Sequentially Right Banach spaces of order p

Mahdi Dehghani, Mohammad B. Dehghani,
Mohammad S. Moshtaghioun

Abstract. We introduce and study two new classes of Banach spaces, the so-called sequentially Right Banach spaces of order p, and those defined by the dual property, the sequentially Right* Banach spaces of order p for $1 \leq p \leq \infty$. These classes of Banach spaces are characterized by the notions of L_p-limited sets in the corresponding dual space and R_p^* subsets of the involved Banach space, respectively. In particular, we investigate whether the injective tensor product of a Banach space X and a reflexive Banach space Y has the sequentially Right property of order p when X enjoys this property.

Keywords: Right topology; sequentially Right Banach space; pseudo weakly compact operator; Pełczyński's property (V) of order p; limited p-converging operator; p-Gelfand–Phillips property; reciprocal Dunford–Pettis property of order p

Classification: 46B20, 47L05, 46B25

1. Introduction and preliminaries

For each Banach space X there is a locally convex topology on X, called the Right topology. This topology is obtained as the restriction to X of the Mackey topology of X^{**}, and it coincides with the topology of uniform convergence on absolutely convex weakly compact subsets of the dual space X^* of X. Sequentially Right Banach spaces are introduced and considered in [28], [29]. The class of Right-to-norm sequentially continuous operators between Banach spaces is called pseudo weakly compact operators in [28]. According to [28], [29], a Banach space X is called sequentially Right if every pseudo weakly compact operator on X is weakly compact. It is shown in the just quoted references that every Banach space possessing Pełczyński's property (V) is sequentially Right. However, M. Kačena proves in [24] that not every sequentially Right Banach space has Pełczyński's property (V). Moreover, the relations of sequentially Right Banach spaces with respect to well-known classes of operators and some properties of Banach spaces such as the Dunford–Pettis property, Reciprocal Dunford–Pettis property, Pełczyński's property (V), Dieudonné property, and the reflexivity are investigated in [24].

DOI 10.14712/1213-7243.2020.011

Let X be a Banach space. A bounded subset K of X^* is called an L-set, if each weakly null sequence (x_n) in X tends to 0 uniformly on K, see [15]. G. Emmanuele in [15] characterized Banach spaces not containing l_1 using L-sets. The L-limited subsets in the dual of a Banach space have been defined in [33]. By applying the notion of L-limited subsets in the dual of a Banach space, I. Ghenciu obtained in [19] some characterizations of those Banach spaces which are sequentially Right in terms of some geometric properties of Banach spaces, such as the Gelfand–Phillips property and the Grothendieck property. In particular, he investigated when the projective tensor product of two Banach spaces is sequentially Right. The notion of Right* subsets of Banach spaces and the sequentially Right* property as a dual property with respect to the sequentially Right Banach spaces have been introduced in [7]. More characterizations and properties of Right* sets have been presented in [18].

A sequence (x_n) in a Banach space X is called weakly p-summable with $1 \leq p < \infty$, if for each $x^* \in X^*$, the sequence $(\langle x_n, x^* \rangle)$ is in l_p and (x_n) is said to be weakly p-convergent to $x \in X$ if $(x_n - x) \in l_p^{\mathrm{weak}}(X)$, where $l_p^{\mathrm{weak}}(X)$ denotes the space of all weakly p-summable sequences in X. The Banach space of all (weakly) bounded sequences in X with supremum norm is denoted by $l_\infty^{\mathrm{weak}}(X)$. Moreover, by $c_0^{\mathrm{weak}}(X)$ we represent the closed subspace of $l_\infty^{\mathrm{weak}}(X)$ which contains all weakly null sequences of X. Also, a bounded set K in a Banach space is said to be relatively weakly p-compact, if every sequence in K has a weakly p-convergent subsequence. If we further assume that the limit point of each weakly p-convergent subsequence is in K, then K is called weakly p-compact. A Banach space X is called weakly p-compact if the closed unit ball B_X of X is a weakly p-compact set; cf. [5], [6], [14].

For Banach spaces X and Y, the Banach space of all bounded linear operators (or compact operators) from X into Y is denoted by $L(X,Y)$ (or $K(X,Y)$, respectively). An operator $T \in L(X,Y)$ is called completely continuous if T takes weakly null sequences to norm null sequences. In [4], [5], J. M. F. Castillo and F. Sánchez introduce the ideal of p-converging operators as those operators which transformed weakly p-summable sequences into norm null sequences to define an alternative Dunford–Pettis property, called the Dunford–Pettis property of order p. The ideal of all p-converging operators from X into Y is denoted by $C_p(X,Y)$. Also, in [16], J. H. Fourie and E. D. Zeekoei introduce the class of limited p-converging operators. An operator $T \in L(X,Y)$ is said to be limited p-converging if it maps limited weakly p-summable sequences into norm null sequences. We denote the space of all limited p-converging operators from X into Y by $C_{lp}(X,Y)$. Using the concepts of p-converging and limited p-converging operators as key notions, some other alternative geometric properties of Banach spaces, such as the Schur property of order p, see [10], [34], p-Gelfand–Phillips property, see [16], Pełczyński's property (V) of order p, see [26], and the reciprocal Dunford–Pettis of order p [21], see [20], have been introduced and studied as generalizations of the respective properties.

The reader is referred to [5], [6], [8], [10], [12], [16], [21], [20], [24], [26], [32], [34] and the references therein for more information about these concepts.

The main aim of this paper is to present and study an appropriate p-variant notion of sequentially Right Banach spaces, the so called sequentially Right Banach spaces of order p for $1 \leq p \leq \infty$. Also, we will investigate its dual counterpart, the sequentially Right* Banach spaces of order p. In Section 2, we consider the class of limited p-converging operators as a generalization of pseudo weakly compact operators to introduce the sequentially Right Banach spaces of order p. We will say that a Banach space X has the sequentially Right property of order p (or sequentially Right* property of order p), if X is sequentially Right of order p (or sequentially Right* of order p, respectively). In addition, we will compare these two properties with other well-known isomorphic properties of Banach spaces such as Pełczyński's property (V) (or (V*)) of order p and the reciprocal Dunford–Pettis property of order p. In particular, in Theorem 2.6 we will obtain some characterizations of those Banach spaces which have the sequentially Right property of order p in terms of the class of L_p-limited sets in their corresponding dual spaces.

It is natural to ask whether there exists a non-reflexive Banach space satisfying the sequentially Right property of order p? To answer this question, we first prove that if a Banach space X has the sequentially Right property of order p and Y is a reflexive Banach space such that $L(X, Y^*) = K(X, Y^*)$, then the injective tensor product $X \widehat{\otimes}_\varepsilon Y$ has the sequentially Right property of order p. Next, we will present a class of non-reflexive Banach spaces with the sequentially Right property of order p for some appropriate p.

Section 3 is devoted to introduce and study the sequentially Right* property of order p, as the dual property of the sequentially Right property of order p. Some characterizations of this property are given, and some examples of Banach spaces satisfying the sequentially Right* property of order p are presented.

2. Sequentially Right Banach spaces of order p

A nonempty bounded subset K of a Banach space X is said to be limited (or Dunford–Pettis (DP)), if for every weak*-null (or weakly null, respectively) sequence (x_n^*) in the dual space X^* of X,

$$\lim_{n\to\infty} \sup_{x\in K} |\langle x, x_n^*\rangle| = 0,$$

where $\langle x, x^*\rangle$ denotes the duality between $x \in X$ and $x^* \in X^*$, see [1], [3]. In particular, a sequence $(x_n) \subset X$ is limited if and only if $\langle x_n, x_n^*\rangle \to 0$ for all weak*-null sequences (x_n^*) in X^*. Clearly, every limited subset of X is Dunford–Pettis.

Let X be a Banach space and let $\mathcal{F}$ be the family of all weakly compact subsets of B_{X^*}. For $F \in \mathcal{F}$, define a semi-norm q_F on X^{**} by

$$q_F(x^{**}) = \sup_{x^*\in F} |\langle x^{**}, x^*\rangle|, \qquad x^{**} \in X^{**}.$$

The locally convex topology of X^{**} generated by the family of semi-norms $\{q_F\colon F \in \mathcal{F}\}$ is called the Mackey topology of X^{**}, and is denoted by $\tau(X^{**}, X^*)$. The restriction to X of the Mackey topology is called the Right topology. In general, the Right topology is stronger than the weak topology and weaker than the norm topology; cf. [28].

A sequence (x_n) in a Banach space X is called Right null if $x_n \to 0$ in the Right topology. The following result gives a characterization of Right null sequences.

Proposition 2.1 ([19, Proposition 1]). *A sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and a* DP *set.*

Definition 2.2 ([28]). Let $T\colon X \to Y$ be a linear mapping between Banach spaces. Then T is called pseudo weakly compact if it is Right-to-norm sequentially continuous, that is, T maps Right convergent sequences to norm convergent sequences.

It is clear that every completely continuous operator is pseudo weakly compact. Also, every pseudo weakly compact operator is limited p-converging. Indeed, if $T\colon X \to Y$ is a pseudo weakly compact operator and (x_n) is a limited weakly p-summable sequence in X, then (x_n) is DP and weakly null, and so is Right null, by Proposition 2.1. Hence $\|Tx_n\| \to 0$. It follows that T is limited p-converging. So, if we denote the spaces of all completely continuous operators and pseudo weakly compact operators between Banach spaces X and Y by $C_c(X,Y)$ and $P_{wc}(X,Y)$, respectively, then we have

$$C_c(X,Y) \subseteq P_{wc}(X,Y) \subseteq C_{lp}(X,Y).$$

According to [28], a Banach space X is said to be sequentially Right (has property SR) if for any Banach space Y every pseudo weakly compact operator $T\colon X \to Y$ is weakly compact. In the following definition we introduce the new property, sequentially Right property of order p, which is stronger than the property of being sequentially Right.

Definition 2.3. Let $1 \leq p \leq \infty$. We say that a Banach space X is sequentially Right of order p (has the sequentially Right property of order p (SR_p property)), if for any Banach space Y, every limited p-converging operator $T\colon X \to Y$ is weakly compact.

It is clear that every Banach space satisfying the SR_p property is sequentially Right. Also, we notice that for $1 \leq p < q \leq \infty$, if X has the SR_p property, then X has the SR_q property. In fact, assume that $T \in C_{lq}(X, Y)$ and $(x_n) \in l_p^{\mathrm{weak}}(X)$ is a limited sequence. Then $(x_n) \in l_q^{\mathrm{weak}}(X)$, and so $\|Tx_n\| \to 0$. It follows that $T \in C_{lq}(X, Y)$. Therefore T is weakly compact, thanks to the SR_p property of X.

A classic property of Banach spaces is the Schur property. A Banach space X has the Schur property if every weakly null sequence in X converges in norm. The simplest Banach space with the Schur property is the sequence space l_1. The notion of the p-Schur property in Banach spaces, as a generalization of the Schur property, has been introduced and studied independently in [10] and [34]. A Banach space X has the p-Schur property, $1 \leq p \leq \infty$, if every weakly p-compact subset of X is compact. In other words, if $1 \leq p < \infty$, X has the p-Schur property if and only if every sequence $(x_n) \in l_p^{\mathrm{weak}}(X)$ is a norm null sequence. Also, X has the ∞-Schur property if and only if every sequence in $c_0^{\mathrm{weak}}(X)$ is norm null. So ∞-Schur property coincides with the Schur property. Moreover, one note that every Schur space has the p-Schur property. The authors of the just quoted references show a wide list of examples of Banach spaces which have the p-Schur property for some $1 \leq p < \infty$, but which do not have the Schur property. For example, l_p has the 1-Schur property.

Example 2.4. (1) If $1 \leq p \leq \infty$, then it is evident that every reflexive Banach space has the SR_p property.

(2) Let $1 \leq p \leq \infty$. If X is a non-reflexive Banach space with the p-Schur property, then X does not have the SR_p property. In fact, in this case, the identity operator on X is p-converging, and so is limited p-converging, while it is not weakly compact. Therefore X does not have the SR_p property.

Our next goal is to present some characterizations of the SR_p property in terms of a class of subsets of dual Banach spaces called L_p-limited sets. Let X be a Banach space. A subset K of a dual space X^* is called an L-limited set if every weakly null limited sequence (x_n) in X converges uniformly on K, see [33]. We can extend this concept to the 'p-sense' in the following way.

Definition 2.5. Let $1 \leq p \leq \infty$. A subset K of the dual space X^* of a Banach space X is an L_p-limited set if

$$\lim_n \sup_{x^* \in K} |\langle x_n, x^* \rangle| = 0$$

for every limited sequence $(x_n) \in l_p^{\text{weak}}(X)$ (or $(x_n) \in c_0^{\text{weak}}(X)$ for $p = \infty$).

It is evident that a sequence (x_n^*) in X^* is an L_p-limited set if and only if $\lim_{n\to\infty} \langle x_n, x_n^* \rangle = 0$ for all limited sequence $(x_n) \in l_p^{\text{weak}}(X)$. Also, note that L_∞-limited sets are called L-limited sets in [33].

The Schur property of l_1 implies that B_{l_∞} is an L_p-limited set. Also, if X has the p-Schur property and $(x_n) \in l_p^{\text{weak}}(X)$, then

$$\lim_n \sup_{x^* \in B_{X^*}} |\langle x_n, x^* \rangle| = \lim_n \|x_n\| = 0.$$

Thus B_{X^*} is an L_p-limited set. In particular, the closed unit ball of each l_p space is an L_1-limited set.

Let us recall that according to [25], a bounded subset K of a Banach space X is called p-limited, $1 \leq p \leq \infty$, if for every $(x_n^*) \in l_p^{\text{weak}}(X^*)$ there exists $(\alpha_n) \in l_p$ such that $|\langle x, x_n^* \rangle| \leq \alpha_n$ for all $x \in K$ and all $n \in \mathbb{N}$. Also, we recall from [26] that a bounded subset K of X is called p-(V*) if

$$\lim_n \sup_{x \in K} |\langle x, x_n^* \rangle| = 0$$

for every $(x_n^*) \in l_p^{\text{weak}}(X^*)$ (or $(x_n^*) \in c_0^{\text{weak}}(X^*)$ for $p = \infty$). It should be noticed that the p-(V*) subsets of Banach spaces are called weakly-p-Dunford–Pettis sets in [21].

It is clear that every p-limited subset of X is a p-(V*) (weakly p-Dunford–Pettis) set. But the converse is false. For example, it is easy to see that B_{c_0} is a p-(V*) set, but it is not weakly compact and so it is not p-limited, since every p-limited subset of a Banach space is relatively weakly compact, see [12, Proposition 2.1]. Also note that every limited set and every Dunford–Pettis set is p-(V*).

We notice that the closed convex hull of an L_p-limited set is also L_p-limited and every L_p-limited set is bounded. In fact, if $K \subseteq X^*$ is an L_p-limited set which is unbounded, then there are (x_n^*) in K and (y_n) in B_X such that $|\langle y_n, x_n^* \rangle| > n^2$ for all n. Let $x_n = y_n/n^2$. Then

$$\sum_{n=1}^{\infty} \|x_n\|^p = \sum_{n=1}^{\infty} \frac{1}{n^{2p}} \|y_n\|^p < \infty.$$

Hence (x_n) is a limited sequence in $l_p^{\text{weak}}(X)$. Therefore

$$0 = \lim_{n\to\infty} \sup_{x_n^*\in K} |\langle x_n, x_n^*\rangle| \geq \lim_{n\to\infty} |\langle x_n, x_n^*\rangle| = \lim_{n\to\infty} \frac{1}{n^2}|\langle y_n, x_n^*\rangle| > 1.$$

This is a contradiction.

The following theorem gives a characterization of the SR_p property.

Theorem 2.6. *Let X be a Banach space and let $1 \leq p \leq \infty$. Then the following are equivalent.*

(1) *The space X has the SR_p property.*
(2) *Every limited p-converging operator $T\colon X \to l_\infty$ is weakly compact.*
(3) *Every L_p-limited subset of X^* is relatively weakly compact.*

Proof: The implication (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Suppose, on the contrary there is a Banach space Y and an operator $T \in C_{lp}(X,Y)$ such that T is not weakly compact. Then there is an operator $U\colon Y \to l_\infty$ such that UT is not weakly compact, see [13, page 114]. Obviously, $UT \in C_{lp}(X, l_\infty)$. But this contradicts (2).

(1) $\Rightarrow$ (3). Let K be an L_p-limited set in X^* and

$$\mathbb{B}(K) = \Big\{f\colon K \to \mathbb{R}\colon \|f\| = \sup_{x^*\in K} |f(x^*)| < \infty\Big\}.$$

Then it is easily seen that the operator $T\colon X \to \mathbb{B}(K)$ defined by $(Tx)(x^*) = \langle x, x^*\rangle$ for any $x \in X$ and $x^* \in K$, is limited p-converging. Indeed, if (x_n) is a limited weakly p-summable sequence in X, then

$$\|Tx_n\| = \sup_{\|x^*\|\leq 1} |(Tx_n)(x^*)| = \sup_{\|x^*\|\leq 1} |\langle x_n, x^*\rangle| \to 0.$$

It follows from (1) that T and so T^* is weakly compact. For any $x^* \in K$, if we define $Q \in \mathbb{B}(K)^*$ by $Q(f) = f(x^*)$, then $\|Q\| \leq 1$ and for any $x \in X$ we have

$$\langle x, T^*Q\rangle = \langle Tx, Q\rangle = Q(Tx) = (Tx)(x^*) = \langle x, x^*\rangle.$$

Hence $T^*(Q) = x^*$. Therefore $K \subset T^*(B_{\mathbb{B}(K)^*})$. On the other hand, $T^*(B_{\mathbb{B}(K)^*})$ is relatively weakly compact. Thus, K is relatively weakly compact.

(3) $\Rightarrow$ (2). Let $T \in C_{lp}(X, l_\infty)$. Then for any limited weakly p-summable sequence (x_n) in X we have

$$\sup\{|\langle x_n, T^*y^*\rangle|\colon y^* \in B_{l_\infty^*}\} = \sup\{|\langle Tx_n, y^*\rangle|\colon y^* \in B_{l_\infty^*}\} = \|Tx_n\| \to 0.$$

It follows that $T^*(B_{l_\infty^*})$ is an L_p-limited set and so it is relatively weakly compact, by (3). Therefore T^* and so T is weakly compact. □

Recall that a Banach space whose limited sets are relatively compact is said to have the Gelfand–Phillips (GP) property.

Corollary 2.7. *A Banach space X has the* GP *and the* SR_p *properties for some* $1 \le p \le \infty$ *if and only if X is reflexive.*

Proof: Let X have the GP and the SR_p properties and let (x_n) be a limited weakly p-summable sequence in X. Then

$$\lim_{n\to\infty} \sup_{x^*\in B_{X^*}} |\langle x_n, x^*\rangle| \le \lim_{n\to\infty} \|x_n\| = 0,$$

because X has the GP property. Hence B_{X^*} is an L_p-limited set and so is weakly compact, by Theorem 2.6. Therefore X is reflexive. □

Remark 2.8. The notion of the p-Gelfand–Phillips property, $1 \le p \le \infty$, as a generalization of the Gelfand–Phillips property and the p-Schur property, has been introduced in [16]. It should be noted that this notion has been called "limited p-Schur property" in [11]. In fact, a Banach space X is said to have the p-Gelfand–Phillips property (p-GP for short) if every limited weakly p-summable sequence (x_n) in X is norm null. Therefore one can refine the above corollary by replacing the GP property of X by the p-GP property of X.

We recall that a Banach space X is said to have Pełczyński's property (V) if every unconditionally converging operator (i.e., an operator mapping weakly unconditionally Cauchy series in X to unconditionally converging series) with domain X is weakly compact, see [27]. In [23] A. Grothendieck introduce the reciprocal Dunford–Pettis (RDP) property. A Banach space X has the RDP property if for every Banach space Y, every completely continuous operator $T\colon X \to Y$ is weakly compact. It is known that a Banach space X has the RDP property if and only if every L-subset of X^* is relatively weakly compact, see [22]. In addition, Banach spaces with Pełczyński's property (V), in particular, reflexive spaces, have the RDP property, see [27].

The concept of p-(V) set in the dual of Banach spaces has been introduced and studied in [26], see also [8], to generalize Pełczyński's property (V). We notice here that the p-(V) subsets of dual Banach spaces are called weakly-p-L-set in [21], [20] to introduce and study the reciprocal Dunford–Pettis property of order p. According to [21], [20], [26], a bounded subset K of X^* is said to be p-(V) set (weakly-p-L-set) if every weakly p-summable sequence (x_n) in X converges uniformly on K, that is

$$\lim_{n} \sup_{x^*\in K} |\langle x_n, x^*\rangle| = 0$$

for all $(x_n) \in l_p^{\mathrm{weak}}(X)$. Note that ∞-(V) sets (weakly ∞-L-sets) are called L-sets in [15].

Obviously, every p-(V) subset (weakly-p-L-subset) of X^* is L_p-limited, but the converse is false. For instance, the closed unit ball of $c_0^* = l_1$ is an L_p-limited set. In fact, c_0 has the GP property and so every limited weakly null sequence in c_0 is norm null. Hence the closed unit ball of c_0^* is an L_p-limited set. However, $B_{c_0^*}$ is not a p-(V) set. Indeed, if $B_{c_0^*}$ is a p-(V) set, then

$$\lim_n \|x_n\| = \lim_n \sup_{x^* \in B_{c_0^*}} |\langle x_n, x^* \rangle| = 0$$

for all $(x_n) \in l_p^{\mathrm{weak}}(X)$. It follows that c_0 has the p-Schur property which is impossible.

Before giving some other results and examples of the SR_p property we remark that in [26] the authors introduce a generalization of Pełczyński's property (V) in the p-sense. The results in [26] prove that a Banach space X has Pełczyński's property (V) of order p (property p-(V) in short) if for every Banach space Y, every p-converging operator $T: X \to Y$ is weakly compact. Also, according to the results in [21], [20], a Banach space X has the reciprocal Dunford–Pettis property of order p (RDP_p property) if every p-(V) subset (weakly-p-L-subset) of X^* is relatively weakly compact. It follows from [26, Theorem 2.4] that the property p-(V) and the RDP_p property are equivalent. For more information about the RDP_p property and the property p-(V), see [20], [21], [26]. It is evident that every p-converging operator is limited p-converging. Therefore, if X has the SR_p property, then it has property p-(V) (or RDP_p property). But in general, the next example shows that not every Banach space with property p-(V) admits the SR_p property.

Example 2.9. The James p-space J_p, $1 < p < \infty$, in [30] is the real Banach space of all sequences $(a_n) \in c_0$ with the norm

$$\|(a_n)\| = \sup\left\{ \left(\sum_{j=2}^{n} |a_{i_j} - a_{i_{j-1}}|^p \right)^{1/p} : \; i_1 < \ldots < i_n, \; n \in \mathbb{N} \right\}.$$

By [26] the James p'-space $J_{p'}$ is a non-reflexive Banach space with property p-(V) for $1 < p < \infty$, where p' is the conjugate of p. But $J_{p'}$ does not have the SR_p property. Indeed, if $J_{p'}$ has the SR_p property, then Corollary 2.7 implies that $J_{p'}$ is reflexive, since $J_{p'}$ is separable, and so has the GP property.

It is easy to see that every Banach space with the SR_p property has the RDP property, but the converse is not true. For example, it is known that the classical Banach space c_0 has the GP and the RDP properties, while it is not SR_p. Indeed, if c_0 has the SR_p property, then it would be reflexive, by Corollary 2.7 which is impossible. The authors of [28] also proved that every C^*-algebra has the SR property. Unlike the SR property, there exist C^*-algebras which do not have the SR_p property. For example, it is known that the C^*-algebra $C(K)$ of all continuous functions on a compact Hausdorff space K has the GP property, while it is not reflexive and so is not SR_p. Similarly, one can see that if H is a separable Hilbert space, then $K(H)$, the C^*-algebra of all compact operators on H, does not have the SR_p property.

To prove our next result, we need the following well known theorem.

Theorem 2.10 ([17, Theorem 3]). *Let X and Y be Banach spaces. Suppose that $L(X,Y) = K(X,Y)$ and M is a subset of $K(X,Y)$ such that*

(1) *$M(x) := \{Tx \colon T \in M\}$ is relatively weakly compact for all $x \in X$.*
(2) *$M^*(y^*) = \{T^*y^* : T \in M\}$ is relatively weakly compact for all $y^* \in Y^*$.*

Then M is relatively weakly compact.

Theorem 2.11. *Let X be a Banach space with the SR_p property and let Y be a reflexive Banach space such that $L(X,Y^*) = K(X,Y^*)$. Then the injective tensor product $X\widehat{\otimes}_\varepsilon Y$ has the SR_p property for all $1 \le p \le \infty$.*

Proof: Suppose that $\mathcal{J}(X,Y^*)$ denotes the Banach space of all integral operators from X into Y^*. It is known that $(X\widehat{\otimes}_\varepsilon Y)^* = \mathcal{J}(X,Y^*)$, see [31, page 67]. Let M be an L_p-limited subset of $(X\widehat{\otimes}_\varepsilon Y)^*$ and let (h_n) be a sequence in M. We will verify the conditions of Theorem 2.10 for the subset $\{h_n \colon n \in \mathbb{N}\}$ of M. We claim that $\{h_n x \colon n \in \mathbb{N}\}$ is an L_p-limited subset of Y^* for all $x \in X$. If (y_n) is a limited weakly p-summable sequence in Y and $T \in (X\widehat{\otimes}_\varepsilon Y)^*$, then

$$\sum_{n=1}^{\infty} |\langle x \otimes y_n, T\rangle|^p = \sum_{n=1}^{\infty} |\langle y_n, Tx\rangle|^p < \infty.$$

Hence $(x \otimes y_n)$ is weakly p-summable. Now, let (T_n) be a weak*-null sequence in $\mathcal{J}(X,Y^*)$. Since the mapping $\varphi_x \colon L(X,Y^*) \to Y^*$ defined by $\varphi_x(T) = Tx$ is a bounded linear operator, we conclude that $(T_n x)$ is a weak*-null sequence in Y^*. It follows that

$$\langle x \otimes y_n, T_n\rangle = \langle T_n x, y_n\rangle \to 0,$$

since (y_n) is a limited sequence in Y. Hence $(x \otimes y_n)$ is limited. Similarly, we can prove that if (x_n) is a limited weakly p-summable sequence in X, then $(x_n \otimes y)$ is limited weakly p-summable for all $y \in Y$. On the other hand, since (h_n) is L_p-limited, we have

$$\langle y_n, h_n x\rangle = \langle x \otimes y_n, h_n\rangle \to 0,$$

and so $\{h_n x\colon n \in \mathbb{N}\}$ is an L_p-limited subset of Y^*. It follows from Theorem 2.6 that $\{h_n x\colon n \in \mathbb{N}\}$ is relatively weakly compact.

Now, fix $y \in Y$. If (x_n) is a limited weakly p-summable sequence in X, then

$$\langle x_n, h_n^* y\rangle = \langle y, h_n x_n\rangle = \langle x_n \otimes y, h_n\rangle \to 0.$$

Therefore $\{h_n^* y\colon n \in \mathbb{N}\}$ is L_p-limited in X^*. It follows that $\{h_n^* y\colon n \in \mathbb{N}\}$ is relatively weakly compact. Thus $\{h_n\colon n \in \mathbb{N}\}$ is relatively weakly compact by Theorem 2.10. Since the sequence (h_n) is arbitrary in M, we conclude that M is relatively weakly compact, and therefore Theorem 2.6 implies that $X \otimes_\varepsilon Y$ has the SR_p property. $\square$

Lemma 2.12 ([31, Corollary 2.24], see also [9])**.** *Let $1 \le p, q < \infty$ and let p' be the conjugate of p. Then $l_p \widehat{\otimes}_\varepsilon l_q$ is reflexive if and only if $p' > q$.*

Finally, we introduce a class of non-reflexive Banach spaces with the SR_p property, and so property p-(V) for some appropriate p.

Example 2.13. Let $1 < p, q < \infty$ such that $p > q$ and $p' \le q$. Then $L(l_p, l_q) = K(l_p, l_q)$, by Pitt's theorem. It follows from Lemma 2.12 and Theorem 2.11 that $l_p \widehat{\otimes}_\varepsilon l_q$ is a non-reflexive Banach space with the SR_p property.

3. Sequentially Right* property of order p

In this section we introduce the sequentially Right* property of order p as a dual notion of the SR_p property, and we give some characterizations of this property. Also, we will present some classes of Banach spaces with the sequentially Right* property of order p.

According to [7], a bounded subset K of a Banach space X is called a Right* set if

$$\lim_n \sup_{x \in K} |\langle x, x_n^*\rangle| = 0$$

for all Right null sequence (x_n^*) in X^*. Also, a Banach space X has the sequentially Right* (SR^*) property if every Right* subset of X is relatively weakly compact. The corresponding p-variant notions of Right* sets and the SR^* property can be defined in the following way.

Definition 3.1. A bounded subset K of a Banach space X is said to be R_p^* set if

$$\lim_n \sup_{x \in K} |\langle x, x_n^*\rangle| = 0$$

for every limited sequence $(x_n^*) \in l_p^{\mathrm{weak}}(X^*)$. We say that a Banach space X is sequentially Right* (has the sequentially Right* property of order p (or SR_p^* property in short)), if every R_p^* set in X is relatively weakly compact.

We notice that an R_p^* subset of a Banach space X is an L_p-limited subset of X^{**}, and thus is bounded. Also, it follows from Proposition 2.1 that every Right* subset of a Banach space X is an R_p^* set for all $1 \leq p \leq \infty$. Therefore every Banach space X with the SR_p^* property has the SR^* property. It is clear that every reflexive Banach space has the SR_p^* property for all $1 \leq p \leq \infty$.

Proposition 3.2. *Let X be a Banach space.*

(1) *If X has the SR_p property, then its dual has the SR_p^* property.*
(2) *If X^* has the SR_p property, then X has the SR_p^* property.*

Proof: (1) If K is an R_p^* subset of X^*, then it is easy to see that K is an L_p-limited subset of X^*, and therefore it is relatively weakly compact, by Theorem 2.6.

(2) If X^* has the SR_p property, then (1) implies that X^{**} and so X has the SR_p^* property. □

As a direct consequence of Example 2.13 and Proposition 3.2 we have the following result.

Corollary 3.3. *Let $1 < p, q < \infty$ such that $p > q$ and $p' \leq q$, where p' is the conjugate of p. Then $(l_p \widehat{\otimes}_\varepsilon l_q)^* = \mathcal{J}(l_p, l_{q'})$ is a non-reflexive Banach space with the SR_p^* property.*

Theorem 3.4. *Let X be a Banach space. Then X^* has the p-GP property if and only if B_X is an R_p^* set in X.*

Proof: Suppose that X^* has the p-GP property. Then every limited weakly p-summable sequence in X^* is norm null. Therefore

$$\lim_n \sup_{x \in B_X} |\langle x, x_n^* \rangle| \leq \lim_n \|x_n^*\| = 0$$

for all limited weakly p-summable (x_n^*) in X^*. It implies that B_X is an R_p^* set. Conversely, assume that X^* does not have the p-GP property. Then there exists a limited sequence $(x_n^*) \in l_p^{\mathrm{weak}}(X^*)$ which is not norm null. Then we may assume that there exists $\varepsilon > 0$ such that $\|x_n^*\| > \varepsilon$ for all $n \in \mathbb{N}$. It follows from the Hahn–Banach theorem that there is a sequence $(x_n) \subseteq B_X$ such that $|\langle x_n, x_n^* \rangle| > \varepsilon$. This is a contradiction, since B_X is an R_p^*-set. □

Corollary 3.5. *If a Banach space X has the SR_p^* property and X^* has the p-GP property, then X is reflexive.*

PROOF: It follows from Theorem 3.4 that B_X is an R_p^* set in X. Therefore B_X is weakly compact, and so X is reflexive. □

Example 3.6. Since $c_0^* = l_1$ has the p-GP property, it follows from Theorem 3.4 that B_{c_0} is an R_p^* set. On the other hand, it is known that B_{c_0} is not weakly compact. Therefore c_0 does not have the SR_p^* property.

A Banach space X has property (V^*) if and only if for every Banach space Y, every unconditionally converging operator $T\colon X \to Y$ is weakly compact, see [27]. Also, a Banach space X has property RDP^* if every DP subset of X is relatively weakly compact, see [2]. The concept of property p-(V^*) (property RDP_p^*) was introduced in [26] (or [21], respectively) as a generalization of property (V^*) (or RDP^*). More precisely, a Banach space X has property p-(V^*) (or RDP_p^*) if every p-(V^*) subset (weakly-p-Dunford–Pettis subset) of X is relatively weakly compact. It is trivial that every p-(V^*) (weakly-p-Dunford–Pettis) subset of a Banach space X is an R_p^* set. Therefore, if X has the SR_p^* property, then X has property p-(V^*) (or RDP_p^*). However, the converse is not true. To provide an appropriate example, we need the following result from [18].

Lemma 3.7 ([18, Corollary 19])**.** *If X is a non-reflexive Banach space with the* SR^* *property, then X contains a copy of l_1.*

Example 3.8. It has been proved in [26, Theorem 2.14] that the James p-space J_p, $1 < p < \infty$, has property p-(V^*). But, J_p does not have the SR_p^* property. In fact, the James p-space, J_p and its dual J_p^* are separable non-reflexive Banach spaces. Then J_p does not contain a copy of l_1 . It follows from Lemma 3.7 that J_p does not have the SR^* property, and so J_p does not have the SR_p^* property.

Theorem 3.9. *Let X and Y be two Banach spaces. Then the following statements are equivalent.*

(1) *If $T \in L(X,Y)$ such that T^* is limited p-converging, then T is weakly compact.*
(2) *If $T \in L(l_1,Y)$ such that T^* is limited p-converging, then T is weakly compact.*
(3) *The space Y has the* SR_p^* *property.*

PROOF: The implication (1) ⇒ (2) is clear.

(2) ⇒ (3). Suppose that K is an R_p^* subset of Y and (y_k) is an arbitrary sequence in K. Define $T\colon l_1 \to Y$ by $T((\alpha_k)) = \sum_{n=1}^{\infty} \alpha_k y_k$. So $T^* y^* = (\langle y_k, y^*\rangle)$ for all $y^* \in Y^*$. Indeed, for $\alpha = (\alpha_k) \in l_1$ we have

$$\langle T^* y^*, \alpha\rangle = \langle y^*, T\alpha\rangle = \left\langle y^*, \sum_{n=1}^{\infty} \alpha_k y_k \right\rangle = \langle (\langle y_k, y^*\rangle), \alpha\rangle.$$

If (y_n^*) is limited weakly p-summable sequence in Y^*, then $\lim_{n\to\infty} \|T^* y_n^*\| = 0$, since K is an R_p^* set, and therefore

$$\lim_n \|T^* y_n^*\| \lim_n \|(\langle y_k, y_n^* \rangle)\| = \lim_n \sup_k |\langle y_k, y_n^* \rangle| = 0.$$

Hence T^* is limited p-converging and so T is weakly compact, by (2). Thus $T((e_k)) = (y_k)$ is a weakly compact sequence. Therefore K is weakly compact, since (y_k) is an arbitrary sequence in K.

(3) $\Rightarrow$ (1). Assume that $T \in L(X,Y)$ and T^* is limited p-converging. If (y_k^*) is a limited weakly p-summable sequence in Y^* and $x \in B_X$, then

$$\begin{aligned}\lim_k \sup_{x\in B_X} |\langle Tx, y_k^* \rangle| &= \lim_k \sup_{x\in B_X} |\langle x, T^* y_k^* \rangle| \\ &\leq \lim_k \|T^* y_k^*\| = 0.\end{aligned}$$

Therefore $T(B_X)$ is an R_p^* subset of Y. It follows from (3) that T is weakly compact. □

Corollary 3.10. *Let X and Y be Banach spaces. If Y has the* SR_p^* *property and X^* has the p-*GP *property, then $L(X,Y) = W(X,Y)$.*

Proof: Let $T \in L(X,Y)$ and let $(y_n^*) \in l_p^{\text{weak}}(Y^*)$ be a limited sequence. Then $(T^* y_n^*)$ is limited weakly p-summable sequence in X^*. So $\|T^* y_n^*\| \to 0$, since X^* has the p-GP property. It yields that $T^* \in C_{lp}(Y^*, X^*)$. Therefore Theorem 3.9 and the SR_p^* property of Y imply that T is weakly compact. □

Theorem 3.11. *If every conjugate limited p-converging operator $T^* \colon Y^* \to X^*$ is weakly compact, then Y has the* SR_p^* *property.*

Proof: Let K be an R_p^* set of Y and (y_n) be an arbitrary sequence in K. Define an operator $T \colon l_1 \to Y$ by

$$T(\alpha) = \sum_{n=1}^{\infty} \alpha_n y_n$$

for all $\alpha = (\alpha_n) \in l_1$. We observe that $T^* y^* = (\langle y_n, y^* \rangle)$ for all $y^* \in Y^*$. We claim that T^* is limited p-converging. Assume, on the contrary, that $(y_n^*) \in l_p^{\text{weak}}(Y^*)$ is limited and $(T^* y_n^*)$ is not norm null. Therefore there is $\varepsilon > 0$ such that

$$\|T^* y_n^*\| > \varepsilon$$

for all $n \in \mathbb{N}$. It follows that there is a subsequence (y_{m_n}) such that

$$|\langle y_{m_n}\, y_n^* \rangle| > \varepsilon, \qquad n \in \mathbb{N}.$$

Since (y_{m_n}) is an R_p^* set, we get a contradiction. Hence T^* is limited p-converging. Then T^{**} and so T is weakly compact, by hypothesis. It follows that $(y_n) = (T(e_n))$ is weakly convergent. Therefore K is relatively weakly compact. $\square$

Theorem 3.12. *If K is an R_p^* subset of a Banach space X and $T\colon X \to l_p$, $1 \le p < \infty$, is a bounded linear operator with limited adjoint, then $T(K)$ is relatively compact.*

Proof: Let $x_n^* = T^*(e_n^*)$, where (e_n^*) is the vector basis of $l_{p'}$ and $1/p+1/p' = 1$. Then $Tx = (\langle x, x_n^*\rangle)$ for all $x \in X$. It is known that the operator $T^*\colon l_{p'} \to X^*$ is weakly p-compact, since $B_{l_{p'}}$ is weakly p-compact. Hence T is weakly compact, and so T^* is (w^*, w)-continuous. It follows that (x_n^*) is weakly null. On the other hand, (x_n^*) is limited weakly p-compact, since T^* is limited. Hence without loss of generality we may assume that $(x_n^*) \in l_p^{\mathrm{weak}}(X^*)$. Since K is an R_p^* set we conclude that

$$\lim_n \sup_{x\in K} |\langle x, x_n^*\rangle| = 0.$$

Therefore $T(K)$ is relatively compact in l_p by the characterization of relatively compact subsets of l_p. $\square$

Theorem 3.13. *Let X be a Banach space and let K be a bounded subset of X. Then the following statements are equivalent.*

(1) *The set K is an R_p^* set.*
(2) *If $T\colon X \to c_0$ is a bounded linear operator with limited weakly p-compact adjoint, then $T(K)$ is relatively compact.*

Proof: The implication (1) $\Rightarrow$ (2) can be proved by the same argument as the proof of Theorem 3.12, using the characterization of relatively compact subsets of c_0.

(2) $\Rightarrow$ (1). Let (x_n^*) be a limited weakly p-summable sequence in X^*. Define $T\colon X \to c_0$ by $Tx = (\langle x, x_n^*\rangle)$, $x \in X$. Then $T^*e_n = x_n^*$ for all $n \in \mathbb{N}$ and $T^*((\alpha_n)) = \sum_{n=1}^\infty \alpha_n x_n^*$ for all $(\alpha_n) \in l_1$. It is clear that $T^*(B_{l_1}) = \left\{\sum_{n=1}^\infty \alpha_n x_n^*\colon (\alpha_n) \in l_1\right\}$ is the closed absolutely convex hull of (x_n^*). Then $T^*(B_{l_1})$ is limited weakly p-compact. Hence T^* is a limited weakly p-compact operator. It follows from (2) that $T(K)$ is relatively compact. Therefore

$$\lim_n \sup_{x\in K} |\langle x, x_n^*\rangle| = 0,$$

by the characterization of relatively compact subsets of c_0, and so K is an R_p^* subset of X. $\square$

Acknowledgment. We would like to thank the referee for his/her careful reading of the manuscript and useful comments.

References

[1] Andrews K. T., *Dunford–Pettis sets in the space of Bochner integrable functions*, Math. Ann. **241** (1979), no. 1, 35–41.

[2] Bator E., Lewis P., Ochoa J., *Evaluation maps, restriction maps, and compactness*, Colloq. Math. **78** (1998), no. 1, 1–17.

[3] Bourgain J., Diestel J., *Limited operators and strict cosingularity*, Math. Nachr. **119** (1984), 55–58.

[4] Castillo J. M. F., *p-converging operators and weakly-p-compact operators in L_p-spaces*, Actas del II Congreso de Análisis Funcional, Jarandilla de la Vera, Cáceres, June 1990, Extracta Math. (1990), 46–54.

[5] Castillo J. M. F., Sanchez F., *Dunford–Pettis-like properties of continuous vector function spaces*, Rev. Mat. Univ. Complut. Madrid. **6** (1993), no. 1, 43–59.

[6] Castillo J. M. F., Sánchez F., *Weakly p-compact, p-Banach–Saks and super-reflexive Banach spaces*, J. Math. Anal. Appl. **185** (1994), no. 2, 256–261.

[7] Cilia R., Emmanuele G., *Some isomorphic properties in $K(X,Y)$ and in projective tensor products*, Colloq. Math. **146** (2017), no. 2, 239–252.

[8] Chen D., Chávez-Domínguez J. A., Li L., *p-converging operators and Dunford–Pettis property of order p*, J. Math. Anal. Appl. **461** (2018), no. 2, 1053–1066.

[9] Defant A., Floret K., *Tensor Norms and Operator Ideals*, North-Holland Mathematics Studies, 176, North-Holland Publishing Co., Amsterdam, 1993.

[10] Dehghani M. B., Moshtaghioun S. M., *On the p-Schur property of Banach spaces*, Ann. Funct. Anal. **9** (2018), no. 1, 123–136.

[11] Dehghani M. B., Moshtaghioun S. M., Dehghani M., *On the limited p-Schur property of some operator spaces*, Int. J. Anal. Appl. **16** (2018), no. 1, 50–61.

[12] Delgado J. M., Piñeiro C., *A note on p-limited sets*, J. Math. Anal. Appl. **410** (2014), no. 2, 713–718.

[13] Diestel J., *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, 92, Springer, New York, 1984.

[14] Diestel J., Jarchow H., Tonge A., *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics, 43, Cambridge University Press, Cambridge, 1995.

[15] Emmanuele G., *A dual characterization of Banach spaces not containing l_1*, Bull. Polish Acad. Sci. Math. **34** (1986), no. 3–4, 155–160.

[16] Fourie J. H., Zeekoei E. D., *On weak-star p-convergent operators*, Quaest. Math. **40** (2017), no. 5, 563–579.

[17] Ghenciu I., *Property (wL) and the reciprocal Dunford–Pettis property in projective tensor products*, Comment. Math. Univ. Carolin. **56** (2015), no. 3, 319–329.

[18] Ghenciu I., *L-sets and property (SR^*) in spaces of compact operators*, Monatsh. Math. **181** (2016), no. 3, 609–628.

[19] Ghenciu I., *A note on some isomorphic properties in projective tensor products*, Extracta Math. **32** (2017), no. 1, 1–24.

[20] Ghenciu I., *A note on Dunford–Pettis like properties and complemented spaces of operators*, Comment. Math. Univ. Carolin. **59** (2018), no. 2, 207–222.

[21] Ghenciu I., *The p-Gelfand–Phillips property in spaces of operators and Dunford–Pettis like sets*, Acta Math. Hungar. **155** (2018), no. 2, 439–457.

[22] Ghenciu I., Lewis P., *The Dunford–Pettis property, the Gelfand–Phillips property, and L-sets*, Colloq. Math. **106** (2006), no. 2, 311–324.

[23] Grothedieck A., *Sur les applications lineaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. **5** (1953), 129–173 (French).

[24] Kačena M., *On sequentially right Banach spaces*, Extracta Math. **26** (2011), no. 1, 1–27.

[25] Karn A. K., Sinha D. P., *An operator summability of sequences in Banach spaces*, Glasg. Math. J. **56** (2014), no. 2, 427–437.

[26] Li L., Chen D., Chávez-Domínguez J. A., *Pełczyński's property (V^*) of order p and its quantification*, Math. Nachr. **291** (2018), no. 2–3, 420–442.

[27] Pełczyński A., *On Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **10** (1962), 641–648.

[28] Peralta A. M., Villanueva I., Wright J. D. M., Ylinen K., *Topological characterisation of weakly compact operators*, J. Math. Anal. Appl. **325** (2007), no. 2, 968–974.

[29] Peralta A. M., Villanueva I., Wright J. D. M., Ylinen K., *Weakly compact operators and the strong* topology for a Banach space*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), no. 6, 1249–1267.

[30] Read C. J., *Strictly singular operators and the invariant subspace problem*, Studia Math. **132** (1999), no. 3, 203–226.

[31] Ryan R. A., *Introduction to Tensor Products of Banach Spaces*, Springer Monographs in Mathematics, Springer, London, 2002.

[32] Salimi M., Moshtaghioun S. M., *The Gelfand–Phillips property in closed subspace of some operator spaces*, Banach J. Math. Anal. **5** (2011), no. 2, 84–92.

[33] Salimi M., Moshtaghioun S. M., *A new class of Banach spaces and its relation with some geometric properties of Banach spaces*, Abstr. Appl. Anal. (2012), Art. ID 212957, 8 pages.

[34] Zeekoei E. D., Fourie J. H., *On p-convergent operators on Banach lattices*, Acta. Math. Sin. (Engl. Ser.) **34** (2018), no. 5, 873–890.

M. Dehghani (corresponding author):
Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, P. O. Box 87317-53153, Kashan, Iran

E-mail: m.dehghani@kashanu.ac.ir

E-mail: e.g.mahdi@gmail.com

M. B. Dehghani, M. S. Moshtaghioun:
Department of Mathematics, Yazd University, P. O. Box 89195-741, Yazd, Iran

E-mail: m.b.deh91@gmail.com

E-mail: moshtagh@yazd.ac.ir

(*Received* November 26, 2018*, revised* April 19, 2019)