Wintgen inequalities on Legendrian submanifolds of generalized Sasakian-space-forms

Shyamal K. Hui, Richard S. Lemence, Pradip Mandal

Abstract. A submanifold M^m of a generalized Sasakian-space-form $\overline{M}^{2n+1}(f_1, f_2, f_3)$ is said to be C-totally real submanifold if $\xi \in \Gamma(T^\perp M)$ and $\varphi X \in \Gamma(T^\perp M)$ for all $X \in \Gamma(TM)$. In particular, if $m = n$, then M^n is called Legendrian submanifold. Here, we derive Wintgen inequalities on Legendrian submanifolds of generalized Sasakian-space-forms with respect to different connections; namely, quarter symmetric metric connection, Schouten–van Kampen connection and Tanaka–Webster connection.

Keywords: generalized Sasakian-space-form; Legendrian submanifold

Classification: 53C25, 53C15

1. Introduction

A generalized Sasakian-space-form is an almost contact metric manifold $\overline{M}(\varphi, \xi, \eta, g)$ whose curvature tensor $\overline{R}$ is of the form, see [1],

$$
\begin{aligned}
\overline{R}(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\
& + f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \\
& + f_3\big[\eta(Z)\{\eta(X)Y - \eta(Y)X\} \\
& + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi\big]
\end{aligned} \tag{1.1}
$$

for all vector fields X, Y, Z on $\overline{M}$, where $f_i \in C^\infty(\overline{M})$, $i = 1, 2, 3$. Such a manifold of dimension $(2n+1)$, $n > 1$, is denoted by $\overline{M}^{2n+1}(f_1, f_2, f_3)$.

In particular, if $f_1 = (c+3)/4$, $f_2 = f_3 = (c-1)/4$ then $\overline{M}^{2n+1}(f_1, f_2, f_3)$ reduces to the notion of Sasakian-space-forms. Many authors studied $\overline{M}^{2n+1}(f_1, f_2, f_3)$ in different context such as ([2]–[5], and references therein).

Beside the Riemannian connection, there exist some other connections on smooth manifolds. In 1975, S. Golab in [6] introduced the idea of quarter symmetric connection. The quarter symmetric connection is called metric connection if the covariant derivative of such connection is zero.

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The Schouten–van Kampen connection (SVKC) introduced for the study of non-holomorphic manifolds, see [11]. In 2006, A. Bejancu in [3] studied SVKC connection on foliated manifolds. Recently Z. Olszak in [10] studied SVKC on almost (para) contact metric structure.

The Tanaka–Webster connection (TWC), see [12], [14], is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold. S. Tanno in [13] defined the TWC for contact metric manifolds. Here we denote quarter symmetric metric connection (QSMC), SVKC and TWC on $\overline{M}^{2n+1}(f_1,f_2,f_3)$ by $\overline{\widetilde{\nabla}}$, $\overline{\widehat{\nabla}}$, $\overline{\overset{*}{\nabla}}$, respectively.

After introducing Wintgen inequality in [15], I. Mihai derived Wintgen inequality for submanifolds of complex-space-form, see [8], and Sasakian-space-form, see [9]. In this paper we derive Wintgen inequality for Legendrian submanifolds of $\overline{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\overline{\widetilde{\nabla}}$, $\overline{\widehat{\nabla}}$ and $\overline{\overset{*}{\nabla}}$.

2. Preliminaries

On an almost contact metric manifold $\overline{M}(\varphi,\xi,\eta,g)$, we have in [4]

$$\varphi^2(X) = -X + \eta(X)\xi, \qquad \varphi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad g(X,\xi) = \eta(X), \qquad \eta(\varphi X) = 0, \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$g(\varphi X, Y) = -g(X, \varphi Y). \tag{2.4}$$

On $\overline{M}^{2n+1}(f_1,f_2,f_3)$, we have in [1]

$$(\overline{\nabla}_X\varphi)(Y) = (f_1 - f_3)\big[g(X,Y)\xi - \eta(Y)X\big]. \tag{2.5}$$

The relations of $\overline{\widetilde{\nabla}}$, $\overline{\widehat{\nabla}}$ and $\overline{\overset{*}{\nabla}}$ with $\overline{\nabla}$ on $\overline{M}^{2n+1}(f_1,f_2,f_3)$ are

$$\overline{\widetilde{\nabla}}_X Y = \overline{\nabla}_X Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \tag{2.6}$$

$$\overline{\widehat{\nabla}}_X Y = \overline{\nabla}_X Y + (f_1 - f_3)\eta(Y)\varphi X - (f_1 - f_3)g(\varphi X, Y)\xi \tag{2.7}$$

and

$$\overline{\overset{*}{\nabla}}_X Y = \overline{\nabla}_X Y + \eta(X)\varphi Y + (f_1 - f_3)\eta(Y)\varphi X - (f_1 - f_3)g(\varphi X, Y)\xi. \tag{2.8}$$

Let $\widetilde{\overline{R}}$ (or $\widehat{\overline{R}}, \overset{*}{\overline{R}}$) be the curvature tensor of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\widetilde{\overline{\nabla}}$ ($\widehat{\overline{\nabla}}, \overset{*}{\overline{\nabla}}$, respectively). Then

$$
\begin{aligned}
\widetilde{\overline{R}}(X,Y,Z,W) = {} & \overline{R}(X,Y,Z,W) + g(\varphi X, Z)g(\varphi Y, W) \\
& - g(\varphi Y, Z)g(\varphi X, W) + (f_1 - f_3)\big[\{\eta(X)g(Y,W) \\
& - \eta(Y)g(X,W)\}\eta(Z) + \{g(X,Z)\eta(Y) \\
& - g(Y,Z)\eta(X)\}\eta(W)\big].
\end{aligned}
\tag{2.9}
$$

Also, we have

$$
\begin{aligned}
\widehat{\overline{R}}(X,Y,Z,W) = {} & f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
& + f_2\{g(X,\varphi Z)g(\varphi Y, W) - g(Y,\varphi Z)g(\varphi X, W) \\
& + 2g(X,\varphi Y)g(\varphi Z, W)\} + \{f_3 + (f_1 - f_3)^2\}\big[\eta(X)\eta(Z)g(Y,W) \\
& - \eta(Y)\eta(Z)g(X,W) + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(W)\big] \\
& + (f_1 - f_3)^2\big[g(X,\varphi Z)g(\varphi Y, W) - g(Y,\varphi Z)g(\varphi X, W)\big],
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
\overset{*}{\overline{R}}(X,Y,Z,W) = {} & f_1\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
& + f_2\{g(X,\varphi Z)g(\varphi Y, W) - g(Y,\varphi Z)g(\varphi X, W) \\
& + 2g(X,\varphi Y)g(\varphi Z, W)\} + \{f_3 + (f_1 - f_3)^2\} \\
& \times \big[\eta(X)\eta(Z)g(Y,W) - \eta(Y)\eta(Z)g(X,W) \\
& + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\eta(W)\big] \\
& + (f_1 - f_3)^2\big[g(X,\varphi Z)g(\varphi Y, W) - g(Y,\varphi Z)g(\varphi X, W)\big] \\
& + 2(f_1 - f_3)g(X,\varphi Y)g(\varphi Z, W),
\end{aligned}
\tag{2.11}
$$

where $(f_1 - f_3)$ is a constant function.

Let M be a submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$. If ∇ and $\nabla^\perp$ are the induced connections on $\Gamma(TM)$ and $\Gamma(T^\perp M)$, respectively, then the Gauss and Weingarten formulas are given by [17]

$$
\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \overline{\nabla}_X V = -A_V X + \nabla^\perp_X V
\tag{2.12}
$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where h and A_V are second fundamental form and shape operator respectively and they are related by [17] $g(h(X,Y),V) = g(A_V X, Y)$.

Let R (or $\widetilde{R}, \widehat{R}, \overset{*}{R}$) be the curvature tensor of M for the induced connection ∇ ($\widetilde{\nabla}$, $\widehat{\nabla}, \overset{*}{\nabla}$, respectively) and $\tilde{h}$ (or $\hat{h}, \overset{*}{h}$) be the second fundamental forms and $\widetilde{A}_V$

(or $\widehat{A}_V, \overset{*}{A}_V$) shape operators with respect to the induced connection $\widetilde{\nabla}$ ($\widehat{\nabla}, \overset{*}{\nabla}$, respectively).

From (2.12), we have the Gauss and Ricci equations as

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)) \tag{2.13}$$

and

$$R^{\perp}(X,Y,\mu,\nu) = \overline{R}(X,Y,\mu,\nu) + g([A_\mu, A_\nu]X,Y) \tag{2.14}$$

where $\mu,\nu \in \Gamma(T^{\perp}M)$. In a similar way, we have

$$\widetilde{R}(X,Y,Z,W) = \widetilde{\overline{R}}(X,Y,Z,W) + g(\tilde{h}(X,W),\tilde{h}(Y,Z)) - g(\tilde{h}(X,Z),\tilde{h}(Y,W)), \tag{2.15}$$

$$\widetilde{R}^{\perp}(X,Y,\mu,\nu) = \widetilde{\overline{R}}(X,Y,\mu,\nu) + g([\widetilde{A}_\mu, \widetilde{A}_\nu]X,Y), \tag{2.16}$$

$$\widehat{R}(X,Y,Z,W) = \widehat{\overline{R}}(X,Y,Z,W) + g(\hat{h}(X,W),\hat{h}(Y,Z)) - g(\hat{h}(X,Z),\hat{h}(Y,W)), \tag{2.17}$$

$$\widehat{R}^{\perp}(X,Y,\mu,\nu) = \widehat{\overline{R}}(X,Y,\mu,\nu) + g([\widehat{A}_\mu, \widehat{A}_\nu]X,Y), \tag{2.18}$$

$$\overset{*}{\overline{R}}(X,Y,Z,W) = \overset{*}{R}(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(\overset{*}{h}(X,Z),\overset{*}{h}(Y,W)), \tag{2.19}$$

$$\overset{*}{R}{}^{\perp}(X,Y,\mu,\nu) = \overset{*}{\overline{R}}(X,Y,\mu,\nu) + g([\overset{*}{A}_\mu, \overset{*}{A}_\nu]X,Y). \tag{2.20}$$

Let $p \in M^m$ and $\{e_1,\dots,e_m\}$ be an orthonormal basis of T_pM and $\{e_{m+1},\dots, e_{2n}, e_{2n+1} = \xi\}$ be an orthonormal basis of $T^{\perp}M^m$. We define the mean curvature vector as

$$H(p) = \frac{1}{m}\sum_{i=1}^{m} h(e_i,e_i).$$

Following [16], we define

$$K_N = -\frac{1}{4}\sum_{r,s=1}^{2n-m+1} \operatorname{Tr}\,[A_r, A_s]^2, \tag{2.21}$$

where $A_r = A_{e_{n+r}}$, $r \in \{1,\dots,2n-m+1\}$ and call it the scalar normal curvature of M^m. The normalized scalar normal curvature is given by $\varrho_N = \frac{2}{m(m-1)}\sqrt{K_N}$.

Since $A_\xi = 0$, it follows that

$$\begin{aligned} K_N &= -\frac{1}{2}\sum_{1\le r<s\le 2n-m} \operatorname{Tr}\,[A_r, A_s]^2 \\ &= \sum_{1\le r<s\le 2n-m}\ \sum_{1\le i<j\le m} (g([A_r,A_s]e_i,e_j))^2. \end{aligned} \tag{2.22}$$

Also we can express K_N as

$$K_N = \sum_{1\le r<s\le 2n-m}\ \sum_{1\le i<j\le m} (h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})^2. \tag{2.23}$$

Again we define

$$\varrho_N = \frac{2}{m(m-1)}\left[\sum_{1\le r<s\le 2n-m}\ \sum_{1\le i<j\le n}\left(\sum_{k=1}^{m}(h^r_{jk}h^s_{ik} - h^r_{ik}h^s_{jk})\right)^2\right]^{1/2}. \tag{2.24}$$

The normalized scalar curvature is given by

$$\varrho = \frac{2\tau}{m(m-1)} = \sum_{1\le i<j\le m}\frac{2}{m(m-1)}R(e_i,e_j,e_j,e_i), \tag{2.25}$$

where τ is the scalar curvature and $\{e_i\colon i=1,2,\dots,m\}$ is an orthonormal basis of TM^m.

The normalized normal scalar curvature is given by

$$\varrho^\perp = \frac{2\tau^\perp}{m(m-1)} = \frac{2}{m(m-1)}\sqrt{\sum_{1\le i<j\le m}\ \sum_{1\le\alpha<\beta\le m}(R^\perp(e_i,e_j,u_\alpha,u_\beta))^2}, \tag{2.26}$$

where R and $R^\perp$ are the curvature tensor and normal curvature of M^m.

In similar of (2.25) and (2.26) we can define $\tilde\varrho, \tilde\varrho^\perp$; $\hat\varrho, \hat\varrho^\perp$ and $\overset{*}{\varrho}, \overset{*}{\varrho}^\perp$ with respect to $\widetilde\nabla$; $\widehat\nabla$ and $\overset{*}{\nabla}$ as

$$\tilde\varrho = \frac{2\,\widetilde\tau}{m(m-1)} = \sum_{1\le i<j\le m}\frac{2}{m(m-1)}\widetilde R(e_i,e_j,e_j,e_i), \tag{2.27}$$

$$\tilde\varrho^\perp = \frac{2\,\widetilde\tau^\perp}{m(m-1)} = \frac{2}{m(m-1)}\sqrt{\sum_{1\le i<j\le m}\ \sum_{1\le\alpha<\beta\le m}(\widetilde R^\perp(e_i,e_j,u_\alpha,u_\beta))^2}, \tag{2.28}$$

$$\hat{\varrho} = \frac{2\,\widehat{\tau}}{m(m-1)} = \sum_{1\leq i<j\leq m} \frac{2}{m(m-1)} \widehat{R}(e_i, e_j, e_j, e_i), \tag{2.29}$$

$$\hat{\varrho}^{\perp} = \frac{2\,\widehat{\tau}^{\perp}}{m(m-1)} = \frac{2}{m(m-1)} \sqrt{\sum_{1\leq i<j\leq m}\ \sum_{1\leq\alpha<\beta\leq m} (\widehat{R}^{\perp}(e_i, e_j, u_\alpha, u_\beta))^2}, \tag{2.30}$$

$$\overset{*}{\varrho} = \frac{2\,\overset{*}{\tau}}{m(m-1)} = \sum_{1\leq i<j\leq m} \frac{2}{m(m-1)} \overset{*}{R}(e_i, e_j, e_j, e_i), \tag{2.31}$$

$$\overset{*}{\varrho}{}^{\perp} = \frac{2\,\overset{*}{\tau}{}^{\perp}}{m(m-1)} = \frac{2}{m(m-1)} \sqrt{\sum_{1\leq i<j\leq m}\ \sum_{1\leq\alpha<\beta\leq m} (\widetilde{R}^{\perp}(e_i, e_j, u_\alpha, u_\beta))^2}. \tag{2.32}$$

A submanifold M^m of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ is said to be C-totally real submanifold if $\xi \in \Gamma(T^{\perp}M)$ and $\varphi X \in \Gamma(T^{\perp}M)$ for all $X \in \Gamma(TM)$. In particular, if $m = n$, then M^n is called Legendrian submanifold.

3. Some basic results

Proposition 3.1. *Let M be a C-totally real submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overline{\widetilde{\nabla}}$. Then following relations hold on M:*

(i) $\tilde{h}(X,Y) = h(X,Y)$, $\widetilde{H} = H$;
(ii) $\widetilde{A}_V X = A_V X$.

Proof: From (2.12), we have

$$\overline{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y + \tilde{h}(X,Y) \tag{3.1}$$

and

$$\overline{\widetilde{\nabla}}_X V = \widetilde{\nabla}^{\perp}_X V - \widetilde{A}_V X. \tag{3.2}$$

From (2.6), (2.12) and (3.1) we have

$$\widetilde{\nabla}_X Y + \tilde{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(Y)\varphi X - g(\varphi X, Y)\xi \tag{3.3}$$

for any $X, Y \in \Gamma(TM)$.

Since $\xi \in \Gamma(T^{\perp}M)$ and $\varphi X \in \Gamma(T^{\perp}M)$ for all X, then from (3.3) we have

$$\widetilde{\nabla}_X Y = \nabla_X Y \quad \text{and} \quad \tilde{h}(X,Y) = h(X,Y).$$

Again from (2.6), (2.12) and (3.2)

$$\widetilde{\nabla}^{\perp}_{X} V - \widetilde{A}_{V} X = \nabla^{\perp}_{X} V - A_{V} X + \eta(V)\varphi X - g(\varphi X, V)\xi. \tag{3.4}$$

Equating tangential and normal part of (3.4) we have

$$\widetilde{A}_{V} X = A_{V} X, \qquad \widetilde{\nabla}^{\perp}_{X} V = \nabla^{\perp}_{X} V + \eta(V)\varphi X - g(\varphi X, V)\xi.$$

□

Proposition 3.2. *Let M be a C-totally real submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\overline{\nabla}}$. Then following relations hold on M:*

(i) $\hat{h}(X,Y) = h(X,Y)$, $\widehat{H} = H$;
(ii) $\widehat{A}_{V} X = A_{V} X$.

Proof: The proof is similar to the proof of Proposition 3.1. □

Proposition 3.3. *Let M be a C-totally real submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\overline{\nabla}}$. Then following relations hold on M:*

(i) $\overset{*}{h}(X,Y) = h(X,Y)$, $\overset{*}{H} = H$;
(ii) $\overset{*}{A}_{V} X = A_{V} X$.

Proof: The proof is similar to the proof of Proposition 3.1. □

4. Wintgen inequality on Legendrian submanifolds of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\widetilde{\overline{\nabla}}$

Proposition 4.1. *Let M^m be a C-totally real submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\widetilde{\overline{\nabla}}$. Then*

$$\|H\|^2 + f_1 \geq \tilde{\varrho} + \varrho_N. \tag{4.1}$$

Proof: We see that

$$\begin{aligned} m^2\|H\|^2 = \sum_{r=1}^{2n-m}\left(\sum_{i=1}^{m} h^r_{ii}\right)^2 &= \frac{1}{m-1}\sum_{r=1}^{2n-m}\sum_{1\leq i<j\leq m}(h^r_{ii} - h^r_{jj})^2 \\ &\quad + \frac{2m}{m-1}\sum_{r=1}^{2n-m}\sum_{1\leq i<j\leq m} h^r_{ii}h^r_{jj}. \end{aligned} \tag{4.2}$$

From [7] we have the inequality

$$\begin{aligned}\sum_{r=1}^{2n-m}\sum_{1\le i<j\le m}(h_{ii}^r-h_{jj}^r)^2+2\sum_{r=1}^{2n-m}\sum_{1\le i<j\le m}h_{ij}^rh_{ij}^r\\ \ge\Big[\sum_{1\le r<s\le 2n-m}\sum_{1\le i<j\le m}\Big(\sum_{k=1}^{m}(h_{jk}^rh_{ik}^s-h_{ik}^rh_{jk}^s)\Big)^2\Big]^{1/2}.\end{aligned}\tag{4.3}$$

From (2.24), (4.2) and (4.3), we get

$$m^2\|H\|^2-m^2\varrho_N\ge\frac{2m}{m-1}\sum_{r=1}^{2n-m}\sum_{1\le i<j\le m}[h_{ii}^rh_{jj}^r-(h_{ij}^r)^2].\tag{4.4}$$

Now from (2.9) and (2.15) we have

$$\widetilde{\tau}=\widetilde{R}(e_i,e_j,e_j,e_i)=\frac{m(m-1)}{2}f_1+\sum_{r=1}^{2n-m}\sum_{1\le i<j\le m}[h_{ii}^rh_{jj}^r-(h_{ij}^r)^2].\tag{4.5}$$

Substituting (4.5) in (4.4) we get (4.1). □

Theorem 4.1. *Let M^n be a Legendrian submanifold of $\overline{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\overline{\widetilde{\nabla}}$. Then*

$$(\tilde{\varrho}^{\perp})^2\le(\|H\|^2-\tilde{\varrho}+f_1)+\frac{2}{n(n-1)}(f_2-1)^2+\frac{4}{n(n-1)}(f_2-1)(\tilde{\varrho}-f_1).\tag{4.6}$$

Proof: Let us consider $\{e_1,\dots,e_n\}$ be an orthonormal basis of TM^n and $\{e_{n+1}=\varphi e_1,\dots,e_{2n}=\varphi e_n,e_{2n+1}=\xi\}$ be orthonormal basis of $T^{\perp}M^n$. Now from (2.9) and (2.16) we have

$$\begin{aligned}&g(\widetilde{R}^{\perp}(e_i,e_j)e_{n+r},e_{n+s})\\ &\quad=f_2[g(\varphi e_i,e_{n+s})g(\varphi e_j,e_{n+r})-g(\varphi e_i,e_{n+r})g(\varphi e_j,e_{n+s})]\\ &\qquad+\{g(\varphi e_i,e_{n+r})g(\varphi e_j,e_{n+s})-g(\varphi e_j,e_{n+r})g(\varphi e_i,e_{n+s})\}\\ &\qquad+g([A_r,A_s]e_i,e_j)\\ &\quad=(f_2-1)(\delta_{is}\delta_{jr}-\delta_{ir}\delta_{js})+g([A_r,A_s]e_i,e_j).\end{aligned}\tag{4.7}$$

Contracting (4.7) and using (2.28) we have

$$
\begin{aligned}
(\widetilde{\tau}^{\perp})^2 &= \sum_{1\le r<s\le n}\ \sum_{1\le i<j\le n} g^2(\widetilde{R}^{\perp}(e_i,e_j)e_{n+r},e_{n+s}) \\
&= \sum_{1\le r<s\le n}\ \sum_{1\le i<j\le n} [(f_2-1)(\delta_{is}\delta_{jr}-\delta_{ir}\delta_{js})+g([A_r,A_s]e_i,e_j)]^2 \\
&= \sum_{1\le r<s\le n}\ \sum_{1\le i<j\le n} \big[g^2([A_r,A_s]e_i,e_j)+(f_2-1)^2(\delta_{is}\delta_{jr}-\delta_{ir}\delta_{js})^2 \\
&\quad + 2(f_2-1)(\delta_{is}\delta_{jr}-\delta_{ir}\delta_{js})g([A_r,A_s]e_i,e_j)\big] \\
&= \frac{n^2(n-1)^2}{4}\varrho_N^2+\frac{n(n-1)(f_2-1)^2}{2}-(f_2-1)\|h\|^2 \\
&\quad + (f_2-1)n^2\|H\|^2.
\end{aligned}
\tag{4.8}
$$

From (2.9), (2.15) and (2.27) we have

$$2\widetilde{\tau}=n^2\|H\|^2-\|h\|^2+n(n-1)f_1$$

or equivalently,

$$n^2\|H\|^2-\|h\|^2=n(n-1)(\tilde{\varrho}-f_1). \tag{4.9}$$

Substituting (4.9) in (4.8) and using (2.28) we get

$$(\tilde{\varrho}^{\perp})^2\le \varrho_N^2+\frac{4}{n(n-1)}(\tilde{\varrho}-f_1)(f_2-1)\frac{2(f_2-1)^2}{n(n-1)}. \tag{4.10}$$

By virtue of Proposition 4.1 and (4.10), we obtain the inequality (4.6). □

5. Wintgen inequality on Legendrian submanifolds of $\overline{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\hat{\overline{\nabla}}$

Proposition 5.1. *Let M^m be a C-totally real submanifold of $\overline{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\hat{\overline{\nabla}}$. Then*

$$\|H\|^2+f_1\ge\hat{\varrho}+\varrho_N. \tag{5.1}$$

Proof: The proof is similar to the proof of Proposition 4.1 □

Theorem 5.1. *Let M^n be a Legendrian submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\overline{\nabla}}$. Then*

$$(\hat{\varrho}^{\perp})^2 \leq (\|H\|^2 - \hat{\varrho} + f_1) + \frac{2}{n(n-1)}(f_2 + (f_1 - f_3)^2)^2 + \frac{4}{n(n-1)}(f_2 + (f_1 - f_3)^2)(\hat{\varrho} - f_1). \tag{5.2}$$

Proof: Now from (2.10) and (2.18) we have

$$\begin{aligned} g(\widehat{R}^{\perp}(e_i, e_j)e_{n+r}, e_{n+s}) &= f_2[g(\varphi e_i, e_{n+s})g(\varphi e_j, e_{n+r}) - g(\varphi e_i, e_{n+r})g(\varphi e_j, e_{n+s})] \\ &\quad + (f_1 - f_3)^2\{g(\varphi e_j, e_{n+r})g(\varphi e_i, e_{n+s}) \\ &\quad - g(\varphi e_i, e_{n+r})g(\varphi e_j, e_{n+s})\} + g([A_r, A_s]e_i, e_j) \\ &= (f_2 + (f_1 - f_3)^2)(\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}) + g([A_r, A_s]e_i, e_j). \end{aligned} \tag{5.3}$$

From (2.30) and (5.3) we have

$$\begin{aligned} (\widehat{\tau}^{\perp})^2 &= \sum_{1\leq r<s\leq n} \sum_{1\leq i<j\leq n} g(\widehat{R}^{\perp}(e_i, e_j)e_{n+r}, e_{n+s})^2 \\ &= \sum_{1\leq r<s\leq n} \sum_{1\leq i<j\leq n} [(f_2 + (f_1 - f_3)^2)(\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}) \\ &\quad + g([A_r, A_s]e_i, e_j)]^2 \\ &= \sum_{1\leq r<s\leq n} \sum_{1\leq i<j\leq n} \big[g^2([A_r, A_s]e_i, e_j) + (f_2 + (f_1 - f_3)^2)^2 \\ &\quad \times (\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js})^2 + 2(f_2 + (f_1 - f_3)^2)(\delta_{is}\delta_{jr} - \delta_{ir}\delta_{js}) \\ &\quad \times g([A_r, A_s]e_i, e_j)\big] \\ &= \frac{n^2(n-1)^2}{4}\varrho_N^2 + \frac{n(n-1)(f_2 + (f_1 - f_3)^2)^2}{2} \\ &\quad - (f_2 + (f_1 - f_3)^2)\|h\|^2 + f_2 n^2\|H\|^2. \end{aligned} \tag{5.4}$$

From (2.10) and (2.17) we have

$$2\,\widehat{\tau} = n^2\|H\|^2 - \|h\|^2 + n(n-1)f_1$$

or equivalently,

$$n^2\|H\|^2 - \|h\|^2 = n(n-1)(\hat{\varrho} - f_1). \tag{5.5}$$

Substituting (5.5) in (5.4) and using (2.30) we get

$$(\hat{\varrho}^{\perp})^2 \leq \varrho_N^2 + \frac{4}{n(n-1)}(\hat{\varrho} - f_1)(f_2 + (f_1 - f_3)^2) + \frac{2(f_2 + (f_1 - f_3)^2)^2}{n(n-1)}. \tag{5.6}$$

By virtue of Proposition 5.1 and (5.6) we have the inequality (5.2). □

6. Wintgen inequality on Legendrian submanifolds of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\overline{\nabla}}$

Proposition 6.1. *Let M^m be a C-totally real submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\overline{\nabla}}$. Then*

$$\|H\|^2 + f_1 \geq \overset{*}{\varrho} + \varrho_N. \tag{6.1}$$

Proof: The proof is similar to the proof of Proposition 4.1. □

Theorem 6.1. *Let M^n be a Legendrian submanifold of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\overset{*}{\overline{\nabla}}$. Then*

$$\begin{aligned}(\overset{*}{\varrho}{}^{\perp})^2 \leq (\|H\|^2 - \hat{\varrho} + f_1) &+ \frac{2}{n(n-1)}(f_2 + (f_1 - f_3)^2)^2 \\ &+ \frac{4}{n(n-1)}(f_2 + (f_1 - f_3)^2)(\hat{\varrho} - f_1).\end{aligned} \tag{6.2}$$

Proof: The proof is similar to the proof of Theorem 5.1. □

7. Summary

Here, we present a summary of the results obtained on submanifolds of $\overline{M}^{2n+1}(f_1, f_2, f_3)$ with respect to the three connections considered; namely, quarter symmetric metric connection $\tilde{\overline{\nabla}}$, Schouten–van Kampen connection $\hat{\overline{\nabla}}$ and Tanaka–Webster connection $\overset{*}{\overline{\nabla}}$.

Connection	**C-totally inequality**	**Wintgen inequality**
$\tilde{\bar{\bar{\nabla}}}$	$\Vert H\Vert^2 + f_1 \geq \tilde{\varrho} + \varrho_N$	$(\tilde{\varrho}^{\perp})^2 \leq (\Vert H\Vert^2 - \tilde{\varrho} + f_1) + \frac{2}{n(n-1)}(f_2 - 1)^2 + \frac{4}{n(n-1)}(f_2 - 1)(\tilde{\varrho} - f_1)$
$\hat{\bar{\bar{\nabla}}}$	$\Vert H\Vert^2 + f_1 \geq \hat{\varrho} + \varrho_N$	$(\hat{\varrho}^{\perp})^2 \leq (\Vert H\Vert^2 - \hat{\varrho} + f_1) + \frac{2}{n(n-1)}(f_2 + (f_1 - f_3)^2)^2 + \frac{4}{n(n-1)}(f_2 + (f_1 - f_3)^2)(\hat{\varrho} - f_1)$
$\overset{*}{\bar{\bar{\nabla}}}$	$\Vert H\Vert^2 + f_1 \geq \overset{*}{\varrho} + \varrho_N$	$(\overset{*}{\varrho}{}^{\perp})^2 \leq (\Vert H\Vert^2 - \overset{*}{\varrho} + f_1) + \frac{2}{n(n-1)}(f_2 + (f_1 - f_3)^2)^2 + \frac{4}{n(n-1)}(f_2 + (f_1 - f_3)^2)(\overset{*}{\varrho} - f_1)$

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S. K. Hui (corresponding author):
Department of Mathematics, The University of Burdwan, Golapbag,
Burdwan 713104, West Bengal, India

E-mail: skhui@math.buruniv.ac.in

R. S. Lemence:
Department of Mathematics, Institute of Arts and Sciences,
Far Eastern University, Nicanor Reyes St, Sampaloc, Manila 1008,
Philippines

E-mail: rlemence@feu.edu.ph

P. Mandal:
Department of Mathematics, The University of Burdwan, Golapbag,
Burdwan 713104, West Bengal, India

E-mail: pm2621994@gmail.com

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