Axiom T_D and the Simmons sublocale theorem

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Dedicated to the memory of Věra Trnková

Abstract. More precisely, we are analyzing some of H. Simmons, S. B. Niefield and K. I. Rosenthal results concerning sublocales induced by subspaces. H. Simmons was concerned with the question when the coframe of sublocales is Boolean; he recognized the role of the axiom T_D for the relation of certain degrees of scatteredness but did not emphasize its role in the relation between sublocales and subspaces. S. B. Niefield and K. I. Rosenthal just mention this axiom in a remark about Simmons' result. In this paper we show that the role of T_D in this question is crucial. Concentration on the properties of T_D-spaces and technique of sublocales in this context allows us to present a simple, transparent and choice-free proof of the scatteredness theorem.

Keywords: frame; locale; sublocale; coframe of sublocales; spatial sublocale; induced sublocale; T_D-separation; covered prime element; scattered space; weakly scattered space

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Introduction

A topological space X, more precisely its associated frame $\Omega(X)$ of open sets, has typically more natural subobjects (sublocales) than subspaces. The first result concerning the question when every sublocale is (induced by) a subspace was presented by H. Simmons in [11]. More precisely, H. Simmons proved a necessary and sufficient condition for the lattice of sublocales being Boolean which is slightly different: if sublocales are in a one-to-one correspondence with subspaces (subsets) they do form a Boolean algebra, while the other implication does not hold. Later, S. B. Niefield and K. I. Rosenthal in [7] treated more directly the question of every sublocale being spatial and gave a characterization of the respective frames. In both cases, however, the question of the one-to-one correspondence between subspaces and sublocales is somehow circumvented. While, as we have already pointed out, typically one has more sublocales than subspaces, there are already

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cases where there are *fewer* sublocales than subspaces. Namely, it turns out that unless the space in question satisfies a certain very weak separation condition T_D, representation of subspaces of X by sublocales of $\Omega(X)$ is imperfect: distinct subspaces can induce the same sublocale (it should be noted that in [11], T_D does appear—under the name of T_B—in the discussion of "degrees of scatteredness"; in [7] it is mentioned in page 267, in a remark about Simmons' paper).

In this paper we present a proof of the fact that for a T_D-space X, the sublocales are in a one-to-one correspondence with subspaces if and only if X is scattered (without T_D it cannot be). Consequent use of properties of T_D-spaces and the sublocale technique makes the proof simpler, and we think more transparent, than those in [11], [7]. Also, since we do not need the concept of a minimal prime (and that of an essential one) we can do it without a choice principle.

1. Preliminaries

1.1 Notation. A join (supremum) of a subset $A \subseteq (X, \leq)$, if it exists, will be denoted by $\bigvee A$, and we write $a \vee b$ for $\bigvee\{a, b\}$; similarly we write $\bigwedge A$ and $a \wedge b$ for meets (infima).

The smallest element of a poset (the supremum $\bigvee \emptyset$), if it exists, will be denoted by 0, and the largest one (the infimum $\bigwedge \emptyset$) will be denoted by 1.

An element $p \in X$ is *prime* if $a \wedge b = p$ implies $a = p$ or $b = p$ (in a distributive lattice this is equivalent with $a \wedge b \leq p$ implying $a \leq p$ or $b \leq p$).

1.1.1 Adjoint maps. If X, Y are posets we say that monotone maps $f\colon X \to Y$ and $g\colon Y \to X$ are *adjoint, f to the left and g to the right*, if

$$f(x) \leq y \;\Leftrightarrow\; x \leq g(y).$$

Recall that this is characterized by the pair of inequalities $fg(y) \leq y$ and $x \leq gf(x)$, and that f or g preserves all the existing suprema or infima, respectively. Furthermore, if X and Y are complete lattices then a monotone map $f\colon X \to Y$ preserves all suprema if and only if it is a left adjoint, and a monotone map $g\colon Y \to X$ preserves all infima if and only if it is a right adjoint.

1.2 The category of frames. Recall that a *frame* is a complete lattice L satisfying the distributivity rule

$$\text{(frm)} \qquad \Big(\bigvee A\Big) \wedge b = \bigvee\{a \wedge b\colon a \in A\}$$

for all $A \subseteq L$ and $b \in L$, and that a *frame homomorphism* $h\colon L \to M$ preserves all joins and all finite meets. The resulting category is denoted by **Frm**.

A *coframe* satisfies (frm) with the roles of joins and meets reversed.

1.2.1 The equality (frm) states, in other words, that for every $b \in L$ the mapping $- \wedge b = (x \mapsto x \wedge b)\colon L \to L$ preserves all joins (suprema). Hence every $- \wedge b$ has a right Galois adjoint resulting in a *Heyting operation* " $\to$ " with

$$a \wedge b \leq c \;\Leftrightarrow\; a \leq b \to c.$$

Thus, each frame is a Heyting algebra (note that, however, the frame homomorphisms do not coincide with the Heyting ones so that **Frm** differs from the category of complete Heyting algebras). The operation " $\to$ " and some of its basic properties (e.g. $a \to a = 1$, $a \to b = 1$ if and only if $a \leq b$, $1 \to a = a$, and $a \to (b \to c) = (a \wedge b) \to c$) will be used in the sequel, see [8, Appendix 1] for more information.

1.3 The concrete category Loc. The functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$ from the category of topological spaces and continuous maps into that of frames ($\Omega(f)$ sending an open set $U \subseteq Y$ to $f^{-1}[U]$ for a continuous map $f\colon X \to Y$ in **Top**) is a full embedding on an important and substantial part of **Top**, the subcategory of *sober* spaces. This justifies to regard frames as a natural generalization of spaces. Since Ω is contravariant, one introduces the *category of locales* **Loc** as the dual of the category of frames. Often one just considers the formal $\mathbf{Frm}^{\mathrm{op}}$ but it is of advantage to represent it as a concrete category with specific maps as morphisms. For this purpose one defines a *localic map* $f\colon L \to M$ as the (unique) right Galois adjoint of a frame homomorphism $h = f^*\colon M \to L$. This can be done since frame homomorphisms preserve suprema; but of course not every mapping preserving infima is a localic one. We refer to [8] for more information about the category of locales.

1.4 Sublocales. A *sublocale* of a frame L is a subset $S \subseteq L$ such that

(1) $M \subseteq S$ implies $\bigwedge M \in S$, and
(2) if $a \in L$ and $s \in S$ then $a \to s \in S$.

The system $\mathsf{S}(L)$ of all sublocales of L is a co-frame, with the lattice operations

$$\bigwedge_{i\in J} S_i = \bigcap_{i\in J} S_i \qquad \text{and} \qquad \bigvee_{i\in J} S_i = \Big\{ \bigwedge A\colon A \subseteq \bigcup_{i\in J} S_i \Big\}.$$

The top element of $\mathsf{S}(L)$ is L and the bottom is the *least sublocale* $\mathsf{O} = \{1\}$.

1.4.1 Sublocales just defined are a natural representation of subobjects in the category of locales (indeed S is a sublocale of L if and only if the imbedding map $j\colon S \subseteq L$ is an extremal monomorphism in the category **Loc**). Equivalently we can represent subobjects of frames (locales) as *frame congruences* E on L (the sublocale as above is then the adjoint to the quotient frame homomorphism $L \to L/E$); yet another representation is that by *nuclei*, see e.g. [5], [8].

1.4.2 Important special sublocales. For any $a \in L$ we have a sublocale

$$\mathfrak{b}(a) = \{x \to a\colon x \in L\}.$$

From the standard properties of the Heyting operation we immediately see that it is really a sublocale; and obviously it is the smallest sublocale containing a. One has that, see e.g. [8, III.10],

the $\mathfrak{b}(a)$'s are precisely the Boolean sublocales of L.

Other sublocales we will work with are the *points*

$$\widetilde{p} = \{p, 1\}$$

with p prime elements of L. These are precisely the sublocales with exactly two elements (with exactly one nontrivial element).

Remark. Typical points of a frame $\Omega(X)$ are the $X \smallsetminus \overline{\{x\}}$. Note that there may be others (a space is *sober* if there are only these), but they suffice for the representations in Section 2.

Proposition 1.5. *Let L be a distributive lattice and let $a \in L$ be complemented. Then, for any supremum $\bigvee x_i$, we have $a \wedge \bigvee x_i = \bigvee (a \wedge x_i)$, and for any infimum $\bigwedge x_i$ we have $a \vee \bigwedge x_i = \bigwedge (a \vee x_i)$.*

In particular, in any co-frame we have for any complemented a,

$$a \wedge \bigvee x_i = \bigvee (a \wedge x_i)$$

although this (frame) distributivity does not generally hold.

Proof: If a' is the complement of a we easily check that $a \wedge x \leq b$ if and only if $x \leq a' \vee b$. Thus for any complemented a, $(x \mapsto a \wedge x)$ is a left adjoint and $(x \mapsto a \vee x)$ is a right adjoint. Use Subsection 1.1.1. $\square$

1.6 The axiom T_D. In [1] the authors studied separation axioms between T_0 and T_1. Among them, particular importance gained the axiom

T_D: *for every $x \in X$ there is an open set $U \ni x$ such that $U \smallsetminus \{x\}$ is still open*

(and hence $U \smallsetminus \{x\} = U \smallsetminus \overline{\{x\}}$). This is equivalent to saying that points are *locally closed*, that is for every $x \in X$ there is an open set $U \ni x$ such that $\{x\} = U \cap \overline{\{x\}}$.

We will need the following two facts from [3].

Lemma 1.6.1. *Let X satisfy T_D. Then every $\big(X \smallsetminus \overline{\{x\}}\big) \cup \{x\}$ is open.*

Proof: Choose an open $U \ni x$ such that $\{x\} = U \cap \overline{\{x\}}$. Then, clearly,

$$\big(X \smallsetminus \overline{\{x\}}\big) \cup \{x\} = \big(X \smallsetminus \overline{\{x\}}\big) \cup U.$$

$\square$

Proposition 1.6.2. *Let X satisfy T_D. Then the primes $p = X \smallsetminus \overline{\{x\}}$ are covered, that is, if $p = \bigwedge_{i \in J} U_i$ then $p = U_k$ for some $k \in J$, not only for finite J but for arbitrary ones, cf.* [4].

Proof: Let $p = X \setminus \overline{\{x\}} \subsetneq U$ for an open U. Then there is a $y \in U \setminus \big(X \setminus \overline{\{x\}}\big)$, hence $y \in U$ and $y \in \overline{\{x\}}$ so that $x \in U$ and $\big(X \setminus \overline{\{x\}}\big) \cup \{x\} \subseteq U$. Hence either $X \setminus \overline{\{x\}} = U_i$ for some i or all the U_i contain the open $\big(X \setminus \overline{\{x\}}\big) \cup \{x\}$ and hence $\bigwedge_{i\in J} U_i = \operatorname{int}\big(\bigcap_{i\in J} U_i\big)$ is not p. □

Concerning terminology, it should be pointed out that the elements p such that $p = \bigwedge_{i\in J} x_i$ implies $p = x_i$ for some $i \in J$ were referred to in [3] as *completely prime.* That term, however, is generally taken to mean that $p \geq \bigwedge_{i\in J} x_i$ implies $p \geq x_i$ for some $i \in J$. Note that in an arbitrary space X, a prime $X \setminus \overline{\{x\}}$ is of the latter type if and only if $x \in \overline{\bigwedge\{U \colon U \in \Omega(X),\ x \in U\}}$. In particular, if x is *isolated* (that is, $\{x\}$ is open) then $X \setminus \overline{\{x\}}$ is always completely prime.

Regarding the relationship between these two notions, any completely prime p is clearly a covered prime, but not conversely: in the topology of a T_1-space X, any $X \setminus \{x\}$, $x \in X$, is obviously a covered prime but the complete primes are only the $X \setminus \{x\}$ with isolated $x \in X$, see [4, Remark 1].

1.7 Scattered and weakly scattered spaces. A space X is said to be *scattered* if for every nonempty closed set A there is an *isolated point* $a \in A$, that is, there is an $a \in A$ and an open $U \ni a$ such that

$$U \cap A = \{a\}.$$

It is *weakly scattered* (or *corrupted*, see [11]), if for every nonempty closed set A there is an $a \in A$ and an open $U \ni a$ such that

$$U \cap A \subseteq \overline{\{a\}}.$$

Observation 1.7.1. *A T_D-space is scattered if and only if it is weakly scattered.*

Proof: Consider an $a \in A$ and an open $U \ni a$ such that $U \cap A \subseteq \overline{\{a\}}$. By T_D, $\{a\} = V \cap \overline{\{a\}}$ for some open V. Then $\{a\} \subseteq V \cap (U \cap A) \subseteq V \cap \overline{\{a\}} = \{a\}$. □

2. Induced sublocales

2.1 Consider a space X and a subspace $Y \subseteq X$. Then the embedding $j \colon Y \subseteq X$ is represented by the frame homomorphism

$$\Omega(j) = (U \mapsto U \cap Y) \colon \Omega(X) \to \Omega(Y)$$

and hence the frame congruence associated with Y is given by

$$\Theta_Y = \{(U, V) \colon U \cap Y = V \cap Y\}.$$

It is easy to see that the localic map adjoint to $\Omega(j)$ is given by

$$k(V) = \operatorname{int}((X \setminus Y) \cup V)$$

(since $U \cap Y \subseteq V$ if and only if $U \subseteq (X \smallsetminus Y) \cap V$ and U is open). Hence the sublocale induced by Y is

$$S_Y = k[\Omega(Y)] = \{\operatorname{int}((X \smallsetminus Y) \cup V) \colon V \text{ open in } Y\} = \\ = \{\operatorname{int}((X \smallsetminus Y) \cup (U \cap Y)) \colon U \in \Omega(X)\}.$$

The sublocale S_Y is said to be *induced* by Y.

2.2 One thinks of frames (locales) as of generalized spaces and this view is basically right; at least for the so called *sober spaces* the frame $\Omega(X)$ contains all the information about X. One can surmise that this concerns also the structure of induced sublocales as above, that is, that when thinking of the locale $\Omega(X)$ as (a representation) of X, the induced sublocales can be thought of as (a representation of) the subspaces (we are not speaking of the fact that there may be also new entities, the non-induced sublocales; they enrich the theory and are very useful). But it is not in general so. Take, e.g., a non-T_D space X and an $x \in X$ such that no $U \smallsetminus \{x\}$ with $U \ni x$ is open. Then $U \cap (X \smallsetminus \{x\}) = V \cap (X \smallsetminus \{x\})$ only if $U = V$ and hence $S_{X \smallsetminus \{x\}} = S_X$.

We say that the representation $Y \mapsto S_Y$ of subspaces is *precise* if it constitutes a one-to-one correspondence between subspaces and induced sublocales. One has the following, see e.g. [8], [2], [6]:

Proposition 2.2.1. *Induced sublocales constitute a precise representation of subspaces of X if and only if X is T_D.*

Note. A mechanism of this fact useful for our purposes will be apparent in Proposition 3.2.1 below.

2.3 Representations of points. Denote by $p_{X,x}$ (briefly p_x) the prime $X \smallsetminus \overline{\{x\}}$ in $\Omega(X)$.

Lemma 2.3.1. *Let Y be a subspace of X. We have $k(p_{Y,y}) = p_{X,y}$.*

Proof: We immediately see that for $\overline{\{y\}}^Y$, the closure in Y,

$$Y \smallsetminus \overline{\{y\}}^Y = Y \smallsetminus \big(\overline{\{y\}} \cap Y\big) = Y \smallsetminus \overline{\{y\}}.$$

Obviously $(X \smallsetminus Y) \cup \big(Y \smallsetminus \overline{\{y\}}\big) \supseteq X \smallsetminus \overline{\{y\}}$ and if for an open U, $U \subseteq (X \smallsetminus Y) \cup \big(Y \smallsetminus \overline{\{y\}}\big)$ then $U \subseteq X \smallsetminus \overline{\{y\}}$ (otherwise there were a $z \in U$ with $z \in \overline{\{y\}}$, but then $y \in U$ and y is neither in $X \smallsetminus Y$ nor in $Y \smallsetminus \overline{\{y\}}$). $\square$

Proposition 2.4. *Let Y be a subspace of X. Then*

$$S_Y = \bigvee \{\widetilde{p}_{X,y} \colon y \in Y\}.$$

Proof: Recall the formula for the joins of sublocales from Subsection 1.4. The elements of the right hand side are the meets

$$U = \bigwedge\{X \smallsetminus \overline{\{y\}} \colon y \in A,\ A \subseteq Y\} = \operatorname{int}\bigcap\{X \smallsetminus \overline{\{y\}} \colon y \in A,\ A \subseteq Y\}.$$

Now if U is the interior as above we have, first, for every $y \in A$ also $U \subseteq X \smallsetminus \overline{\{y\}}$, which is the same as $y \notin U$, and, hence, whatever the A was, U is also the interior of $\bigcap\{X \smallsetminus \overline{\{y\}} \colon y \notin U\}$. Now we have

$$\bigcap\{X \smallsetminus \overline{\{y\}} \colon y \notin U\} \subseteq \bigcap\{X \smallsetminus \{y\} \colon y \notin U\}$$

but for an open V we have $V \subseteq X \smallsetminus \overline{\{y\}}$ if and only if $V \subseteq X \smallsetminus \{y\}$ and hence

$$\begin{aligned}\operatorname{int}\bigcap\{X \smallsetminus \overline{\{y\}} \colon y \in Y \smallsetminus U\} &= \operatorname{int}\bigcap\{X \smallsetminus \{y\} \colon y \in Y \smallsetminus U\} = \\ &= \operatorname{int}(X \smallsetminus (Y \smallsetminus U)) = \operatorname{int}((X \smallsetminus Y) \cup (Y \cap U)).\end{aligned}$$

Compare this with the formula for k in Subsection 2.1. □

2.5 Now from Subsection 2.3 and Proposition 2.4 we can conclude that

A sublocale S of $\Omega(X)$ is induced if and only if $S = \bigvee\{\widetilde{p}_{X,x} \colon p_{X,x} \in S\}$.

3. The main theorem

3.1 Recall the notation $p_x = X \smallsetminus \overline{\{x\}}$ from Subsection 2.3 (the X in $p_{X,x}$ will be always the same and hence we can use the shorter notation), and $\widetilde{p}$ from Subsection 1.4.2. Also recall from Proposition 1.6.2 that if X is T_D then every p_x is a covered prime element.

Definition 3.2. A prime p in a frame L is *a-regular* if $p = (p \to a) \to a$.

Proposition 3.2.1. *The following are equivalent for $L = \Omega(X)$ with X a T_D-space:*

1. *$\mathsf{S}(L)$ is Boolean.*
2. *Each Boolean (that is, minimal) sublocale is complemented.*
3. *For every $a \neq 1$, there is an a-regular element of the form p_x.*
4. *All sublocales S are induced, i.e., $S = \bigvee\{\widetilde{p}_x \colon p_x \in S\}$, and precisely represent subspaces of X.*
5. *The map $\mu \colon \mathsf{S}(L) \to \mathfrak{P}(X)$, given by $\mu(S) = \{x \colon p_x \in S\}$, is a poset isomorphism, with inverse $\gamma \colon \mathfrak{P}(X) \to \mathsf{S}(L)$ given by $\gamma(Y) = \bigvee\{\widetilde{p}_y \colon y \in Y\}$.*

Proof: (1) ⇒ (2) and (5) ⇒ (1) are clear.

(2) ⇒ (3): Each $\mathfrak{b}(a)$ (recall Subsection 1.4.2) is complemented hence, by Proposition 1.5,

$$\mathfrak{b}(a) = \mathfrak{b}(a) \cap \bigvee\{\widetilde{p}_x \colon p_x \in L\} = \bigvee\{\widetilde{p}_x \colon p_x \in \mathfrak{b}(a)\}.$$

Since $a \in \mathfrak{b}(a)$, we conclude that a is a meet of a-regular elements, and since $a \neq 1$, this meet is non-void.

(3) $\Rightarrow$ (4): Let $S \subseteq L$ be an arbitrary sublocale. For an $a \in S$ set

$$a' = \bigwedge \{p_x \colon a \leq p_x \in S\}.$$

We have to prove that $a = a'$ (then $S = \bigvee \{\widetilde{p}_x \colon p_x \in S\}$ holds).

If not, we have $a < a'$ and $b = a' \to a \neq 1$, and there exists a b-regular element p. Since $a \in \mathfrak{b}(a)$, $b = a' \to a \in \mathfrak{b}(a)$ and as $p \in \mathfrak{b}(b) \subseteq \mathfrak{b}(a)$ we have $a \leq p$ and by the definition of a', $a' \leq p$. Thus,

$$p \to b = p \to (a' \to a) = (p \wedge a') \to a = a' \to a = b,$$

and $p = (p \to b) \to b = b \to b = 1$, a contradiction.

Finally, the fact that the representation is precise follows from Proposition 2.2.1.

(4) $\Rightarrow$ (5): First, $\gamma(\mu(S)) = S$ is in the equality $S = \bigvee \{\widetilde{p}_x \colon p_x \in S\}$. Next, obviously $Y \subseteq \mu(\gamma(Y))$; on the other hand, if $p_x \in \gamma(Y) = \bigvee \{\widetilde{p}_y \colon y \in Y\}$ then $p_x = \bigwedge_{y \in A} p_y$ for some $A \subseteq Y$ and therefore there is an $y \in A \subseteq Y$ such that $p_x = p_y$. $\square$

Lemma 3.2.2. *The following are equivalent for a space X:*

(1) *Space X is weakly scattered.*

(2) *For $U \neq X$ in $\Omega(X)$ and $A = X \smallsetminus U$, there is an $x \in A$ such that*

$$x \in X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}.$$

(3) *For $U \neq X$ in $\Omega(X)$, there is a U-regular element $X \smallsetminus \overline{\{x\}}$.*

Proof: (1) $\Rightarrow$ (2): Let X be weakly scattered and let $\emptyset \neq A \subseteq X$. Choose an open V such that $x \in V \cap A \subseteq \overline{\{x\}}$. Then we have

$$\emptyset = V \cap A \cap (X \smallsetminus \overline{\{x\}}) = V \cap (A \smallsetminus \overline{\{x\}}),$$

and hence $x \in (X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}) \cap A$.

(2) $\Rightarrow$ (1): If $x \in (X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}) \cap A$ for a nonempty closed A set $V = X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}$ to obtain $x \in V \cap A \subseteq \overline{\{x\}}$.

(2) $\Rightarrow$ (3): Let $U \neq X$ in $\Omega(X)$ and $A = X \smallsetminus U$. We need an x such that

$$U \subseteq X \smallsetminus \overline{\{x\}} \qquad \text{and} \qquad \big((X \smallsetminus \overline{\{x\}}\,) \to U\big) \to U \subseteq X \smallsetminus \overline{\{x\}}$$

(the latter is the essential inclusion from the equality). In other words,

$$(*) \qquad x \notin U \text{ that is, } x \in A \qquad \text{and} \qquad x \notin ((X \smallsetminus \overline{\{x\}}) \to U) \to U.$$

We have $V \to U = \mathrm{int}((X \smallsetminus V) \cup U)$, hence $\big(X \smallsetminus \overline{\{x\}}\,\big) \to U = \mathrm{int}\big(\,\overline{\{x\}} \cup U\big)$ and

$$\begin{aligned}
\big(\big(X \smallsetminus \overline{\{x\}}\,\big) \to U\big) \to U &= \mathrm{int}\big(\,\overline{\{x\}} \cup U\big) \to U \\
&= \mathrm{int}\big(X \smallsetminus \mathrm{int}\big(\,\overline{\{x\}} \cup U\big) \cup U\big) \\
&= \mathrm{int}\Big(\overline{X \smallsetminus (\overline{\{x\}} \cup U)} \cup U\Big) \\
&= \mathrm{int}\Big(\overline{(X \smallsetminus \overline{\{x\}}) \cap (X \smallsetminus U)} \cup U\Big) \\
&= \mathrm{int}\Big(\overline{A \smallsetminus \overline{\{x\}}} \cup (X \smallsetminus A)\Big).
\end{aligned}$$

Thus, $(*)$ transforms to stating that there is an $x \in A$ such that

$$x \notin \mathrm{int}\Big(\overline{A \smallsetminus \overline{\{x\}}} \cup (X \smallsetminus A)\Big),$$

that is,

$$x \in \overline{\big(X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}\big) \cap A},$$

which is guaranteed by the hypothesis.

(3) $\Rightarrow$ (2): By the proof of the previous implication, we know that there is an $x \in A$ such that

$$x \in \overline{\big(X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}\big) \cap A}.$$

We will show now that the x is in fact in the set under closure. First, observe that

$$\Big(X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}\Big) \cap A \subseteq \overline{\{x\}}.$$

Indeed, if $a \in \big(X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}\big)$ then $a \notin \overline{A \smallsetminus \overline{\{x\}}}$ and hence $a \notin (A \smallsetminus \overline{\{x\}})$, that is, $a \in \overline{\{x\}}$ (and A is closed). Now denote for a moment $V = X \smallsetminus \overline{A \smallsetminus \overline{\{x\}}}$. We have $V \cap A \neq \emptyset$ (since it has a nonempty closure); and hence $x \notin V$ makes $\overline{\{x\}} \cap V = \emptyset$ and a contradiction $\emptyset \neq A \cap V \subseteq \overline{\{x\}} \subseteq X \smallsetminus V$. $\square$

Theorem 3.3. *The following are equivalent for a T_D-space X:*

(1) $\mathsf{S}(\Omega(X))$ *is Boolean.*
(2) *All sublocales of $\Omega(X)$ are induced and precisely represent subspaces of X.*
(3) *Space X is scattered.*
(4) *Each Boolean* (*that is, minimal*) *sublocale is complemented.*

Proof: This follows directly from Observation 1.7.1, Proposition 3.2.1 and Lemma 3.2.2. $\square$

Note 3.3.1. Since T_D is a necessary condition for the precise representation, the preceding theorem can be reformulated to a (perhaps more elegant) statement:

> *All sublocales of $\Omega(X)$ are induced and precisely represent subspaces of X if and only if X is T_D and scattered.*

The results of [11] and [7] concern the booleanness even for the non-T_D case and the not necessarily precise representation of subspaces by sublocales. Thus, the scope is broader while, on the other hand, the nature of the representation of subspaces by sublocales is not quite specified. If we wish to have this precise, T_D is a condition sine qua non. Furthermore, however, having to assume this axiom makes the situation much simpler because of the covered primeness of the $X \smallsetminus \overline{\{x\}}$.

Let us point out that the importance of the axiom T_D in point-free topology, in particular in fitting together spatial and point-free facts, is sometimes underestimated. It appeared, first, in [1], in a technical context. But in the same year, in [12], one of the authors proved that under this condition the lattice of open sets determined the space (one of the first results of this kind). It can be claimed that the importance of T_D is in the rank of that of sobriety. The two properties are closely related, in fact they are, in a sense, dual to each other, see [3], [6] and also the exercise in [5, II.1.7]: while sobriety states that one cannot *add* a point without changing the topology, T_D asserts that one cannot *subtract* a point. And the fact crucial in this paper, namely that T_D is equivalent with precise representation of subspaces by sublocales can be viewed in the general setting as similarly important as the sobriety standing for precise representation of continuous maps by localic ones, see [5], [9].

Note. While working on the present paper we learned the sad news that Harold Simmons, the author of the fundamental theorem discussed here, passed away. He was a great and resourceful mathematician, and a very nice person. Since we are working, mostly, in point-free topology, we would like to mention, besides the scatteredness theorems, and among his many other achievements, also his role in the development of separation theory, notably in subfitness ("conjunctivity", see [10]). He will be missed.

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