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A short note on f-biharmonic hypersurfaces

Selcen Y. Perktaş, Bilal E. Acet, Adara M. Blaga

Abstract. In the present paper we give some properties of f-biharmonic hypersurfaces in real space forms. By using the f-biharmonic equation for a hypersurface of a Riemannian manifold, we characterize the f-biharmonicity of constant mean curvature and totally umbilical hypersurfaces in a Riemannian manifold and, in particular, in a real space form. As an example, we consider f-biharmonic vertical cylinders in $S^2 \times \mathbb{R}$.

Keywords: f-biharmonic maps; f-biharmonic hypersurface

Classification: 58E20, 53C25, 53C43

1. Introduction

In the latest years, the interest in biharmonic maps theory and its applications to other areas has considerably increased. For some recent geometric studies of general biharmonic maps and biharmonic submanifolds, see [13], [1], [12], [15], [16], [8], [17], [18] and the references therein. Harmonic maps are generalizations of geodesics and minimal immersions, defined as critical points of the energy functional.

J. Eells and J. H. Sampson in [5] introduced the notion of biharmonic map as a critical point of the bienergy functional. In [7], G. Y. Jiang derived the Euler–Lagrange equations whose solutions are the biharmonic maps, from where it is clear that any harmonic map is biharmonic.

On the other hand, B. Y. Chen in [2] defined biharmonic submanifolds of Euclidean spaces by $\Delta H = 0$, more precisely, any submanifold in an Euclidean space whose mean curvature H is harmonic is called a biharmonic submanifold, where Δ is the Laplace operator of the submanifold. If one uses the definition of biharmonic maps to Riemannian immersions into Euclidean spaces, it is easy to see that Chen's definition for biharmonic submanifold coincides with the definition given by using the bienergy functional. There are many results on the nonexistence of biharmonic submanifolds in manifolds with nonpositive sectional

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curvature. These nonexistence consequences, see [1], [6], [11], led the studies to spheres and other nonnegatively curved spaces.

Recall that f-harmonic maps between Riemannian manifolds were firstly introduced and studied by A. Lichnerowicz in 1970, see also [4], as critical points of the f-energy functional. If f is a constant function, then f-harmonic maps are harmonic. Thus, proper f-harmonic maps (i.e. for f a nonconstant function) are more interesting to study. f-harmonic maps have also some physical meanings by considering them as solutions of the continuous spin systems and inhomogenous Heisenberg spin systems, see [9], [3]. Moreover, there is a strong relationship between f-harmonic maps and the gradient Ricci solitons, see [19].

The notion of f-biharmonic map has been introduced by W.-J. Lu in [10] as a critical point of the f-bienergy functional. If $f = 1$, then f-biharmonic maps are biharmonic.

2. f-biharmonic hypersurfaces

Let $\Psi\colon (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and f a positive smooth function on M.

The map Ψ is called *f-biharmonic map* if it is a critical point of the f-bienergy functional $E_{2,f}$ defined by

$$E_{2,f}(\Psi) := \frac{1}{2}\int_{\Omega} f|\tau(\Psi)|^2\vartheta_g,$$

where Ω is a compact domain of M and $\tau(\Psi) := \mathrm{trace}\nabla \mathrm{d}\Psi$ is the tension field of Ψ. The Euler–Lagrange equation gives the f-biharmonic map equation, see [10],

$$\tau_{2,f}(\Psi) \equiv f\tau_2(\Psi) + (\triangle f)\tau(\Psi) + 2\nabla^{\Psi}_{\mathrm{grad} f}\tau(\Psi) = 0,$$

where $\tau_2(\Psi) \equiv -\triangle\tau(\Psi) - \mathrm{trace}R^N(\mathrm{d}\Psi, \tau(\Psi))\mathrm{d}\Psi$ is the bitension field of Ψ, with $\Delta = -\mathrm{trace}(\nabla^{\Psi}\nabla^{\Psi} - \nabla^{\Psi}_{\nabla})$ the rough Laplace operator on sections of $\Psi^{-1}TN$ and R^N the curvature operator of N.

Definition 2.1 ([14])**.** A submanifold in a Riemannian manifold is called an *f-biharmonic submanifold* if the isometric immersion defining the submanifold is an f-biharmonic map.

The minimal submanifolds are well known basic examples of biharmonic submanifolds and all the biharmonic submanifolds are f-biharmonic with the function $f = 1$. f-biharmonic submanifolds which are neither minimal nor biharmonic submanifolds will be called proper f-biharmonic submanifolds.

The f-biharmonic equation for a hypersurface of a Riemannian manifold is given in the following.

Theorem 2.1 ([14]). *Let $\Psi\colon M^m \to N^{m+1}$ be an isometric immersion of codimension one with mean curvature vector $\eta = H\xi$. Then Ψ is an f-biharmonic map if and only if*

$$\text{(2.1)} \qquad \begin{cases} \triangle H - \big(|A|^2 - \mathrm{Ric}^N(\xi,\xi) - \frac{\triangle f}{f}\big)H + 2(\mathrm{grad}\ln f)(H) = 0, \\ 2A(\mathrm{grad}H) + \frac{m}{2}\mathrm{grad}H^2 - 2H\left(\mathrm{Ric}^N(\xi)\right)^T + 2HA(\mathrm{grad}\ln f) = 0, \end{cases}$$

where Ric^N denotes the Ricci operator of the ambient space, A is the shape operator of the hypersurface with respect to the unit normal vector ξ, and Δ and grad *are the Laplace and the gradient operator of the hypersurface, respectively.*

For hypersurfaces with constant mean curvature, by using (2.1) we have

Proposition 2.1. *Let M be a constant mean curvature hypersurface in a Riemannian manifold N. Then M is an f-biharmonic hypersurface if and only if it is minimal or*

$$\text{(2.2)} \qquad \begin{cases} \mathrm{Ric}^N(\xi,\xi) = |A|^2 - \frac{\triangle f}{f}, \\ \left(\mathrm{Ric}^N(\xi)\right)^T = A(\mathrm{grad}\ln f). \end{cases}$$

Corollary 2.1. *If f is a harmonic function, then a constant mean curvature hypersurface in a Riemannian manifold N with $\mathrm{Ric}^N(\xi,\xi) \le 0$ is f-biharmonic if and only if it is minimal.*

It is well known that a Riemannian manifold (N^{m+1}, h) is called an *Einstein manifold* if its Ricci tensor Ric^N is of the form $\mathrm{Ric}^N = ah$, where a is constant. For an Einstein manifold (N^{m+1}, h), we have $a = r/(m+1)$, where r is the scalar curvature of (N^{m+1}, h). Then we get

$$\left(\mathrm{Ric}^N(\xi)\right)^T = 0 \qquad \text{and} \qquad \mathrm{Ric}^N(\xi,\xi) = \frac{r}{m+1}.$$

By using the last equations in (2.1) we get

Proposition 2.2. *Let M be a hypersurface in an Einstein manifold (N^{m+1}, h). Then M is an f-biharmonic hypersurface if and only if its mean curvature H satisfies*

$$\text{(2.3)} \qquad \begin{cases} \triangle H - \big(|A|^2 - \frac{r}{m+1} - \frac{\triangle f}{f}\big)H + 2(\mathrm{grad}\ln f)(H) = 0, \\ 2A(\mathrm{grad}H) + \frac{m}{2}\mathrm{grad}H^2 + 2HA(\mathrm{grad}\ln f) = 0, \end{cases}$$

where r is the scalar curvature of the ambient space.

If $N^{m+1}(c)$ is a real space form (i.e. a Riemannian manifold with constant sectional curvature c), then it is an Einstein manifold with scalar curvature $r = m(m+1)c$. From the previous Proposition, we recover the following result of [14].

Theorem 2.2 ([14]). *A hypersurface in a real space form $N^{m+1}(c)$ is f-biharmonic if and only if its mean curvature H satisfies*

$$\begin{cases} \triangle H - \big(|A|^2 - mc - \frac{\triangle f}{f}\big)H + 2(\operatorname{grad}\ln f)(H) = 0, \\ 2A(\operatorname{grad}H) + \frac{m}{2}\operatorname{grad}H^2 + 2HA(\operatorname{grad}\ln f) = 0. \end{cases} \tag{2.4}$$

For hypersurfaces with constant mean curvature, by using (2.4) we have

Proposition 2.3. *Let M be a constant mean curvature hypersurface in a real space form $N^{m+1}(c)$. Then M is an f-biharmonic hypersurface if and only if it is minimal or*

$$\begin{cases} \triangle f = (|A|^2 - mc)f, \\ A(\operatorname{grad}\ln f) = 0. \end{cases} \tag{2.5}$$

As a consequence we get

Corollary 2.2. *Let M be a nonzero constant mean curvature f-biharmonic hypersurface in the $(m+1)$-dimensional unit Euclidean sphere. If $|A|^2 = m$, then f is a harmonic function.*

Newt we shall characterize the f-biharmonic totally umbilical hypersurfaces in an Einstein manifold.

Theorem 2.3. *Let M be a totally umbilical hypersurface in an Einstein manifold (N^{m+1}, h). Then M is an f-biharmonic submanifold if and only if it is totally geodesic or*

$$\begin{cases} \triangle H - \big(|A|^2 - \frac{r}{m+1} - \frac{\triangle f}{f}\big)H + 2(\operatorname{grad}\ln f)(H) = 0, \\ (m+2)\operatorname{grad}H = -2A(\operatorname{grad}\ln f). \end{cases} \tag{2.6}$$

Proof: Let $\{e_1, \dots, e_m, \xi\}$ be an orthonormal frame of N adapted to the hypersurface M such that $Ae_i = \lambda_i e_i$, where A is the Weingarten map of the hypersurface and λ_i is the principal curvature in the direction e_i. Since M is totally umbilical, all the principal curvatures at any point $p \in M$ are equal to a $\lambda(p)$. It follows that

$$H = \frac{1}{m}\sum_{i=1}^{m}\langle Ae_i, e_i\rangle = \lambda, \tag{2.7}$$

$$A(\operatorname{grad}H) = A\bigg(\sum_{i=1}^{m} e_i(\lambda)e_i\bigg) = \frac{1}{2}\operatorname{grad}\lambda^2, \tag{2.8}$$

$$|A|^2 = m\lambda^2. \tag{2.9}$$

Using (2.7)–(2.9) in (2.3) we get

$$\begin{cases} \triangle\lambda - m\lambda^3 + \left(\frac{r}{m+1} + \frac{\triangle f}{f}\right)\lambda + 2(\operatorname{grad}\ln f)(\lambda) = 0, \\ \frac{m+2}{2}\operatorname{grad}\lambda^2 + 2\lambda A(\operatorname{grad}\ln f) = 0, \end{cases} \tag{2.10}$$

which is equivalent to

$$\begin{cases} \triangle\lambda - m\lambda^3 + \left(\frac{r}{m+1} + \frac{\triangle f}{f}\right)\lambda + 2(\operatorname{grad}\ln f)(\lambda) = 0, \\ \lambda[(m+2)\operatorname{grad}\lambda + 2A(\operatorname{grad}\ln f)] = 0. \end{cases} \tag{2.11}$$

Solving the equation we have either $\lambda = 0$ and hence $H = 0$, or (2.6) holds. □

3. Example. f-biharmonic cylinders in $S^2 \times \mathbb{R}$

Assume that $\varphi\colon (M^3, g) \to (N^2, h)$ is a Riemannian submersion with totally geodesic fibres from a complete manifold and $\alpha\colon I \to (N^2, h)$ is an immersed regular curve parametrized by arclength. In this case, the disjoint union $S = \bigsqcup_{t\in I}\varphi^{-1}(\alpha(t))$ of all horizontal lifts of the curve α, is a surface in M. Let $\beta\colon I \to (M^3, g)$ be a horizontal lift of α and let $\{X, \xi, V\}$ be an orthonormal frame of M adapted to the surface with ξ being the unit normal vector of surface and V the unit vector field tangent to the fibres of the submersion φ. Note that the restriction of this frame to the curve β is the Frenet frame along β and the Frenet formulas along β are

$$\begin{aligned} \nabla_X X &= \kappa\xi, \\ \nabla_X \xi &= -\kappa X + \tau V, \\ \nabla_X V &= -\tau\xi, \end{aligned} \tag{3.1}$$

where ∇ denotes the Levi–Civita connection of (M^3, g).

Denoting by b the second fundamental form, $b(X, Y) := \langle A(X), Y\rangle$, by using (3.1) we have the following equations, see also [12], which will be used in the f-biharmonic equation:

$$\begin{gathered} A(X) = -\langle\nabla_X \xi, X\rangle X - \langle\nabla_X \xi, V\rangle V = \kappa X - \tau V, \\ A(V) = -\langle\nabla_V \xi, X\rangle X - \langle\nabla_V \xi, V\rangle V = -\tau X, \\ b(X, X) = \langle A(X), X\rangle = \kappa, \\ b(X, V) = \langle A(X), V\rangle = -\tau = \langle A(V), X\rangle = b(V, X), \\ b(V, V) = \langle A(V), V\rangle = 0, \\ H = \frac{1}{2}(b(X, X) + b(V, V)) = \frac{\kappa}{2}, \end{gathered}$$

$$A(\mathrm{grad}H) = A\Big(X\Big(\frac{\kappa}{2}\Big)X + V\Big(\frac{\kappa}{2}\Big)V\Big) = \frac{\kappa'}{2}(\kappa X - \tau V),$$

$$\Delta H = X(X(H)) - (\nabla_X X)(H) + V(V(H)) - (\nabla_V V)(H) = \frac{\kappa''}{2},$$

$$|A|^2 = (b(X,X))^2 + (b(X,V))^2 + (b(V,X))^2 + (b(V,V))^2 = \kappa^2 + 2\tau^2.$$

Writing all these terms in the f-biharmonic equation for hypersurfaces given by (2.1), we obtain that S is an f-biharmonic surface in (M^3, g) if and only if

$$\text{(3.2)} \qquad \begin{cases} \frac{\kappa''}{2} - \frac{\kappa}{2}(\kappa^2 + 2\tau^2) + \frac{\kappa}{2}\mathrm{Ric}^M(\xi,\xi) + \frac{\kappa}{2}\frac{\triangle f}{f} + \kappa' X(\ln f) = 0, \\ \kappa\big[\frac{7}{4}\kappa' + \kappa X(\ln f) - \tau V(\ln f) - \mathrm{Ric}^M(\xi, X)\big]X \\ \qquad +[-\kappa'\tau - \kappa\mathrm{Ric}^M(\xi, V) - \kappa\tau X(\ln f)]V = 0, \end{cases}$$

or equivalently

$$\text{(3.3)} \qquad \begin{cases} \kappa'' - \kappa(\kappa^2 + 2\tau^2) + \kappa\mathrm{Ric}^M(\xi,\xi) + \kappa\frac{\triangle f}{f} + 2\kappa' X(\ln f) = 0, \\ \kappa\big[\frac{7}{4}\kappa' + \kappa X(\ln f) - \tau V(\ln f) - \mathrm{Ric}^M(\xi, X)\big] = 0, \\ -\kappa'\tau - \kappa\mathrm{Ric}^M(\xi, V) - \kappa\tau X(\ln f) = 0. \end{cases}$$

Now we consider the product space $S^2 \times \mathbb{R}$, where S^2 is the unit sphere with the standard metric. Let $\pi\colon S^2 \times \mathbb{R} \to S^2$ be the Riemannian submersion with totally geodesic fibers and integrable horizontal distribution. Then we have

Proposition 3.1. *The vertical cylinder $S = \bigsqcup_{t\in I} \varphi^{-1}(\alpha(t))$ is an f-biharmonic surface in $S^2 \times \mathbb{R}$ if and only if $\alpha\colon I \to (S^2, h)$ is a geodesic or a regular curve with the geodesic curvature satisfying*

$$\text{(3.4)} \qquad \frac{\kappa''}{k} - \frac{7(\kappa')^2}{2k^2} - k^2 = -\Big(\frac{\triangle f}{f} + 1\Big).$$

Proof: Let $S^2 \times \mathbb{R}$ be the product space endowed with the metric $h = \mathrm{d}r^2 + \sin^2 r\mathrm{d}\theta^2 + \mathrm{d}z^2$ with respect to the coordinates (r, θ, z). It is easy to see that

$$\text{(3.5)} \qquad \Big\{E_1 = \frac{\partial}{\partial r}, E_2 = \frac{1}{\sin r}\frac{\partial}{\partial\theta}, E_3 = \frac{\partial}{\partial z}\Big\}$$

is a local orthonormal frame on $S^2 \times \mathbb{R}$. The coefficients of the Levi–Civita connection are

$$\text{(3.6)} \qquad \begin{array}{ccc} \nabla_{E_1}E_1 = 0, & \nabla_{E_1}E_2 = 0, & \nabla_{E_1}E_3 = 0, \\ \nabla_{E_2}E_1 = \cot r\, E_2, & \nabla_{E_2}E_2 = -\cot r\, E_1, & \nabla_{E_2}E_3 = 0, \\ \nabla_{E_3}E_1 = 0, & \nabla_{E_3}E_2 = 0, & \nabla_{E_3}E_3 = 0. \end{array}$$

Hence we have

$$[E_1, E_2] = -\cot r\, E_2, \qquad [E_1, E_3] = [E_2, E_3] = 0, \tag{3.7}$$

and

$$\mathrm{Ric}(E_1, E_1) = \mathrm{Ric}(E_2, E_2) = 1, \qquad \text{all the other } \ \mathrm{Ric}(E_i, E_j) = 0. \tag{3.8}$$

Now let $\alpha\colon I \to (S^2, h)$ with $\alpha(s) = x(s)E_1 + y(s)E_2$ be an immersed regular curve parametrized by arclength. The tangent vector of the curve is given by

$$X = \alpha'(s) = x'(s)E_1 + y'(s)E_2,$$

and its principal normal is

$$\xi = y'(s)E_1 - x'(s)E_2,$$

while

$$V = E_3.$$

Then using (3.8) we have

$$\begin{aligned} \mathrm{Ric}(\xi, \xi) &= (x')^2 + (y')^2 = 1, \\ \mathrm{Ric}(\xi, X) &= x'y' - y'x' = 0, \\ \mathrm{Ric}(\xi, V) &= 0. \end{aligned} \tag{3.9}$$

By replacing (3.9) in (3.3) and using the fact that $\tau = 0$, we see that the vertical cylinder S is an f-biharmonic surface if and only if

$$\begin{cases} \kappa'' - \kappa\big(\kappa^2 - \frac{\triangle f}{f} - 1\big) + 2\kappa' X(\ln f) = 0, \\ \kappa[7\kappa' + 4\kappa X(\ln f)] = 0. \end{cases} \tag{3.10}$$

If we solve the equation above we have $\kappa = 0$ which implies that the vertical cylinder is minimal or the geodesic curvature of α satisfies $\frac{\kappa''}{k} - \frac{7(\kappa')^2}{2k^2} - k^2 = -\big(\frac{\triangle f}{f} + 1\big)$. □

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References

[1] Caddeo R., Montaldo S., Oniciuc C., *Biharmonic submanifolds of S^3*, Internat. J. Math. **12** (2001), no. 8, 867–876.

[2] Chen B.-Y., *Some open problems and conjectures on submanifolds of finite type*, Soochow J. Math. **17** (1991), no. 2, 169–188.

[3] Cieśliński J., Sym A., Wesselius W., *On the geometry of the inhomogeneous Heisenberg ferromagnet: nonintegrable case*, J. Phys. A. **26** (1993), no. 6, 1353–1364.

[4] Eells J., Lemaire L., *A report on harmonic maps*, Bull. London Math. Soc. **10** (1978), no. 1, 1–68.
[5] Eells J. Jr., Sampson J. H., *Harmonic mappings of the Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
[6] Jiang G. Y., *2-harmonic isometric immersions between Riemannian manifolds*, Chinese Ann. Math. Ser. A **7** (1986), no. 2, 130–144 (Chinese); English summary in Chinese Ann. Math. Ser. B **7** (1986), no. 2, 255.
[7] Jiang G. Y., *2-harmonic maps and their first and second variation formulas*, Chinese Ann. Math. Ser. A. **7** (1986), no. 4, 389–402 (Chinese); English summary in Chinese Ann. Math. Ser. B **7** (1986), no. 4, 523.
[8] Keleş S., Perktaş S. Y., Kılıç E., *Biharmonic curves in Lorentzian para-Sasakian manifolds*, Bull. Malays. Math. Sci. Soc. (2) **33** (2010), no. 2, 325–344.
[9] Li Y., Wang Y., *Bubbling location for F-harmonic maps and inhomogeneous Landau-Lifshitz equations*, Comment. Math. Helv. **81** (2006), no. 2, 433–448.
[10] Lu W.-J., *On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds*, Sci. China Math. **58** (2015), no. 7, 1483–1498.
[11] Montaldo S., Oniciuc C., *A short survey on biharmonic maps between Riemannian manifolds*, Rev. Un. Mat. Argentina **47** (2006), no. 2, 1–22.
[12] Ou Y.-L., *Biharmonic hypersurfaces in Riemannian manifolds*, Pacific J. Math. **248** (2010), no. 1, 217–232.
[13] Ou Y.-L., *Some constructions of biharmonic maps and Chen's conjecture on biharmonic hypersurfaces*, J. Geom. Phys. **62** (2012), no. 4, 751–762.
[14] Ou Y.-L., *On f-biharmonic maps and f-biharmonic submanifolds*, Pacific J. Math. **271** (2014), no. 2, 461–477.
[15] Ou Y.-L., Tang L., *On the generalized Chen's conjecture on biharmonic submanifolds*, Michigan Math. J. **61** (2012), no. 3, 531–542.
[16] Ou Y.-L., Wang Z.-P., *Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries*, J. Geom. Phys. **61** (2011), no. 10, 1845–1853.
[17] Perktaş S. Y., Kılıç E., *Biharmonic maps between doubly warped product manifolds*, Balkan J. Geom. Appl. **15** (2010), no. 2, 159–170.
[18] Perktaş S. Y., Kılıç E., Keleş S., *Biharmonic hypersurfaces of LP-Sasakian manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N.S.) **57** (2011), no. 2, 387–408.
[19] Rimoldi M., Veronelli G., *Topology of steady and expanding gradient Ricci solitons via f-harmonic maps*, Differetial. Geom. Appl. **31** (2013), no. 5, 623–638.

S. Y. Perktaş, B. E. Acet:
Faculty of Arts and Sciences, Department of Mathematics,
Adiyaman University, Altinşehir, Atatürk Blv. No:1,
02040 Merkez/Adiyaman, Turkey

E-mail: sperktas@adiyaman.edu.tr

E-mail: eacet@adiyaman.edu.tr

A. M. Blaga:
Department of Mathematics, West University of Timişoara,
Bld. V. Pârvan nr. 4, 300223, Timişoara, România

E-mail: adarablaga@yahoo.com

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