One Erdős style inequality

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Dedicated to the memory of Věra Trnková

Abstract. One unusual inequality is examined.

Keywords: inequality

Classification: 11D75

In 1951, P. Erdős in [1] investigated the diophantine equation

$$\binom{n}{k} = x^l, \qquad k \ge 2,\;\; n \ge 2k,\;\; x > 1,\;\; l > 1 \tag{1}$$

and he showed that this equation has no solution for $k > 3$ (there are infinitely many solutions if $k = l = 2$, and for $k = 3$, $l = 2$, equation (1) has only one solution $n = 50$, $x = 140$). The remaining cases $k = 2, 3$ and $l > 2$ were settled by K. Győry in [2]. The proof in [1] is making use of some quite unusual inequalities and one of them, namely the inequality $(h - g)^3 > h$, is carefully examined and generalized in this ultrashort note. Needless to say that our approach is fully calculus-free.

First of all, let a, b, c be positive integers such that $a < c$ and $ac = b^2$. Then $a < b < c$ and the well-known relation of arithmetic and geometric means yields $a+c > 2b$. Put $m = c-b$, $n = b-a$ and $p = m-n = a+c-2b$. Then $m, n, p \ge 1$, $m \ge n + 1$ and $bm = b(c - b) = bc - b^2 = bc - ac = (b - a)c = nc$. Hence

$$bp = b(m - n) = bm - bn = nc - bn = nm. \tag{2}$$

Since $m \ge n+1$ and $p \ge 1$, (2) implies $m^2 \ge (n+1)m = nm+m = bp+m \ge b+m$, and consequently $m^2 - m \ge b$. From this,

$$m^2 - (m + n) = m^2 - m - n \ge b - n = a. \tag{3}$$

As $m + n = c - a$, we have $m^2 - (m + n) = (c - b)^2 - c + a$. By (3), $(c - b)^2 \ge c$, and hence

$$(c - a)^2 > c. \tag{4}$$

DOI 10.14712/1213-7243.2019.025

Now, let g, h be positive integers such that $g \le a$, $c \le h$ and put $\delta = h - c$. Using (4), we obtain $(h-g)^2 \ge (h-a)^2 = (c-a+\delta)^2 \ge (c-a)^2 + \delta > c + \delta = h$.

Let $a, b, c, d, e, f, g, h, t, \alpha, \beta, \gamma$ be positive integers satisfying $a \ne b \ne c \ne a$, $g \le \min(a, b, c)$, $\max(a, b, c) \le h$, $5h \le 6g$, $t \ge 3$, $\beta^2 = \alpha\gamma$, $a = \alpha d^t$, $b = \beta e^t$, $c = \gamma f^t$. We aim to show that $(h-g)^3 > h$.

The case $b^2 = ac$ is settled down in the above-mentioned part, where we got $(h-g)^2 > h$. In view of this, we can restrict ourselves to the case $b^2 > ac$ (the other case, $ac > b^2$, being quite analogous). We can assume $a < c$ as well. Then, of course, $g \le a < b \le h$, $g \le a < c \le h$ and

$$g^2 < ac. \tag{5}$$

Furthermore, $b^2 - ac = \beta^2 e^{2t} - \alpha\gamma(df)^t = \beta^2(e^{2t} - (df)^t) > 0$, hence $e^2 \ge df + 1$ and $b^2 - ac \ge \beta^2((df+1)^t - (df)^t) \ge \beta^2 t (df)^{t-1}$. Thus

$$df(b^2 - ac) \ge \beta^2 t (df)^t = t\alpha d^t \gamma f^t = tac. \tag{6}$$

Now, $2(h-g)h = (h-g)^2 + h^2 - g^2 > (h-g)^2 + b^2 - ac$ by (5). Using (6) and (5), we see that $2(h-g)hdf > (h-g)^2 df + (b^2 - ac)df \ge (h-g)^2 df + tac > (h-g)^2 df + tg^2 > tg^2$. Since $t \ge 3$ and $5h \le 6g$, we have $tg^2 \ge 3(h - (h-g))^2 = 3h^2 - 6h(h-g) + 3(h-g)^2 = 2h^2 + h(h - 6(h-g)) + 3(h-g)g^2 > 2h^2$, and therefore

$$(h-g)df > h. \tag{7}$$

Let s be an integer such that $4 \le s \le t+2$. We have $(h-g)^{s-2}h^s > (h-g)^{s-2}h^{s-2}ac = \beta^2(h-g)^{s-2}h^{s-2}d^t f^t \ge \beta^2(h-g)^{s-2}h^{s-2}d^{s-2}f^{s-2} = \beta^2((h-g)df)^{s-2}h^{s-2} > \beta^2 h^{2s-4}$ by (7), and hence

$$(h-g)^{s-2} > \beta^2 h^{s-4} \ge h^{s-4}.$$

For $s = t+2$ we get $(h-g)^t > h^{t-2}$. For $s = 5$, we get $(h-g)^3 > h$. If $\bar{g} = \min(a, b, c)$ and $\bar{h} = \max(a, b, c)$ then $5\bar{h} \le 6\bar{g}$ and $(\bar{h} - \bar{g})^3 > \bar{h}$.

We have shown that the inequality $(h-g)^3 > h$ holds if $5h \le 6g$ and some unusual additional conditions are satisfied. On the other hand, $5 \cdot 18 < 6 \cdot 16$, but $(18-16)^3 < 18$. If $5h > 6g$ and $h \ge 15$ then $6^3(h-g)^3 > h^3 \ge 6^3 h$ and the inequality holds.

Now, let us have a look at the inequality $(h-g)^3 > h$ from another point of view. Let H, g, h, Δ be positive integers such that $H \ge 3$ and $\Delta \ge 2$. Put $G_H = H - 1 - \left[\sqrt[3]{H}\right]$ (here $[\alpha]$ denotes the integer part of α). Then $H - G_H \ge 2$, $(H - G_H)^3 > H$ and $(H-g)^3 \le H$ for $g > G_H$. If $g \le G$, $h \ge H$ and $\delta = h - H$ then $(h-g)^3 \ge (h - G_H)^3 = (H - G_H + d)^3 \ge (H - G_H)^3 + \delta > H + \delta = h$.

Let $(\Delta - 1)^3 \le H \le \Delta^3 - 1$. Then $G_H = H - \Delta$ and, moreover, $5H \le 6G_H$ if and only if $6\Delta \le H$. Since $6\Delta < (D-1)^3$ for $\Delta \ge 4$, we see that $5H \le 6G_H$ if and only if $H \ge 18$. Finally, g such that $g \le H - 2$, $(H-g)^3 \le H$ exists if and only if $H \ge 8$. If g is so then $5H > 6g$ for $H \le 11$ and $5H \le 6g$ otherwise.

Acknowledgement. The authors would like to express their thanks to the anonymous referee for her/his fruitful comments which helped to improve the readability of this note.

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(*Received* December 29, 2018, *revised* January 10, 2019)