1} = s^j s^{i-2} d^{k-1};$$

(4) $k = i, i+1$:

$$d^k s^i s^j = s^j,$$
$$d^k s^j s^{i-1} = s^j d^{k-1} s^{j-1} = s^j;$$

(5) $k > i+1$:

$$d^k s^i s^j = s^i d^{k-1} s^j = s^i s^j d^{k-2} = s^j s^{i-1} d^{k-2},$$
$$d^k s^j s^{i-1} = s^j d^{k-1} s^{i-1} = s^j s^{i-1} d^{k-2}.$$

Now suppose the original fragment is contractible so far. We extend the contraction to $t\colon X_{n-1} \longrightarrow X_n$ by

$$tx = (x, td^0x, td^1x, \ldots, td^{n-1}x).$$

We must show that this is an element of X_n. Suppose $i < j$. When $i = 0$, $d^0 d^j tx = d^{j-1} d^0 tx = d^{j-1} x = d^{j-1} x$. For $0 < i < j$, we have

$$d^i d^j tx = d^i t d^{j-1} x = t d^{i-1} d^{j-1} x = t d^{j-1} d^{i-2} x,$$
$$d^j d^{i-1} tx = d^j t d^{i-2} x = t d^{j-1} d^{i-2} x.$$

We will show that the above extension of a contraction t is a contraction on $\mathrm{Cosk}(X)$, and, if the original contraction t is strong, then the extension will be strong. $s^i t = t s^{i-1}$ for $i > 0$. Again we compose with all the d^k.

(1) $1 < k < i$:

$$d^k s^i t = s^{i-1} d^k t = s^{i-1} t d^{k-1} = t s^{i-2} d^{k-1},$$
$$d^k t s^{i-1} = t d^{k-1} s^{i-1} = t s^{i-2} d^{k-1};$$

(2) $1 < k = i, i+1$: $\quad d^k s^i t = t = t d^{k-1} s^{i-1} = d^k t s^{i-1};$

(3) $k > i+1$:

$$d^k s^i t = s^i d^{k-1} t = s^i t d^{k-2} = t s^{i-1} d^{k-2},$$
$$d^k t s^{i-1} = t d^{k-1} s^{i-1} = t s^{i-1} d^{k-2}.$$

Finally, we show that if the original contraction t is a strong contraction, then so is its extension.

(1) $k = 0$: $\quad d^0 s^0 t = t = d^0 t t$;
(2) $k = 1$: $\quad d^1 s^0 t = t = t d^0 t = d^1 t t$;
(3) $k > 1$: $\quad d^k s^0 t = s^0 d^{k-1} t = s^0 t d^{k-2} = t t d^{k-2} = d^k t t$.

Appendix D. The simplicial homotopy equations

Proof of Theorem 7.1.

Proof of RH-1:

$$r^0 u(a_0, \ldots, a_n) = H(u(a_0, \ldots, a_n), 0) = f u(a_0, \ldots, a_n) = \mathrm{SS}(f)(u).$$

Proof of RH-2:

$$r^{n+1}(u(a_0, \ldots, a_n) = H(u(a_0, \ldots, a_n), 1) = g u(a_0, \ldots, a_n) = \mathrm{SS}(g)(u).$$

Proof of RH-3: For $i < j$

$$\begin{aligned} d^i r^j u(a_0, \ldots, a_{n-1}) &= r^j(u)(a_0, \ldots, a_{i-1}, 0, a_i, \ldots, a_j, \ldots, a_{n-1}) \\ &= H(u(a_0, \ldots, a_{i-1}, 0, a_i, \ldots, a_j, \ldots, a_{n-1}), a_0 + \cdots + 0 + \cdots + a_{j-2}) \end{aligned}$$

while

$$\begin{aligned} r^{j-1} d^i u(a_0, \ldots, a_{n-1}) &= H(d^i u(a_0, \ldots, a_{n-1}), a_0 + \cdots + a_{j-2}) \\ &= H(u(a_0, \ldots, a_{i-1}, 0, a_i, \ldots, a_j, \ldots, a_{n-1}), a_0 + \cdots + a_{j-2}). \end{aligned}$$

For $i \geq j$

$$\begin{aligned} d^i r^j u(a_0, \ldots, a_{n-1}) &= r^j(u)(a_0, \ldots, a_{j-1}, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1}) \\ &= H(u(a_0, \ldots, a_{j-1}, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1}), a_0 + \cdots + a_{j-1}) \end{aligned}$$

while

$$\begin{aligned} r^j d^i u(a_0, \ldots, a_{n-1}) &= H(d^i u(a_0, \ldots, a_{n-1}), a_0 + \cdots + a_{j-1}) \\ &= H(u(a_0, \ldots, a_{j-1}, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1}) a_0 + \cdots + a_{j-1}). \end{aligned}$$

Proof of RH-4: For $i < j$

$$\begin{aligned} s^i r^j u(a_0,\ldots,a_{n+1}) &= r^j u(a_0,\ldots,a_{i-1},a_i+a_{i+1},\ldots,a_{j+1},\ldots,a_{n+1}) \\ &= H(u(a_0,\ldots,a_{i-1},a_i+a_{i+1},\ldots,a_{j+1},\ldots,a_{n+1}), \\ &\qquad a_0+\cdots+(a_i+a_{i+1})+\cdots+a_j) \end{aligned}$$

while

$$\begin{aligned} r^{j+1}s^i u(a_0,\ldots,a_{n+1}) &= r^{j+1}s^i u(a_0,\ldots,a_i,\ldots,a_j\ldots,a_{n+1}) \\ &= H(s^i u(a_0,\ldots,a_{n+1}),a_0+\cdots+a_j), \end{aligned}$$

which is clearly the same as $s^i r^j u$. For $i \geq j$

$$\begin{aligned} s^i r^j u(a_0,\ldots,a_{n+1}) &= r^j u(a_0,\ldots,a_{j-1},\ldots,a_{i-1},a_i+a_{i+1},\ldots,a_{n+1}) \\ &= H(u(a_0,\ldots,a_{j-1},\ldots,a_{i-1},a_i+a_{i+1},\ldots,a_{n+1}),a_0+\cdots+a_{j-1}) \end{aligned}$$

while

$$\begin{aligned} r^j s^i u(a_0,\ldots,a_{n+1}) &= r^j s^i u(a_0,\ldots,a_j,\ldots,a_i,\ldots,a_{n+1}) \\ &= H(s^i u(a_0,\ldots,a_{n+1}),a_0+\cdots+a_{j-1}) \end{aligned}$$

which is clearly the same as $s^i r^j u$.

It seems worth pointing out that Meyer's condition is not satisfied. For example, recall that $h^i = r^{i+1}s^i$ then:

$$d^1h^0u(a_0,a_1) = h^0u(a_0,0,a_1) = H(u(a_0,a_1),a_0)$$

while

$$h^0d^0u(a_0,a_1) = H(d^0u(a_0+a_1),a_0) = H(u(0,a_0+a_1),a_0).$$

The result is that, while as shown above, a space homotopic to a point gives a contractible simplicial complex, Meyer's construction does not do the job in this case. □

Proof of Theorem 7.2. First we show that tu is well-defined and continuous. Since that $u(a_1/(1-a_0),\ldots,a_{n+1}/(1-a_0))$ is always defined when $a_0 \neq 1$, it suffices to show that for $1 \leq i \leq n+1$ we have $0 \leq a_i/(1-a_0) \leq 1$. But this readily follows as $1-a_0 = a_1+\cdots+a_{n+1} \geq a_i$.

As for continuity, it is clear that tu is continuous at every point of Δ_{n+1} except possibly when $a_0 = 1$. Let $q = (1,0,\ldots,0)$ be the only such point in Δ_{n+1}. It suffices to show that if $p^1,p^2,\ldots,p^i,\ldots$ is a sequence of points of Δ_{n+1} that converges to q, then $*$ is in the closure of the set $\{tu(p^i)\colon i=1,2,\ldots\}$.

Now write $p^i = (x_0^i,x_1^i,\ldots,x_{n+1}^i)$. Note that we may as well assume that $x_0^i \neq 1$ for all i as otherwise $q \in \{tu(p^i)\colon i=1,2,\ldots\}$. But the sequence $x_0^1,x_0^2,\ldots,x_0^i,\ldots$ must converge to 1 as the projection from $\Delta_{n+1}\longrightarrow[0,1]$, which sends $(a_0,a_1,\ldots,a_{n+1})$ to a_0 is continuous.

Now let $Y \subseteq X$ be the image Δ_{n+1} under u. Note that Y is compact. Define $\varphi(p^i) = (u(x_1^i/(1-x_0^i),\ldots,x_{n+1}^i/(1-x_0^i)),1-x_0^i) \in Y\times[0,1]$. Since $Y\times[0,1]$ is compact, the sequence $\varphi(p^1),\ldots,\varphi(p^i),\ldots$ must have a *cluster point* (meaning a point $(y,b)\in Y\times[0,1]$ such that every neighborhood of (y,b) contains infinitely many members of the sequence). It clearly follows that $b=0$.

Finally, since $tu(p^i) = H(\varphi(p^i))$ and since H is continuous, and therefore preserves cluster points, we see that $H(y,b)$ is a cluster point of, and thus in the closure of, $\{tu(p^i)\colon i = 1,2,\ldots\}$. But as shown above, $b = 0$ so $H(y,b) = *$.

Next we show that t is a contraction.

$$d^0tu(a_0,\ldots,a_n) = tu(0,a_0,\ldots,a_n) = 1u(a_0,\ldots,a_n) = u(a_0,\ldots,a_n)$$

so that $d^0t = \mathrm{id}$. For $i > 0$

$$\begin{aligned} d^itu(a_0,\ldots,a_n) &= tu(a_0,\ldots,a_{i-1},0,a_i,\ldots,a_n) \\ &= (1-a_0)u\Big(\frac{a_1}{1-a_0},\ldots,\frac{a_{i-1}}{1-a_0},0,\frac{a_i}{1-a_0},\ldots,\frac{a_n}{1-a_0}\Big) \end{aligned}$$

while

$$\begin{aligned} td^{i-1}u(a_0,\ldots,a_n) &= (1-a_0)d^{i-1}u\Big(\frac{a_1}{1-a_0},\ldots,\frac{a_n}{1-a_0}\Big) \\ &= (1-a_0)u\Big(\frac{a_1}{1-a_0},\ldots,\frac{a_{i-1}}{1-a_0},0,\frac{a_i}{1-a_0},\ldots,\frac{a_n}{1-a_0}\Big). \end{aligned}$$

The argument with the degeneracies is similar. For $i > 0$

$$\begin{aligned} s^itu(a_0,\ldots,a_{n+2}) &= tu(a_0,\ldots,a_i+a_{i+1},\ldots,a_n) \\ &= (1-a_0)u\Big(\frac{a_1}{1-a_0},\ldots,\frac{a_i+a_{i+1}}{1-a_0},\ldots,\frac{a_n}{1-a_0}\Big) \end{aligned}$$

while

$$\begin{aligned} ts^{i-1}u(a_0,\ldots,a_{n+2}) &= (1-a_0)s^{i-1}u\Big(\frac{a_1}{1-a_0},\ldots,\frac{a_{n+2}}{1-a_0}\Big) \\ &= (1-a_0)u\Big(\frac{a_1}{1-a_0},\ldots,\frac{a_i}{1-a_0}+\frac{a_{i+1}}{1-a_0},\ldots,\frac{a_{n+2}}{1-a_0}\Big). \end{aligned}$$

Assuming there is a regular contraction on X we will show that the t, as defined above, is a strong contraction on $\mathrm{SS}(X)$. We have that

$$\begin{aligned} ttu(a_0,\ldots,a_{n+2}) &= (1-a_0)tu\Big(\frac{a_1}{1-a_0},\ldots,\frac{a_{n+2}}{1-a_0}\Big) \\ &= (1-a_0)\Big(1-\frac{a_1}{1-a_0}\Big)u\Big(\frac{a_2/(1-a_0)}{1-a_1/(1-a_0)},\ldots,\frac{a_{n+2}/(1-a_0)}{1-a_1/(1-a_0)}\Big) \\ &= (1-a_0)\Big(\frac{1-a_0-a_1}{1-a_0}\Big)u\Big(\frac{a_2}{1-a_0-a_1},\ldots,\frac{a_{n+2}}{1-a_0-a_1}\Big) \\ &= (1-a_0-a_1)u\Big(\frac{a_2}{1-a_0-a_1},\ldots,\frac{a_{n+2}}{1-a_0-a_1}\Big) \\ &= tu(a_0+a_1,a_2,\ldots,a_{n+2}) \\ &= s^0tu(a_0,\ldots,a_{n+2}). \end{aligned}$$

$\square$

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