Orthomodular lattices that are horizontal sums of Boolean algebras

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Abstract. The paper deals with orthomodular lattices which are so-called horizontal sums of Boolean algebras. It is elementary that every such orthomodular lattice is simple and its blocks are just these Boolean algebras. Hence, the commutativity relation plays a key role and enables us to classify these orthomodular lattices. Moreover, this relation is closely related to the binary commutator which is a term function. Using the class $\mathcal{H}$ of horizontal sums of Boolean algebras, we establish an identity which is satisfied in the variety generated by $\mathcal{H}$ but not in the variety of all orthomodular lattices. The concept of ternary discriminator can be generalized for the class $\mathcal{H}$ in a modified version. Finally, we present several results on varieties generated by finite subsets of finite members of $\mathcal{H}$.

Keywords: orthomodular lattice; horizontal sum; commuting elements; Boolean algebra

Classification: 06C15, 06C20, 06E75

Orthomodular lattices form an algebraic semantics of the logic of quantum mechanics and hence it is important to study their properties. The aim of our paper is to get some insight in the structure and properties of those orthomodular lattices which are so-called horizontal sums of Boolean algebras. These orthomodular lattices are simple and hence also subdirectly irreducible. Denote by $\mathcal{H}$ the class of horizontal sums of Boolean algebras. Some orthomodular lattices which do not belong to $\mathcal{H}$ are pastings of Boolean algebras, i.e. contrary to horizontal sums, some pairs of Boolean subalgebras have some atoms and the corresponding coatoms in common. It is shown that also such orthomodular lattices may belong to the variety $\mathrm{Var}(\mathcal{H})$ generated by $\mathcal{H}$. Because of this fact one would expect that $\mathrm{Var}(\mathcal{H})$ is really large and the question arises if it coincides with the variety of all orthomodular lattices. We apply the commutativity relation as well as the commutator in order to characterize the class $\mathcal{H}$. Moreover, using the commutator

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which is binary term, we obtain an identity which is satisfied in $\mathrm{Var}(\mathcal{H})$ but not in every orthomodular lattice showing that the variety of orthomdular lattices is not generated by $\mathcal{H}$. We, finally, show that the variety generated by a finite set of finite members of $\mathcal{H}$ is semisimple and directly representable.

Recall that an *orthomodular lattice* is an algebra $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $\mathbf{L}$ satisfies the following identities:

$$
\begin{aligned}
x \vee x' &\approx 1,\\
x \wedge x' &\approx 0,\\
(x \vee y)' &\approx x' \wedge y',\\
(x \wedge y)' &\approx x' \vee y',\\
(x')' &\approx x,\\
x \vee y &\approx x \vee ((x \vee y) \wedge x').
\end{aligned}
$$

The last identity is called the *orthomodular law*. Let $\mathcal{O}$ denote the variety of orthomodular lattices and $\mathcal{B}$ the variety of Boolean algebras.

Of course, Boolean algebras are exactly the distributive orthomodular lattices. On the other hand, a kind of distributivity can be observed in every orthomodular lattice. Namely, if we consider the so-called *Sasaki operation*, see [1], $x \odot y := (x \vee y') \wedge y$ then every orthomodular lattice satisfies the identity

$$(x \vee y) \odot z \approx (x \odot z) \vee (y \odot z)$$

as can be seen by applying the so-called *Foulis–Holland theorem*, see e.g. [6].

Moreover, we can show that the original definition of an orthomodular lattice given above is redundant. Namely, we need not assume that the underlying lattice is bounded or that the unary operation "$'$" is a complementation. Both of these properties in fact follow from the orthomodular law, see the following

Lemma 1. *Let $\mathbf{L} = (L, \vee, \wedge, ')$ be a nonempty lattice with an antitone involution "$'$" satisfying the orthomodular law. Then $\mathbf{L}$ is an orthomodular lattice.*

Proof: Let $a, b \in L$. Then the orthomodular law implies

$$a \le a \vee b = b \vee ((a \vee b) \wedge b') \le b \vee b'.$$

This shows that $b \vee b'$ is the greatest element of $\mathbf{L}$. Since "$'$" is an antitone involution, the dual form of the orthomodular law is valid in $\mathbf{L}$, too and using this law we obtain similarly that $b \wedge b'$ is the least element of $\mathbf{L}$. Hence $\mathbf{L}$ is bounded and b' a complement of b. It is elementary to check the De Morgan laws. $\square$

For orthomodular lattices, the concept of commuting elements is introduced as follows: Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and $a, b \in L$. Then a and b are said to *commute* with each other, in signs $a \mathrel{\mathrm{C}} b$, if $a = (a \wedge b) \vee (a \wedge b')$.

This concept enables us to consider the so-called blocks of an orthomodular lattice $\mathbf{L}$. A *block* is a maximal subset of mutually commuting elements. It turns out that the blocks are exactly the maximal Boolean subalgebras, cf. e.g. [6]. We have the following lemma:

Lemma 2 (cf. e.g. [6]). *If* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *is an orthomodular lattice and* $a, b \in L$ *then*

(i) $a \mathrel{\mathrm{C}} b$ *if and only if there exists a Boolean subalgebra* $\mathbf{B} = (B, \vee, \wedge, ', 0, 1)$ *of* $\mathbf{L}$ *with* $a, b \in B$*;*
(ii) *if* $a \leq b$ *then* $a \mathrel{\mathrm{C}} b$*;*
(iii) *the lattice* $\mathbf{L}$ *is a Boolean algebra if and only if* $x \mathrel{\mathrm{C}} y$ *for all* $x, y \in L$*.*

Hence for all $x, y \in L$ we have $0 \mathrel{\mathrm{C}} x$ and $1 \mathrel{\mathrm{C}} x$ and, moreover, that $x \mathrel{\mathrm{C}} y$ implies $y \mathrel{\mathrm{C}} x$ and $x \mathrel{\mathrm{C}} y'$.

The *center* $\mathrm{C}(\mathbf{L})$ of an orthomodular lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ is defined by

$$\mathrm{C}(\mathbf{L}) := \{x \in L \colon x \mathrel{\mathrm{C}} y \text{ for all } y \in L\}.$$

We have the following

Lemma 3 (cf. e.g. [6]). *If* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *is an orthomodular lattice then:*

- L *is the set-theoretic union of the blocks of* $\mathbf{L}$*.*
- *Every Boolean subalgebra of* $\mathbf{L}$ *is included in a block of* $\mathbf{L}$*.*

The central concept of the present paper is the following one:

Definition 4. Let $(\mathbf{L}_i;\, i \in I) = ((L_i, \vee_i, \wedge_i, {}'_i, 0, 1);\, i \in I)$ be a nonempty family of orthomodular lattices such that $L_i \cap L_j = \{0, 1\}$ for all $i, j \in I$ with $i \neq j$ and put $L := \bigcup\limits_{i \in I} L_i$. Define binary operations $\vee, \wedge$ and a unary operation "$\,'\,$" on L as follows:

$$x \vee y := \begin{cases} x \vee_i y & \text{if there exists some } i \in I \text{ with } x, y \in L_i, \\ 1 & \text{otherwise;} \end{cases}$$

$$x \wedge y := \begin{cases} x \wedge_i y & \text{if there exists some } i \in I \text{ with } x, y \in L_i, \\ 0 & \text{otherwise;} \end{cases}$$

$$x' := x'_i \text{ if } x \in L_i.$$

Then "$\vee$", "$\wedge$" and "$\,'\,$" are well-defined and $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ is an orthomodular lattice, called the *horizontal sum* of the $\mathbf{L}_i$, $i \in I$. Moreover, every $\mathbf{L}_i$ is a subalgebra of $\mathbf{L}$. Let $\mathcal{H}$ denote the class of all horizontal sums of Boolean algebras and for every class $\mathcal{K}$ of algebras of the same type $\mathrm{Var}(\mathcal{K})$ the variety generated by $\mathcal{K}$.

It is easy to check that if an orthomodular lattice $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ belongs to $\mathcal{H}$, $a \in L\setminus\{1\}$ and $b \in L\setminus\{0\}$ then $([0,a], \vee, \wedge)$ and $([b,1], \vee, \wedge)$ are distributive.

The following theorem provides a characterization of members of $\mathcal{H}$.

Theorem 5. *Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice. Then $\mathbf{L}$ is a horizontal sum of Boolean algebras if and only if* (i) *and* (ii) *hold for all $x, y, z \in L$:*

(i) *if $x \mathrel{\mathrm{C}} y$, $y \mathrel{\mathrm{C}} z$ and not $x \mathrel{\mathrm{C}} z$ then $y \in \{0, 1\}$;*
(ii) *if not $x \mathrel{\mathrm{C}} y$ then $x \vee y = 1$ and $x \wedge y = 0$.*

Proof: Let $a, b, c \in L$.

First assume $\mathbf{L}$ to be the horizontal sum of the Boolean algebras $\mathbf{B}_i = (B_i, \vee, \wedge, ', 0, 1)$, $i \in I$.

(i) Assume $a \mathrel{\mathrm{C}} b \mathrel{\mathrm{C}} c$ and not $a \mathrel{\mathrm{C}} c$. Suppose there exists no $i \in I$ with $a, b \in B_i$. Then there exists no $i \in I$ with $a, b' \in B_i$ and hence $a \wedge b = a \wedge b' = 0$ from which we conclude $a \neq 0 = 0 \vee 0 = (a \wedge b) \vee (a \wedge b')$, i.e. not $a \mathrel{\mathrm{C}} b$, a contradiction. This shows that there exists some $j \in I$ with $a, b \in B_j$. Analogously, there exists some $k \in I$ with $b, c \in B_k$. Since not $a \mathrel{\mathrm{C}} c$ we have $j \neq k$ which yields $b \in B_j \cap B_k = \{0, 1\}$.

(ii) If not $a \mathrel{\mathrm{C}} b$ then there exists no $i \in I$ with $a, b \in B_i$ and hence $a \vee b = 1$ and $a \wedge b = 0$.

Conversely, assume (i) and (ii) to hold. Let B and D be two different blocks of $\mathbf{L}$, $d \in B \setminus D$ and $e \in B \cap D$. Then there exists some $f \in D$ such that not $d \mathrel{\mathrm{C}} f$. Now we have $d \mathrel{\mathrm{C}} e \mathrel{\mathrm{C}} f$ and not $d \mathrel{\mathrm{C}} f$ and hence $e \in \{0, 1\}$. This shows $B \cap D = \{0, 1\}$. If there exists some block E of $\mathbf{L}$ with $a, b \in E$ then $a \vee_L b = a \vee_E b$ and $a \wedge_L b = a \wedge_E b$. If there exists no block F of $\mathbf{L}$ with $a, b \in F$ then $a \mathrel{\mathrm{C}} b$ does not hold and hence $a \vee b = 1$ and $a \wedge b = 0$. $\square$

We want to mention another formulation of condition (ii) of Theorem 5. For this purpose we first prove

Lemma 6. *Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and $a, b \in L$. Then $a \mathrel{\mathrm{C}} b$ is equivalent to any single distributivity condition of the form $(c \vee d) \wedge e = (c \wedge e) \vee (d \wedge e)$ or $(c \wedge d) \vee e = (c \vee e) \wedge (d \vee e)$ where c, d, e are three of the four elements a, a', b, b'.*

Proof: According to the definition of C, $a \mathrel{\mathrm{C}} b$ is equivalent to $(b \vee b') \wedge a = (b \wedge a) \vee (b' \wedge a)$. If $(a \vee b) \wedge a' = (a \wedge a') \vee (b \wedge a')$ then, according to orthomodularity, $a' = (a' \wedge b') \vee (a' \wedge (a \vee b)) = (a' \wedge b) \vee (a' \wedge b')$, i.e. $a' \mathrel{\mathrm{C}} b$ which implies $a \mathrel{\mathrm{C}} b$. Hence also $(a \vee b) \wedge a' = (a \wedge a') \vee (b \wedge a')$ is equivalent to $a \mathrel{\mathrm{C}} b$. Using the fact that $a \mathrel{\mathrm{C}} b$ implies $b \mathrel{\mathrm{C}} a$ and $a \mathrel{\mathrm{C}} b'$, we obtain that $a \mathrel{\mathrm{C}} b$ is equivalent to any single distributivity condition of the form $(c \vee d) \wedge e = (c \wedge e) \vee (d \wedge e)$ where c, d, e

are three of the four elements a, a', b, b'. Applying De Morgan laws completes the proof of the lemma. □

Remark 7. Condition (ii) of Theorem 5 can now be reformulated in the following way:

(ii') If either $x \vee y \neq 1$ or $x \wedge y \neq 0$ then $x' \vee (x \wedge y) = x' \vee y$ (or $x' \wedge (x \vee y) = x' \wedge y$).

From Lemma 6 we conclude that an orthomodular lattice is a Boolean algebra if and only if it satisfies the following identities:

$$x' \vee (x \wedge y) \approx x' \vee y, \tag{1}$$

$$x' \wedge (x \vee y) \approx x' \wedge y. \tag{2}$$

In [4] the following more general result was proved: A nonempty lattice $(L, \vee, \wedge, ')$ with a unary operation "$'$" is a Boolean algebra if and only if it satisfies (1) and (2). For similar characterizations of Boolean algebras cf. [3].

Now we are going to study properties of the class $\mathcal{H}$. The following lemma turns out to be very useful.

Lemma 8. *Let the orthomodular lattice* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *be the horizontal sum of orthomodular lattices* $\mathbf{L}_i = (L_i, \vee, \wedge, ', 0, 1)$, $i \in I$, $j, k \in I$ *with* $j \neq k$, $a \in L_j \setminus \{0, 1\}$ *and* $b \in L_k \setminus \{0, 1\}$. *Then there exists no* $i \in I$ *with* $a, b \in L_i$.

Proof: If there would exist some $i \in I$ with $a, b \in L_i$ then either $i \neq j$ or $i \neq k$. In the first case we would obtain $a \in L_i \cap L_j = \{0, 1\}$ contradicting $a \in L_j \setminus \{0, 1\}$ whereas the second case would yield $b \in L_i \cap L_k = \{0, 1\}$ contradicting $b \in L_k \setminus \{0, 1\}$. This completes the proof. □

Note that $\mathcal{H}$ is not closed under direct products (see Theorem 12) and hence $\mathcal{H}$ is a proper subclass of $\mathrm{Var}(\mathcal{H})$.

It is well-known that lattices are congruence distributive and orthomodular lattices are congruence permutable and congruence regular.

It is clear that $\mathcal{H}$ is closed with respect to forming subalgebras. That it is also closed with respect to forming homomorphic images follows from the following proposition.

Proposition 9. *Let* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *be the horizontal sum of the orthomodular lattices* $\mathbf{L}_i = (L_i, \vee, \wedge, ', 0, 1)$, $i \in I$, *and assume that at least two* L_i *have more than two elements. Then* $\mathbf{L}$ *is simple.*

Proof: Let $\Theta \in \operatorname{Con} \mathbf{L}$ with $\Theta \neq \{(x, x) : x \in L\}$. Then there exists some $(a, b) \in \Theta$ with $a \neq b$. If there exists no $i \in I$ with $a, b \in L_i$ then $a \vee b = 1$ and $a \wedge b = 0$ and hence $0 = a \wedge b \,\Theta\, a \wedge a = a = a \vee a \,\Theta\, a \vee b = 1$ which implies $x = x \vee 0 \,\Theta\, x \vee 1 = 1$ for all $x \in L$, i.e. $\Theta = L^2$. Now assume there exists some

$j \in I$ with $a, b \in L_j$. Since $\mathbf{L}_j$ is congruence regular there exists some $c \in L_j \setminus \{1\}$ with $(c, 1) \in \Theta$. If $c = 0$ then $\Theta = L^2$ as before. Now assume $c \neq 0$. Because of $(c, 1) \in \Theta$ we have $(c', 0) \in \Theta$. According to our assumption there exists some $k \in I \setminus \{j\}$ with $|L_k| > 2$. Let $d \in L_k \setminus \{0, 1\}$. According to Lemma 8 there exists no $i \in I$ with $c, d \in L_i$ which shows $c \wedge d = 0$ and $c' \vee d = 1$. Now we have

$$0 = c \wedge d \,\Theta\, 1 \wedge d = d = 0 \vee d \,\Theta\, c' \vee d = 1$$

which implies $\Theta = L^2$ as before, i.e. in any case $\mathbf{L}$ is simple. $\square$

Remark 10. Proposition 9 implies that every member of $\mathcal{H} \setminus \mathcal{B}$ is simple and hence also subdirectly irreducible.

The class $\mathcal{H}$ is not closed with respect to forming direct products as can be seen from the following Proposition.

Proposition 11. *A direct product* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *of a nonempty family* $(\mathbf{L}_i;\ i \in I) = ((L_i, \vee, \wedge, ', 0, 1); i \in I)$ *of members of* $\mathcal{H}$ *belongs to* $\mathcal{H}$ *if and only if either there exists some* $j \in I$ *with* $|L_i| = 1$ *for all* $i \in I \setminus \{j\}$ *or if* $\mathbf{L}_i \in \mathcal{B}$ *for all* $i \in I$*.*

Proof: If there exists some $j \in I$ with $|L_i| = 1$ for all $i \in I \setminus \{j\}$ then $\mathbf{L} \cong \mathbf{L}_j \in \mathcal{H}$ and hence $\mathbf{L} \in \mathcal{H}$. If $\mathbf{L}_i \in \mathcal{B}$ for all $i \in I$ then $\mathbf{L} \in \mathcal{B} \subseteq \mathcal{H}$. Now assume there exist $k, s \in I$ with $k \neq s$, $|L_k| > 1$ and $\mathbf{L}_s \notin \mathcal{B}$. Then L_s has two elements not commuting with each other. Let $a = (a_i; i \in I), b = (b_i; i \in I) \in L$ such that $a_k = b_k = 0$ and not $a_s \,\mathrm{C}\, b_s$. Then $a \vee b \neq 1$ since $a_k \vee b_k = 0 \vee 0 = 0 \neq 1$. Now $a \,\mathrm{C}\, b$ would imply $a_s \,\mathrm{C}\, b_s$, a contradiction. Hence we do not have $a \,\mathrm{C}\, b$, i.e. $\mathbf{L} \notin \mathcal{H}$ according to Theorem 5. $\square$

We can summarize the above results as follows:

Theorem 12. *The class* $\mathcal{H}$ *is closed with respect to forming subalgebras and homomorphic images, but not with respect to forming direct products.*

Let us recall the concept of a commutator which is a binary term function closely related to the commutativity relation C.

Definition 13. Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice. Then the term function

$$c(x, y) := (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

is called the *commutator*.

Lemma 14 (cf. e.g. [6])**.** *Let* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *be an orthomodular lattice and* $a, b \in L$*. Then* $a \,\mathrm{C}\, b$ *if and only if* $c(a, b) = 1$*.*

It is clear that if $\mathbf{L} = (L, \vee, \wedge, ', 0, 1) \in \mathcal{H}$, $|L| > 1$ and $a, b \in L$ then $a \, \mathrm{C} \, b$ does not hold if and only if $c(a, b) = 0$.

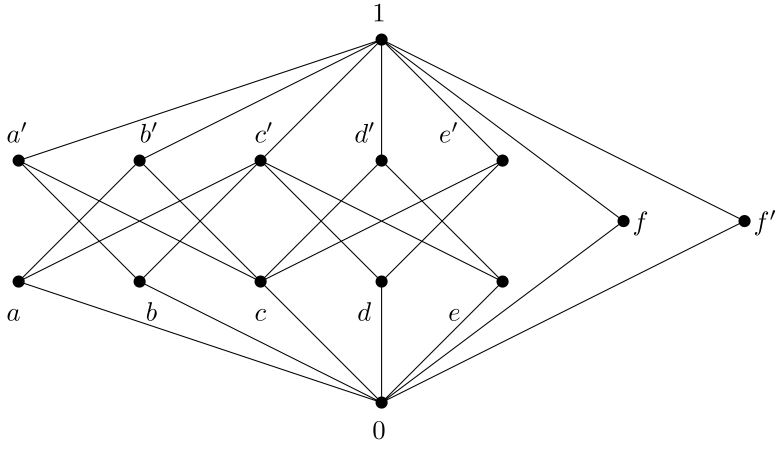
The commutator enables us to distinguish the variety $\mathrm{Var}(\mathcal{H})$ from $\mathcal{O}$. Namely, we present a simple identity which is valid in $\mathcal{H}$ and hence in $\mathrm{Var}(\mathcal{H})$, but not in $\mathcal{O}$. This shows that $\mathrm{Var}(\mathcal{H})$ is a proper subvariety of $\mathcal{O}$. Let us recall that the members of $\mathcal{H} \setminus \mathcal{B}$ are subdirectly irreducible members of $\mathrm{Var}(\mathcal{H})$.

Theorem 15. *The variety* $\mathrm{Var}(\mathcal{H})$ *satisfies the identity* $c(c(x, y), z) \approx 1$.

Proof: Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1) \in \mathcal{H}$ and $x, y, z \in L$. If $x \, \mathrm{C} \, y$ then $c(c(x, y), z) = c(1, z) = 1$. If not $x \, \mathrm{C} \, y$ then $c(c(x, y), z) = c(0, z) = 1$. Hence $\mathbf{L}$ and therefore $\mathcal{H}$ and the variety $\mathrm{Var}(\mathcal{H})$ satisfies the identity $c(c(x, y), z) \approx 1$. □

Corollary 16. *The class* $\mathcal{H}$ *does not generate the variety* $\mathcal{O}$ *and hence* $\mathrm{Var}(\mathcal{H}) \neq \mathcal{O}$.

Proof: The orthomodular lattice $\mathbf{L}$ with the Hasse diagram



is not a member of $\mathcal{H}$ since it does not satisfy the identity $c(c(x, y), z) \approx 1$ because of $c(c(a, e), f) = c(c, f) = 0 \neq 1$. Hence, $\mathbf{L} \in \mathcal{O}$, but $\mathbf{L} \notin \mathrm{Var}(\mathcal{H})$. □

We can generalize the example from the previous proof in the following way:

Lemma 17. *Assume that the orthomodular lattice* $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$ *is the horizontal sum of the orthomodular lattices* $\mathbf{L}_1 = (L_1, \vee, \wedge, ', 0, 1)$ *and* $\mathbf{L}_2 = (L_2, \vee, \wedge, ', 0, 1)$, *that* $\mathbf{L}_1$ *does not satisfy condition* (ii) *of Theorem* 5 *and that* $|L_2| > 2$. *Then* $\mathbf{L}$ *does not satisfy the identity* $c(c(x, y), z) \approx 1$.

Proof: Since $\mathbf{L}_1$ does not satisfy condition (ii) of Theorem 5 there exist $a, b \in L_1$ not satisfying $a \, \mathrm{C} \, b$ such that either $a \vee b \neq 1$ or $a \wedge b \neq 0$. If $a \vee b \neq 1$ then $c(a, b) \geq a' \wedge b' > 0$ and if $a \wedge b \neq 0$ then $c(a, b) \geq a \wedge b > 0$. Since $a \, \mathrm{C} \, b$ does not hold we have $c(a,b) \neq 1$. Together we obtain $c(a,b) \in L_1 \setminus \{0,1\}$. Because of $|L_2| > 2$ there exists some $d \in L_2 \setminus \{0,1\}$. According to Lemma 8 there exists no $i \in \{1,2\}$ with $c(a,b), d \in L_i$. This shows $c(c(a,b),d) = 0 \neq 1$, i.e. $\mathbf{L}$ does not satisfy the identity $c(c(x,y),z) \approx 1$. □

Using the term $c(x,y)$ we can establish a connection between $\mathcal{H}$ and the so-called ternary discriminator. However, we must modify the concept of a ternary discriminator in such a way that the equality relation is replaced by the commutativity relation C.

Recall that a ternary function t satisfying

$$t(x,y,z) = \begin{cases} z & \text{if } x = y, \\ x & \text{otherwise,} \end{cases}$$

is called a *ternary discriminator.*

From the above considerations we obtain

Proposition 18. *If in a member of $\mathcal{H}$ one defines*

$$t(x,y,z) := ((c(x,y))' \wedge x) \vee (c(x,y) \wedge z)$$

then t satisfies

$$t(x,y,z) = \begin{cases} z & \text{if } x \,\mathrm{C}\, y, \\ x & \text{otherwise.} \end{cases}$$

In particular, t satisfies the identity $t(x,x,z) \approx z$.

It is well-known that a nontrivial Boolean algebra has a ternary discriminator as a term function if and only if it has two elements, i.e. it is subdirectly irreducible. Such a discriminator is given by

$$t(x,y,z) = ((x \oplus y) \wedge x) \vee ((x \oplus y \oplus 1) \wedge z)$$

where $\oplus$ denotes the symmetric difference (i.e. $x \oplus y := (x \wedge y') \vee (x' \wedge y)$). Hence, the variety of Boolean algebras is a so-called discriminator variety.

As mentioned above, all algebras belonging to $\mathcal{H} \setminus \mathcal{B}$ are subdirectly irreducible members of the variety $\mathrm{Var}(\mathcal{H})$. It is well-known that the variety of orthomodular lattices is not a discriminator variety, i.e. there does not exist a ternary discriminator as a term function on its subdirectly irreducible members. Hence, it is a natural question if a certain modification of the notion of a ternary discriminator on algebras $\mathbf{L}$ belonging to $\mathcal{H}$ can be introduced. Using the ternary function $t(x,y,z)$ from Proposition 18 we obtain such a function. Although this term function assumes the constant value z if x and y belong to the same block (which is a Boolean algebra) of $\mathbf{L}$, it assumes the value x if x and y are taken from different blocks of $\mathbf{L}$. Hence, this generalized discriminator discerns different blocks in a similar way as the ternary discriminator discerns different elements.

In the sequel we use the following corollary of the famous Jónsson's lemma:

Proposition 19 (cf. [5]). *If $\mathcal{K}$ is a finite set of finite algebras of the same type and* $\mathrm{Var}(\mathcal{K})$ *is congruence distributive then every subdirectly irreducible member of* $\mathrm{Var}(\mathcal{K})$ *belongs to* $\mathrm{H\,S}(\mathcal{K})$.

Let $\mathcal{H}_{\mathrm{fin}}$ denote the class of all finite members of $\mathcal{H}$. Recall that a variety is called *semisimple* if every of its members is isomorphic to a subdirect product of simple algebras, it is called *finitely generated* if it is generated by finitely many finite members and it is called *directly representable* if it is finitely generated and has (up to isomorphism) only finitely many finite directly indecomposable members.

Theorem 20. *Let $\mathcal{K}$ be a finite subset of $\mathcal{H}_{\mathrm{fin}}$ and $\mathbf{L}$ a nontrivial subdirectly irreducible member of* $\mathrm{Var}(\mathcal{K})$. *Then* (i)–(iii) *hold:*

(i) *The member $\mathbf{L}$ is a two-element Boolean algebra or $\mathbf{L} \in \mathcal{H} \setminus \mathcal{B}$.*
(ii) *The variety* $\mathrm{Var}(\mathcal{K})$ *is semisimple.*
(iii) *The variety* $\mathrm{Var}(\mathcal{K})$ *is directly representable.*

Proof: (i) Let $\mathbf{L} = (L, \vee, \wedge, ', 0, 1)$. According to Proposition 19, $\mathbf{L} \in \mathrm{H\,S}(\mathcal{K})$ and hence $\mathbf{L} \in \mathrm{H\,S}(\mathcal{H})$. Because of the remark before Proposition 9, $\mathrm{H\,S}(\mathcal{H}) \subseteq \mathcal{H}$. Hence $\mathbf{L} \in \mathcal{H}$. But then either $\mathbf{L} \in \mathcal{B}$ and in this case $|L| = 2$, or $\mathbf{L} \in \mathcal{H} \setminus \mathcal{B}$.

(ii) This follows from Proposition 9, from (i) and from the fact that a variety is semisimple if and only if every of its subdirectly irreducible members is simple, cf. [2].

(iii) This follows from (ii) and from the fact that a finitely generated congruence distributive variety is directly representable if and only if it is congruence permutable and semisimple, cf. [2]. □

Recall that a class of finite algebras of the same type closed under forming of subalgebras, homomorphic images and finite direct products is a *pseudovariety*. For every class $\mathcal{K}$ of finite algebras of the same type let $\mathrm{PVar}(\mathcal{K})$ denote the pseudovariety generated by $\mathcal{K}$.

Theorem 21. *Let $\mathbf{L}$ be a nontrivial subdirectly irreducible member of the pseudovariety* $\mathrm{PVar}(\mathcal{H}_{\mathrm{fin}})$. *Then $\mathbf{L}$ is a two-element Boolean algebra or $\mathbf{L} \in \mathcal{H}_{\mathrm{fin}} \setminus \mathcal{B}$.*

Proof: Since $\mathbf{L} \in \mathrm{PVar}(\mathcal{H}_{\mathrm{fin}})$ there exist $\mathbf{L}_1, \dots, \mathbf{L}_n \in \mathcal{H}_{\mathrm{fin}}$ such that $\mathbf{L} \in \mathrm{PVar}(\{\mathbf{L}_1, \dots, \mathbf{L}_n\})$. Now $\mathrm{PVar}(\{\mathbf{L}_1, \dots, \mathbf{L}_n\}) \subseteq \mathrm{Var}(\{\mathbf{L}_1, \dots, \mathbf{L}_n\})$ and we can apply Theorem 20. □

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