Relative weak derived functors

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Abstract. Let R be a ring, n a fixed non-negative integer, $\mathscr{WI}$ the class of all left R-modules with weak injective dimension at most n, and $\mathscr{WF}$ the class of all right R-modules with weak flat dimension at most n. Using left (right) $\mathscr{WI}$-resolutions and the left derived functors of Hom we study the weak injective dimensions of modules and rings. Also we prove that $-\otimes-$ is right balanced on $\mathscr{M}_R \times {}_R\mathscr{M}$ by $\mathscr{WF} \times \mathscr{WI}$, and investigate the global right $\mathscr{WI}$-dimension of ${}_R\mathscr{M}$ by right derived functors of $\otimes$.

Keywords: weak injective module; weak flat module; weak injective dimension; weak flat dimension

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1. Introduction

Throughout this paper, R denotes an associative ring with identity and all modules are unitary. For a left R-module M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+ and for a class of R-modules $\mathcal{C}$, we denote by $\mathcal{C}^+ = \{C^+ \colon C \in \mathcal{C}\}$. Denote by ${}_R\mathscr{M}$ the category of all left R-modules and by $\mathscr{M}_R$ the category of right R-modules. For unexplained concepts and notations, we refer the reader to [2], [7], [9].

We first recall some known notions and facts needed in the sequel.

Let $\mathcal{C}$ be a class of left R-modules and M a left R-module. Following [2], we say that a map $f \in \mathrm{Hom}_R(C, M)$ with $C \in \mathcal{C}$ is a $\mathcal{C}$-*precover* of M, if the group homomorphism $\mathrm{Hom}_R(C', f) \colon \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $f \in \mathrm{Hom}_R(C, M)$ of M is called a $\mathcal{C}$-*cover* of M if f is right minimal, that is, if $fg = f$ implies that g is an automorphism for each $g \in \mathrm{End}_R(C)$. Dually, we have the definition of $\mathcal{C}$-*preenvelope* (or $\mathcal{C}$-*envelope*). In general, $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist, if exists, they are unique up to isomorphism.

A pair $(\mathcal{F}, \mathcal{C})$ of classes of left R-modules is called a *cotorsion theory*, see [2], if $\mathcal{F}^\perp = \mathcal{C}$ and ${}^\perp\mathcal{C} = \mathcal{F}$, where $\mathcal{F}^\perp = \{M \in {}_R\mathscr{M} \colon \ \mathrm{Ext}^1_R(F, M) = 0 \ \ \forall\, F \in \mathcal{F}\}$ and ${}^\perp\mathcal{C} = \{M \in {}_R\mathscr{M} \colon \ \mathrm{Ext}^1_R(M, C) = 0 \ \ \forall\, C \in \mathcal{C}\}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called *perfect*, see [6], if every left R-module has a $\mathcal{C}$-envelope and a $\mathcal{F}$-cover.

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Let C, D and E be abelian categories and $T\colon C \times D \to E$ an additive functor contravariant in the first variable and covariant in the second. Let $\mathcal{F}$ and $\mathcal{G}$ be classes of objects of C and D, respectively. Then T is said to be right (or left) balanced by $\mathcal{F} \times \mathcal{G}$ [2, Definition 8.2.13] if for every object M of C, there is a $T(-, \mathcal{G})$-exact complex

$$\cdots \to F_1 \to F_0 \to M \to 0 \qquad (\text{or } \; 0 \to M \to F^0 \to F^1 \to \cdots)$$

with each F_i (or F^i) in $\mathcal{F}$, and for every object N of D, there is a $T(\mathcal{F}, -)$-exact complex

$$0 \to N \to G^0 \to G^1 \to \cdots \qquad (\text{or } \; \cdots \to G_1 \to G_0 \to N \to 0)$$

with each G^i (or G_i, respectively) in $\mathcal{G}$.

In [8], B. Stenström defined and studied FP-injective modules. A left R-module M is called *FP-injective* (or *absolutely pure*) if $\mathrm{Ext}^1_R(F, M) = 0$ for all finitely presented left R-modules F. The *FP-injective dimension* of M, denoted by FP-id(M), is defined to be the smallest non-negative integer n such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for every finitely presented left R-module F (if no such n exists, set FP-id$(M) = \infty$).

A left R-module M is called *super finitely presented*, see [4], if there exists an exact sequence of left R-modules: $\cdots \to F_1 \to F_0 \to M \to 0$, where each F_i is finitely generated and projective. Recently, Z. Gao and F. Wang introduced the notion of weak injective and weak flat modules, see [4]. A left R-module M is called *weak injective* if $\mathrm{Ext}^1_R(F, M) = 0$ for any super finitely presented left R-module F. A right R-module N is called *weak flat* if $\mathrm{Tor}^R_1(N, F) = 0$ for any super finitely presented left R-module F. The class of all weak injective (or weak flat) left (or right) R-modules is denoted by $\mathcal{WI}$ (or $\mathcal{WF}$, respectively).

Accordingly, the *weak injective dimension* of a left R-module M, denoted by $\mathrm{wid}_R(M)$, is defined to be the smallest $n \geq 0$ such that $\mathrm{Ext}^{n+1}_R(F, M) = 0$ for all super finitely presented left R-modules F. If no such n exists, set $\mathrm{wid}_R(M) = \infty$. The *weak flat dimension* of a right R-module N, denoted by $\mathrm{wfd}_R(N)$, is defined to be the smallest $n \geq 0$ such that $\mathrm{Tor}^R_{n+1}(N, F) = 0$ for all super finitely presented left R-modules F. If no such n exists, set $\mathrm{wfd}_R(N) = \infty$. The *left super finitely presented dimension*, denoted by l.sp.gldim(R), of a ring R is defined as l.sp.gldim$(R) = \sup\{pd_R(M)\colon M$ is a super finitely presented left R-module$\}$.

Let n be a fixed non-negative integer. In what follows, the symbols $\mathscr{F}$, $\mathscr{WI}$ and $\mathscr{WF}$ denotes the classes of all left R-modules with FP-injective dimension at most n, left R-modules with weak injective dimension at most n and right R-modules with weak flat dimension at most n, respectively.

In [10], Y. Zeng and J. Chen proved all left R-modules over a left coherent ring R have $\mathscr{F}$-preenvelope and $\mathscr{F}$-cover and they investigated the derived functors of Hom using $\mathscr{F}$-resolutions. In [5], Z. Gao and Z. Huang investigated the derived functors of Hom and $\otimes$ using $\mathcal{WI}$ and $\mathcal{WF}$-resolutions. Recently, T. Zhao in [12] proved that over any ring R, $\mathscr{WI}$ and $\mathscr{WF}$ are preenveloping and covering classes. Inspired by the above works and by [11], in this paper we investigate the derived functors of Hom and $\otimes$ using $\mathscr{WI}$ and $\mathscr{WF}$-resolutions. This paper is organized as follows.

In Section 2, we investigate the $\mathscr{WI}$-dimensions of modules and rings in terms of left or right $\mathscr{WI}$-resolutions. We give some characterizations of right $\mathscr{WI}$-dim $M \leq m$ and right $\mathscr{WI}$-dim ${}_RR \leq m$. Also, we obtain some equivalent conditions concerning the weak injective dimension of a module N.

In Section 3, we first show that $-\otimes-$ is right balanced on $\mathscr{M}_R \times {}_R\mathscr{M}$ by $\mathscr{WF} \times \mathscr{WI}$. Then we investigate the global right $\mathscr{WI}$-dimension of ${}_R\mathscr{M}$ in terms of the properties of the right derived functors of "$\otimes$".

The following results proved by T. Zhao in [12] will be used frequently in this paper.

Proposition 1.1 ([12, Corollary 2.4])**.**

(1) *For a left R-module M, we have* $\mathrm{wid}_R(M) = \mathrm{wfd}_R(M^+)$.
(2) *For a right R-module M, we have* $\mathrm{wfd}_R(M) = \mathrm{wid}_R(M^+)$.

Theorem 1.2 ([12, Theorem 4.8 and Theorem 4.9])**.** *The class $\mathscr{WI}$ is preenveloping and covering.*

Theorem 1.3 ([12, Theorem 4.4 and Theorem 4.5])**.** *The class $\mathscr{WF}$ is preenveloping and covering.*

2. Left derived functors of Hom and right $\mathscr{WI}$-dimension

By Theorem 1.2, all left R-modules have $\mathscr{WI}$-preenvelopes and $\mathscr{WI}$-covers. Hence $\mathrm{Hom}(-,-)$ is left balanced on ${}_R\mathscr{M} \times {}_R\mathscr{M}$ by $\mathscr{WI} \times \mathscr{WI}$. Let $\mathrm{Ext}_m(-,-)$ denote the mth left derived functor of $\mathrm{Hom}(-,-)$ with respect to the pair $\mathscr{WI} \times \mathscr{WI}$. Then, for any two left R-modules M and N, $\mathrm{Ext}_m(M,N)$ can computed by using a right $\mathscr{WI}$-resolution of M or a left $\mathscr{WI}$-resolution of N. For a left $\mathscr{WI}$-resolution of M: $\cdots \to F_1 \to F_0 \to M \to 0$ with each $F_i \in \mathscr{WI}$, write

$$K_0 = M, \quad K_1 = \ker(F_0 \to M), \quad \text{and} \quad K_i = \ker(F_{i-1} \to F_{i-2}) \qquad \text{for } \ i \geq 2.$$

The mth kernel K_m, $m \geq 0$, is called the mth $\mathscr{WI}$-syzygy of M.

Let $0 \to M \overset{g}{\to} F^0 \overset{f}{\to} F^1 \to \cdots$ be a right $\mathscr{WI}$-resolution of M in ${}_R\mathscr{M}$. Applying $\mathrm{Hom}_R(-,N)$ to the sequence, we get the deleted complex

$$\cdots \to \mathrm{Hom}(F^1,N) \overset{f^*}{\to} \mathrm{Hom}(F^0,N) \to 0.$$

Then $\mathrm{Ext}_m(M,N)$ is exactly the mth homology of the complex above. There is a canonical map

$$\sigma\colon \mathrm{Ext}_0(M,N) = \mathrm{Hom}(F^0,N)/\mathrm{im}(f^*) \to \mathrm{Hom}(M,N),$$

which is defined by $\sigma(\alpha + \mathrm{im}(f^*)) = \alpha g$ for each $\alpha \in \mathrm{Hom}(F^0,N)$.

Following [2], the left $\mathscr{WI}$-dimension of a left R-module M, denoted by left $\mathscr{WI}$-dim M, is defined as $\inf\{m\colon$ there is a left $\mathscr{WI}$-resolution of the form $0 \to F_m \to \cdots \to F_0 \to M \to 0\}$. If there is no such m, set left $\mathscr{WI}$-dim $M = \infty$. The global left $\mathscr{WI}$-dimension of ${}_R\mathscr{M}$, denoted by gl.left $\mathscr{WI}$-dim ${}_R\mathscr{M}$, is defined to be sup {left $\mathscr{WI}$-dim $M\colon M \in {}_R\mathscr{M}$}. The right versions can be defined similarly, and they are denoted by right $\mathscr{WI}$-dim M and gl.right $\mathscr{WI}$-dim ${}_R\mathscr{M}$.

Definition 2.1. Let R be a ring and M a left R-module. Then $\mathscr{WI}$-dim(M) is defined to be the smallest non-negative integer m such that $\mathrm{Ext}^{m+n+1}(F,M) = 0$ for every super finitely presented left R-module F. If no such m exists, set $\mathscr{WI}$-dim$(M) = \infty$.

Remark 2.2. We note that if $n = 0$, then $\mathscr{WI}$-dim(M) coincides with wid(M) and if R is coherent ring then $\mathscr{WI}$-dim(M) coincides with $\mathscr{F}$-dim(M), see [10, Definition 3.1]. Moreover, if R is a coherent ring and $n = 0$, then $\mathscr{WI}$-dim(M) is coincide with FP-id(M).

Lemma 2.3. *The following statements are equivalent for any $M \in {}_R\mathscr{M}$ and $m \geq 0$:*

(1) $\mathscr{WI}$-dim$(M) \leq m$;
(2) $\mathrm{Ext}^{n+m+1}(N,M) = 0$ *for any super finitely presented left R-module N;*
(3) *if the sequence $0 \to M \to F^0 \to \cdots F^m \to 0$ is exact with each $F^0, \cdots, F^{m-1} \in \mathscr{WI}$ then $F^m \in \mathscr{WI}$;*
(4) $\mathrm{wid}_R(M) \leq m + n$.

Proof: (1) $\Rightarrow$ (2). We will proceed by induction on m. If $\mathscr{WI}$-dim$(M) = 0$, then it is clear. Suppose that $m \geq 1$ and N is a super finitely presented left R-module. Let $0 \to K \to P \to N \to 0$ be a projective resolution of N with P finitely generated projective. Then K is super finitely presented, and $\mathrm{Ext}^{n+m+1}(N,M) \cong \mathrm{Ext}^{n+m}(K,M) = 0$ by induction.

(2) $\Rightarrow$ (1) is trivial.

(2) $\Leftrightarrow$ (4) follows from [4, Proposition 3.3].

(2) $\Rightarrow$ (3). Note that $\mathrm{Ext}^{n+m+1}(N,M) \cong \mathrm{Ext}^{n+1}(N,F^m)$ for all super finitely presented left R-module N. Then the implication follows by [4, Proposition 3.3].

(3) $\Rightarrow$ (2). Let $0 \to M \to E^0 \to \cdots \to E^{m-1} \to \cdots$ be an injective resolution of M. Then we have $K = \mathrm{coker}(E^{m-2} \to E^{m-1}) \in \mathscr{WI}$. From the isomorphism $\mathrm{Ext}^{n+m+1}(N,M) \cong \mathrm{Ext}^{n+1}(N,K)$, it follows that $\mathrm{Ext}^{n+m+1}(N,M) = 0$ for all super finitely presented left R-module N. □

Proposition 2.4. *Let R be a ring. Then $\mathscr{WI}$-$\mathrm{dim}(M)$ = right $\mathscr{WI}$-$\mathrm{dim}\ M$ for any left R-module M. Moreover right $\mathscr{WI}$-$\mathrm{dim}\ M \leq m$ if and only if $\mathrm{wid}_R(M) \leq m+n$.*

PROOF: It is trivial by Lemma 2.3. □

Proposition 2.5. *The following statements are equivalent for any $M \in {}_R\mathscr{M}$:*

(1) $\mathrm{wid}_R(M) \leq n$;
(2) *the canonical map $\sigma\colon \mathrm{Ext}_0(M,N) \to \mathrm{Hom}(M,N)$ is an isomorphism for any $N \in {}_R\mathscr{M}$;*
(3) *the canonical map $\sigma\colon \mathrm{Ext}_0(M,M) \to \mathrm{Hom}(M,M)$ is an isomorphism;*
(4) *the canonical map $\sigma\colon \mathrm{Ext}_0(M,N) \to \mathrm{Hom}(M,N)$ is an epimorphism for any $N \in {}_R\mathscr{M}$;*
(5) *the canonical map $\sigma\colon \mathrm{Ext}_0(M,M) \to \mathrm{Hom}(M,M)$ is an epimorphism.*

PROOF: (1) $\Rightarrow$ (2) is clear by setting $F^0 = M$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are trivial.

(5) $\Rightarrow$ (1). By (5), there exists $\alpha \in \mathrm{Hom}(F^0,M)$ such that $\sigma(\alpha + \mathrm{im}(f^*)) = \alpha g = 1_M$. So M is isomorphism to a direct summand of F^0, and hence $\mathrm{wid}_R(M) \leq n$. □

Corollary 2.6. *The following statements are equivalent:*

(1) $\mathrm{wid}_R({}_RR) \leq n$;
(2) *the canonical map $\sigma\colon \mathrm{Ext}_0(R,N) \to \mathrm{Hom}(R,N)$ is an isomorphism for any $N \in {}_R\mathscr{M}$;*
(3) *the canonical map $\sigma\colon \mathrm{Ext}_0(R,R) \to \mathrm{Hom}(R,R)$ is an isomorphism;*
(4) *the canonical map $\sigma\colon \mathrm{Ext}_0(R,N) \to \mathrm{Hom}(R,N)$ is an epimorphism for any $N \in {}_R\mathscr{M}$;*
(5) *the canonical map $\sigma\colon \mathrm{Ext}_0(R,R) \to \mathrm{Hom}(R,R)$ is an epimorphism.*

PROOF: It follows from Proposition 2.5. □

Proposition 2.7. *The following statements are equivalent for any $M \in {}_R\mathscr{M}$:*

(1) *right $\mathscr{WI}$-dim $M \leq 1$;*
(2) *the canonical map $\sigma\colon \mathrm{Ext}_0(M,N) \to \mathrm{Hom}(M,N)$ is a monomorphism for any left R-module N.*

Proof: (1) $\Rightarrow$ (2). By assumption, M has a right $\mathscr{WI}$-resolution $0 \to M \to F^0 \to F^1 \to 0$. Thus we get an exact sequence $0 \to \mathrm{Hom}(F^1, N) \to \mathrm{Hom}(F^0, N) \to \mathrm{Hom}(M, N)$ for any left R-module N. Hence σ is a monomorphism.

(2) $\Rightarrow$ (1). Let $0 \to M \to E \to L \to 0$ be an exact sequence of left R-modules with $M \to E$ being a $\mathscr{WI}$-preenvelope of M. It is enough to prove that $L \in \mathscr{WI}$. By [2, Theorem 8.2.3], we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & \mathrm{Ext}_0(L,L) \arrow[r] \arrow[d, "\sigma_1"'] & \mathrm{Ext}_0(E,L) \arrow[r] \arrow[d, "\sigma_2"'] & \mathrm{Ext}_0(M,L) \arrow[r] \arrow[d, "\sigma_3"'] & 0 \\
0 \arrow[r] & \mathrm{Hom}(L,L) \arrow[r] & \mathrm{Hom}(E,L) \arrow[r] & \mathrm{Hom}(M,L). &
\end{tikzcd}$$

Note that σ_2 is an epimorphism by Proposition 2.5 and σ_3 is a monomorphism by (2). Hence σ_1 is an epimorphism by the Snake lemma. Thus $L \in \mathscr{WI}$ by Proposition 2.5. $\square$

Proposition 2.8. *The following statements are equivalent for any $M \in {}_R\mathscr{M}$ and any $m \geq 2$:*

(1) *right $\mathscr{WI}$-dim $M \leq m$;*
(2) $\mathrm{Ext}_{m+k}(M,N) = 0$ *for any $N \in {}_R\mathscr{M}$ and $k \geq -1$;*
(3) $\mathrm{Ext}_{m-1}(M,N) = 0$ *for any $N \in {}_R\mathscr{M}$.*

Proof: (1) $\Rightarrow$ (2). Let $0 \to M \to F^0 \to \cdots \to F^m \to 0$ be a right $\mathscr{WI}$-resolution of M. Then we have an exact sequence

$$0 \to \mathrm{Hom}(F^m, N) \to \mathrm{Hom}(F^{m-1}, N) \to \mathrm{Hom}(F^{m-2}, N)$$

for all left R-modules N. Hence $\mathrm{Ext}_m(M,N) = \mathrm{Ext}_{m-1}(M,N) = 0$. It is clear that $\mathrm{Ext}_{m+k}(M,N) = 0$ for all $k \geq -1$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Assume that $0 \to M \to F^0 \to F^1 \to \cdots \to F^m \to \cdots$ is a right $\mathscr{WI}$-resolution of M with $L^m = \mathrm{coker}(F^{m-2} \to F^{m-1})$. It suffices to show that $L^m \in \mathscr{WI}$. Clearly, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & F^0 \arrow[r] & \cdots \arrow[r] & F^{m-2} \arrow[r, "f"] & F^{m-1} \arrow[rr, "g"] \arrow[dr, "\pi"'] & & F^m \arrow[r] & \cdots . \\
 & & & & & & L^m \arrow[ur, "\lambda"'] \arrow[dr] & & \\
 & & & & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

By (3), we have $\mathrm{Ext}_{m-1}(M, L^m) = 0$. The sequence

$$\mathrm{Hom}(F^m, L^m) \xrightarrow{g^*} \mathrm{Hom}(F^{m-1}, L^m) \xrightarrow{f^*} \mathrm{Hom}(F^{m-2}, L^m)$$

is exact. Since $f^*(\pi) = \pi f = 0$, $\pi \in \ker(f^*) = \mathrm{im}(g^*)$. Thus there exists $h \in \mathrm{Hom}(F^m, L^m)$ such that $\pi = g^*(h) = hg = h\lambda\pi$, and hence $h\lambda = 1$ since π is epic. Thus $L^m \in \mathscr{WI}$. $\square$

Lemma 2.9. *Let R be a ring. Then the following hold.*

(1) *If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules with $A, B \in \mathscr{WI}$, then $C \in \mathscr{WI}$.*
(2) *If $0 \to A \to B \to C \to 0$ is an exact sequence of right R-modules with $B, C \in \mathscr{WF}$, then $A \in \mathscr{WF}$.*

Proof: (1). If $0 \to A \to B \to C \to 0$ is an exact sequence, then we have a long exact sequence

$$\cdots \to \mathrm{Ext}^{n+1}(F, B) \to \mathrm{Ext}^{n+1}(F, C) \to \mathrm{Ext}^{n+2}(F, A) \to \cdots$$

for any super finitely presented left R-module F. Because $A, B \in \mathscr{WI}$, $\mathrm{Ext}^{n+1}(F, B) = 0 = \mathrm{Ext}^{n+2}(F, A)$. This implies that $\mathrm{Ext}^{n+1}(F, C) = 0$ and hence $C \in \mathscr{WI}$ by [4, Proposition 3.3].

(2). By hypothesis, the sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ is exact with $C^+, B^+ \in \mathscr{WI}$ by Proposition 1.1. Then by (1), we have $A^+ \in \mathscr{WI}$. Hence $A \in \mathscr{WF}$ by Proposition 1.1 again. $\square$

Theorem 2.10. *The following are equivalent for a left R-module N and any $m \geq 2$:*

(1) *left $\mathscr{WI}$-dim $N \leq m-2$;*
(2) $\mathrm{Ext}_{m+k}(M, N) = 0$ *for any $M \in {}_R\mathscr{M}$ and $k \geq -1$;*
(3) $\mathrm{Ext}_{m-1}(M, N) = 0$ *for any $M \in {}_R\mathscr{M}$.*

Proof: (1) $\Rightarrow$ (2). By (1), N has a left $\mathscr{WI}$-resolution $0 \to F_{m-2} \to \cdots \to F_1 \to F_0 \to N \to 0$. Then for any left R-module M, we have the following complex

$$0 \to \mathrm{Hom}(M, F_{m-2}) \to \mathrm{Hom}(M, F_{m-3}) \to \cdots \to \mathrm{Hom}(M, F_0) \to 0.$$

Hence, $\mathrm{Ext}_{m+k}(M, N) = 0$ for all left R-module M and all $k \geq -1$.

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). By Theorem 1.2, N has a left minimal $\mathscr{WI}$-resolution

$$\cdots \longrightarrow F_m \xrightarrow{f_m} F_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} N \longrightarrow 0$$

with each $F_i \in \mathscr{WI}$. Put $K_m = \ker(F_{m-1} \to F_{m-2})$ and $H = F_{m-1}/K_m$. Let $\lambda\colon K_m \to F_{m-1}$ be the inclusion and $\pi\colon F_{m-1} \to H$ the canonical projection.

Then there exists $p\colon F_m \to K_m$ such that $f_m = \lambda p$, and there exists a monomorphism $\alpha\colon H \to F_{m-2}$ such that $f_{m-1} = \alpha\pi$. Put $L = F_{m-2}/\mathrm{im}(\alpha)$ and let $\beta\colon F_{m-2} \to L$ be the canonical projection. Then there exits a homomorphism $i\colon L \to F_{m-3}$ via $i(x + \mathrm{im}(\alpha)) = f_{m-2}(x)$ such that $f_{m-2} = i\beta$. So we have the following commutative diagram:

$$\begin{tikzcd}
F_m \arrow[rr, "f_m"] \arrow[dr, "p"] & & F_{m-1} \arrow[rr, "f_{m-1}"] \arrow[dr, "\pi"] & & F_{m-2} \arrow[rr, "f_{m-2}"] \arrow[dr, "\beta"] & & F_{m-3} \\
& K_m \arrow[ur, "\lambda"] \arrow[dr, dashed] & & H \arrow[ur, "\alpha"] \arrow[dr] & & L \arrow[ur, "i"] \arrow[dr] & \\
0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & 0
\end{tikzcd} .$$

By (3), $\mathrm{Ext}_{m-1}(K_m, N) = 0$. Thus, the sequence

$$\mathrm{Hom}(K_m, F_m) \xrightarrow{f_{m*}} \mathrm{Hom}(K_m, F_{m-1}) \xrightarrow{f_{m-1*}} \mathrm{Hom}(K_m, F_{m-2})$$

is exact. Since $f_{m-1*}(\lambda) = f_{m-1}\lambda = 0$ and $\lambda \in \ker(f_{m-1*}) = \mathrm{im}(f_{m*})$, we have $\lambda = f_{m*}(l) = f_m l$ for some $l \in \mathrm{Hom}(K_m, F_m)$. But $f_m = \lambda p$, and hence $\lambda = \lambda p l$. We obtain $pl = 1$ since λ is monic, and so $K_m \in \mathscr{WI}$. Since $0 \to K_m \to F_{m-1} \to H \to 0$ is an exact sequence, $H \in \mathscr{WI}$ by Lemma 2.9. Similarly, $L \in \mathscr{WI}$.

Next we will show that the complex

$$0 \to F_{m-2} \to F_{m-3} \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a left $\mathscr{WI}$-resolution of N. First we show that $\beta\colon F_{m-2} \to L$ is an isomorphism. Let $T = \ker(f_{m-3})$, $\varphi\colon F_{m-2} \to T$ be an $\mathscr{WI}$-cover of T and $\psi\colon T \to F_{m-3}$ the inclusion mapping. Then $f_{m-2} = \psi\varphi$. Consider the following commutative diagram:

$$\begin{tikzcd}
F_{m-1} \arrow[rr, "f_{m-1}"] \arrow[dr, "\pi"] & & F_{m-2} \arrow[rrrr, "f_{m-2}"] \arrow[dr, "\beta"] \arrow[drrr, "\varphi"] & & & & F_{m-3} \arrow[r, "f_{m-3}"] & F_{m-4} \\
& H \arrow[ur, "\alpha"] \arrow[dr] & & L \arrow[urrr, "i"] \arrow[rr, dashed, "\sigma"] \arrow[dr] & & T \arrow[ur, "\psi"] & & \\
0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & &
\end{tikzcd} .$$

Set $\sigma\colon L \to T$ via $x + \mathrm{im}(\alpha) \mapsto f_{m-2}(x)$. It is easy to verify that σ is well defined and $i = \psi\sigma$. We have $\psi\varphi = f_{m-2} = i\beta = \psi\sigma\beta$, and $\varphi = \sigma\beta$ since ψ is monic. Hence, there exists a homomorphism $\eta\colon L \to F_{m-2}$ such that $\sigma = \varphi\eta$ for φ is an

$\mathscr{WI}$-cover and $L \in \mathscr{WI}$. So we have $\varphi = \sigma\beta = \varphi\eta\beta$ and $\eta\beta$ is an automorphism of F_{m-2} for $\varphi\colon F_{m-2} \to T$ is an $\mathscr{WI}$-cover. Hence, β is a monomorphism and so $F_{m-2} \cong L$. Consider the exact sequence

$$0 \to H \xrightarrow{\alpha} F_{m-2} \xrightarrow{\beta} L \to 0,$$

then $\alpha = 0$ and $H \cong 0$. So the complex

$$0 \to F_{m-2} \to F_{m-3} \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a left $\mathscr{WI}$-resolution of N, as desired. □

Remark 2.11. We note that Theorem 2.10 is a generalization of [5, Proposition 4.10] and [10, Theorem 4.2]. In fact, if $n = 0$, then this is [5, Proposition 4.10] and if R is a coherent ring, then this is [10, Theorem 4.2].

Theorem 2.12. *The following are equivalent for $m \geq 2$:*

(1) *gl.right* $\mathscr{WI}$-dim ${}_R\mathscr{M} \leq m$;
(2) *gl.left* $\mathscr{WI}$-dim ${}_R\mathscr{M} \leq m-2$;
(3) $\mathrm{Ext}_{m+k}(M, N) = 0$ *for all left R-modules M, N and $k \geq -1$;*
(4) $\mathrm{Ext}_{m-1}(M, N) = 0$ *for all left R-modules M, N;*
(5) $\mathrm{l.sp.gldim}(R) \leq m + n$.

Proof: By Proposition 2.8 and Theorem 2.10 the statements (1)–(4) are equivalent and (1) $\Leftrightarrow$ (5) follows from Lemma 2.3 and Proposition 2.4. □

Corollary 2.13. *For any ring R we have gl.left* $\mathscr{WI}$-dim ${}_R\mathscr{M}$ = *gl.right* $\mathscr{WI}$-dim ${}_R\mathscr{M} - 2$, *and is zero if gl.right* $\mathscr{WI}$-dim ${}_R\mathscr{M} \leq 2$.

Lemma 2.14. *The following statements are equivalent for any $M \in {}_R\mathscr{M}$ and $m \geq 0$:*

(1) $\mathrm{wid}_R(M) \leq m + n$;
(2) *for any left* $\mathscr{WI}$*-resolution* $\cdots \to F_m \to F_{m-1} \to F_{m-2} \to \cdots \to F_1 \to F_0 \to N \to 0$ *for each $N \in {}_R\mathscr{M}$,* $\mathrm{Hom}_R(M, F_m) \to \mathrm{Hom}(M, K_m) \to 0$ *is exact, where K_m is the mth* $\mathscr{WI}$*-syzygy of N.*

Proof: We proceed by induction on m. For $m \geq 1$, we consider the exact sequence $0 \to M \to F \to H \to 0$, where F is an $\mathscr{WI}$-preenvelope of M. Then we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(F,F_m) \arrow[r] \arrow[d] & \mathrm{Hom}(F,K_m) \arrow[r] \arrow[d] & 0 \\
\mathrm{Hom}(M,F_m) \arrow[r] \arrow[d] & \mathrm{Hom}(M,K_m) & \\
0 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(H,K_m) \arrow[r] \arrow[d] & \mathrm{Hom}(H,F_{m-1}) \arrow[r] \arrow[d] & \mathrm{Hom}(H,K_{m-1}) \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(F,K_m) \arrow[r] \arrow[d] & \mathrm{Hom}(F,F_{m-1}) \arrow[r] \arrow[d] & \mathrm{Hom}(F,K_{m-1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(M,K_m) \arrow[r] & \mathrm{Hom}(M,F_{m-1}) \arrow[r] \arrow[d] & \mathrm{Hom}(M,K_{m-1}) & \\
 & & 0 & &
\end{tikzcd}$$

Hence $\mathrm{wid}_R(M) \leq m+n$ if and only if $\mathrm{wid}_R(H) \leq m+n-1$ by Lemma 2.3 if and only if $\mathrm{Hom}(H,F_{m-1}) \to \mathrm{Hom}(H,K_{m-1})$ is surjective by induction if and only if $\mathrm{Hom}(F,K_m) \to \mathrm{Hom}(M,K_m)$ is surjective by the second diagram if and only if $\mathrm{Hom}(M,F_m) \to \mathrm{Hom}(M,K_m)$ is surjective by the first diagram.

For $m=0$, let $K_0 = M$ in the first diagram. Then $\mathrm{Hom}(M,F_0) \to \mathrm{Hom}(M,K_0)$ is surjective. Thus $F_0 \to M$ splits, and hence $M \in \mathscr{WI}$. If $M \in \mathscr{WI}$, it is clear that $\mathrm{Hom}(M,F_0) \to \mathrm{Hom}(M,K_0)$ is surjective. $\square$

Corollary 2.15. *The following conditions are equivalent for any $m \geq 0$:*

1. *if $\cdots \to F_1 \to F_0 \to M \to 0$ is a left $\mathscr{WI}$-resolution of a left R-module M, then the sequence is exact at F_k for $k \geq m-1$, where $F_{-1} = M$;*
2. *right $\mathscr{WI}$-dim ${}_RR \leq m$;*
3. $\mathrm{wid}_R({}_RR) \leq m+n$;
4. *if K_m is the mth syzygy of M, then the $\mathscr{WI}$-precover $F_m \to K_m$ is surjective.*

Proof: (1) $\Rightarrow$ (4). By the assumption, $\cdots \to F_1 \to F_0 \to M \to 0$ is exact at F_{m-1}. Thus $F_m \to K_m$ is surjective.

(4) $\Leftrightarrow$ (2). It follows by Lemma 2.14.

(3) $\Leftrightarrow$ (2) is clear.

(2) $\Rightarrow$ (1). Suppose $m \geq 2$, and let $0 \to R \to F^0 \to F^1 \to \cdots \to F^m \to 0$ be a right $\mathscr{WI}$-resolution of R. Then $\mathrm{Ext}_k(R, M) = 0$ for $k \geq m-1$. Computing $\mathrm{Ext}_k(R, M)$ by using a left $\mathscr{WI}$-resolution $\cdots \to F_1 \to F_0 \to M \to 0$, we see that the sequence is exact at F_k for any $k \geq m-1$.

If $m = 1$ and $0 \to R \to F^0 \to F^1 \to 0$ is a right $\mathscr{WI}$-resolution of R, then $0 \to \mathrm{Hom}(F^1, M) \to \mathrm{Hom}(F^0, M) \to \mathrm{Hom}(R, M)$ is exact. Thus $\mathrm{Ext}_k(R, M) = 0$ for $k \geq 1$ and $\mathrm{Ext}_0(R, M) \to M$ is a monomorphism. But computing $\mathrm{Ext}_0(R, M)$ by using a left $\mathscr{WI}$-resolution $\cdots \to F_1 \to F_0 \to M \to 0$, we see that the sequence is exact at F_0. So $\cdots \to F_1 \to F_0 \to M \to 0$ is exact at F_k for any $k \geq 0$.

Now let $m = 0$. Then ${}_RR \in \mathscr{WI}$, and so every $\mathscr{WI}$-precover is surjective. Thus $\cdots \to F_1 \to F_0 \to M \to 0$ is exact. $\square$

3. Right derived functors of $\otimes$ and right $\mathscr{WI}$-dimension

In this section, we prove that $- \otimes -$ is right balanced on $\mathscr{M}_R \times_R \mathscr{M}$ by $\mathscr{WF} \times \mathscr{WI}$.

Proposition 3.1. *The following hold for any ring R:*

1. *If $f\colon A \to B$ be a $\mathscr{WI}$-preenvelope of a module A in ${}_R\mathscr{M}$, then $f^*\colon B^+ \to A^+$ is a $\mathscr{WF}$-precover of A^+ in $\mathscr{M}_R$.*
2. *If $f\colon A \to B$ be a $\mathscr{WF}$-preenvelope of a module A in $\mathscr{M}_R$, then $f^*\colon B^+ \to A^+$ is a $\mathscr{WI}$-precover of A^+ in ${}_R\mathscr{M}$.*

Proof: By Proposition 1.1, we have $\mathscr{WI}^+ \subseteq \mathscr{WF}$ and $\mathscr{WF}^+ \subseteq \mathscr{WI}$. Now both the assertions follows from [3, Theorem 3.1]. $\square$

The following proposition is the generalization of [5, Proposition 5.1] and [2, Example 8.3.9].

Proposition 3.2. *$- \otimes -$ is right balanced on $\mathscr{M}_R \times {}_R\mathscr{M}$ by $\mathscr{WF} \times \mathscr{WI}$.*

Proof: Assume that $M \in \mathscr{M}_R$ and $0 \to M \to F^0 \to F^1 \to \cdots$ is a right $\mathscr{WF}$-resolution of M in $\mathscr{M}_R$. Let $E \in \mathscr{WI}$. Then $E^+ \in \mathscr{WF}$ by Proposition 1.1. So we get the exact sequence:

$$\cdots \to \mathrm{Hom}(F^1, E^+) \to \mathrm{Hom}(F^0, E^+) \to \mathrm{Hom}(M, E^+) \to 0$$

which gives the exact sequence:

$$\cdots \to (F^1 \otimes E)^+ \to (F^0 \otimes E)^+ \to (M \otimes E)^+ \to 0.$$

Thus we get the exact sequence $0 \to M \otimes E \to F^0 \otimes E \to F^1 \otimes E \to \cdots$.

On the other hand, let $N \in {}_R\mathscr{M}$ and let $0 \to N \to E^0 \to E^1 \to \cdots$ be a right $\mathscr{WI}$-resolution of N. Then $\cdots \to E^{1+} \to E^{0+} \to N^+ \to 0$ is a left $\mathscr{WF}$-resolution of N^+ by Proposition 1.1. Hence

$$\cdots \to \mathrm{Hom}(F, E^{1+}) \to \mathrm{Hom}(F, E^{0+}) \to \mathrm{Hom}(F, N^+) \to 0$$

is exact for any right R-module $F \in \mathscr{WF}$, this is equivalent to the sequence

$$\cdots \to (F \otimes E^1)^+ \to (F \otimes E^0)^+ \to (F \otimes N)^+ \to 0$$

being exact. So $0 \to F \otimes N \to F \otimes E^0 \to F \otimes E^1 \to \cdots$ is exact for any right R-module $F \in \mathscr{WF}$, as desired. □

We denote by $\mathrm{Tor}^n(-,-)$ the nth right derived functor of $- \otimes -$ with respect to $\mathscr{WF} \times \mathscr{WI}$.

Proposition 3.3. *The following are equivalent for a left R-module N and $m \geq 2$:*

(1) *right $\mathscr{WI}$-dim $N \leq m$;*
(2) $\mathrm{Tor}^{m+k}(M, N) = 0$ *for all $M \in \mathscr{M}_R$ and $k \geq -1$;*
(3) $\mathrm{Tor}^{m}(M, N) = \mathrm{Tor}^{m-1}(M, N) = 0$ *for all $M \in \mathscr{M}_R$;*
(4) $\mathrm{Tor}^{m-1}(M, N) = 0$ *for any finitely presented right R-module M.*

Proof: (1) ⇒ (2). Assume $0 \to N \to F^0 \to F^1 \to \cdots \to F^m \to 0$ is a right $\mathscr{WI}$-resolution of N. Then the sequence

$$M \otimes F^{m-2} \to M \otimes F^{m-1} \to M \otimes F^m \to 0$$

is exact for any $M \in \mathscr{M}_R$. It follows that $\mathrm{Tor}^m(M, N) = \mathrm{Tor}^{m-1}(M, N) = 0$. It is clear that $\mathrm{Tor}^{m+k}(M, N) = 0$ for any $k \geq 1$. Hence, (2) holds.

(2) ⇒ (3) ⇒ (4) are trivial.

(4) ⇒ (1). Let $0 \to N \to F^0 \to F^1 \to \cdots$ be a right $\mathscr{WI}$-resolution of N. Then for any finitely presented right R-module P,

$$P \otimes F^{m-2} \to P \otimes F^{m-1} \to P \otimes F^m \to P \otimes F^{m+1}$$

is exact by (4). Hence, $K = \ker(F^m \to F^{m+1})$ is pure in F^m by [2, Lemma 8.4.23], and $K \in \mathscr{WI}$ by [12, Corollary 4.7]. So $0 \to N \to F^0 \to F^1 \to \cdots \to F^{m-1} \to K \to 0$ is a right $\mathscr{WI}$-resolution of N and hence (1) follows. □

Theorem 3.4. *The following are equivalent for a ring R and $m \geq 2$:*

(1) *gl.right $\mathscr{WI}$-dim ${}_R\mathscr{M} \leq m$;*
(2) $\mathrm{Tor}^{m+k}(M, N) = 0$ *for all $N \in {}_R\mathscr{M}$ and $M \in \mathscr{M}_R$ and $k \geq -1$;*
(3) $\mathrm{Tor}^{m}(M, N) = \mathrm{Tor}^{m-1}(M, N) = 0$ *for all $N \in {}_R\mathscr{M}$ and $M \in \mathscr{M}_R$;*
(4) $\mathrm{Tor}^{m-1}(M, N) = 0$ *for all $N \in {}_R\mathscr{M}$ and all finitely presented right R-module M.*

Proof: The result follows from Proposition 3.3. □

Theorem 3.5. *Let R be a ring and $m \geq 0$. Then the following are equivalent:*

(1) *for every flat left R-module F, there is an exact sequence $0 \to F \to A^0 \to A^1 \to \cdots \to A^m \to 0$ with each $A^i \in \mathscr{WI}$;*
(2) *there is an exact sequence $0 \to R \to A^0 \to A^1 \to \cdots \to A^m \to 0$ of left R-modules with each $A^i \in \mathscr{WI}$;*
(3) *if $0 \to M \to F^0 \to F^1 \to \cdots$ is a right $\mathscr{WF}$-resolution of a right R-module M, then the sequence is exact at F^k for $k \geq m-1$, where $F^{-1} = M$.*

Proof: (1) $\Rightarrow$ (2) is immediate.

(2) $\Rightarrow$ (3). By Proposition 3.2, we know that $-\otimes-$ is right balanced on $\mathscr{M}_R \times_R \mathscr{M}$ by $\mathscr{WF} \times \mathscr{WI}$ with right derived functor $\mathrm{Tor}^k(-,-)$.

If $m \geq 2$, there is a right $\mathscr{WI}$-resolution $0 \to R \to B^0 \to B^1 \to \cdots \to B^m \to \cdots$ with $B^i \in \mathscr{WI}$. Moreover the above sequence is exact. Let $K = \mathrm{coker}(B^{m-2} \to B^{m-1})$. Since there is an exact sequence $0 \to R \to A^0 \to A^1 \to \cdots \to A^m \to 0$ with each $A^i \in \mathscr{WI}$ by (2), we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, equal] & B^0 \arrow[r] \arrow[d] & \cdots \arrow[r] & B^{m-2} \arrow[r] \arrow[d] & B^{m-1} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] & A^0 \arrow[r] & \cdots \arrow[r] & A^{m-2} \arrow[r] & A^{m-1} \arrow[r] & A^m \arrow[r] & 0
\end{tikzcd}$$

Hence, there is an exact complex:

$$0 \to R \to B^0 \oplus R \to B^1 \oplus A^0 \to \cdots \to B^{m-1} \otimes A^{m-2} \to K \oplus A^{m-1} \to A^m \to 0$$

with exact subcomplex $0 \to R \to R \to 0 \to \cdots \to 0$. We have the exact quotient complex:

$$0 \to B^0 \to B^1 \oplus A^0 \to \cdots \to B^{m-1} \otimes A^{m-2} \to K \oplus A^{m-1} \to A^m \to 0.$$

Since $\mathscr{WI}$ is closed under cokernels of monomorphisms, extensions and direct summands. It follows that $K \in \mathscr{WI}$. Hence, there is a right $\mathscr{WI}$-resolution $0 \to R \to B^0 \to B^1 \to \cdots \to B^{m-1} \to K \to 0$ with B^i, $K \in \mathscr{WI}$. It is easy to check that $\mathrm{Tor}^k(M,R) = 0$ for $k \geq m-1$. Computing by $0 \to M \to F^0 \to F^1 \to \cdots$, as in (3), we see that $\mathrm{Tor}^k(M,R)$ is just the kth homology group of this complex, giving the desired result.

If $m = 1$, we can assume that $0 \to R \to A^0 \to A^1 \to 0$ is a right $\mathscr{WI}$-resolution of R by the proof above. Hence, $\mathrm{Tor}^1(M,R) = 0$, so that $F^0 \to F^1 \to F^2$ is exact and $M \otimes R \to \mathrm{Tor}^0(M,R)$ is onto. Computing the later morphism using $0 \to M \to F^0 \to F^1$, we obtain that $M \to F^0 \to F^1$ is exact.

If $m = 0$, then (2) means that $\mathrm{wid}_R({}_RR) \leq n$. But we have the exact sequence $0 \to M \otimes R \to F^0 \otimes R \to F^1 \otimes R \to \cdots$ since the functor $- \otimes -$ is right balanced. That is, $0 \to M \to F^0 \to F^1 \to \cdots$ is exact.

(3) $\Rightarrow$ (1). Assume (3) with $m \geq 2$. Let F be a flat left R-module and $0 \to F \to A^0 \to A^1 \to \cdots$ a right $\mathscr{WI}$-resolution of F. Obviously, this complex is exact. Then by (3), we get $\mathrm{Tor}^k(M, F) = 0$ for $k \geq m - 1$ since F is flat. Computing using $0 \to A^0 \to A^1 \to \cdots$ and using [5, Lemma 5.6], we get $K = \ker(A^m \to A^{m+1})$ is pure in A^m, so $K \in \mathscr{WI}$. Hence $0 \to F \to A^0 \to A^1 \to \cdots \to A^{m-1} \to K \to 0$ gives the desired exact sequence.

Now let $m = 1$. Then (3) says $M \to F^0 \to F^1 \to \cdots$ is exact, so $\mathrm{Tor}^k(M, F) = 0$ for $k > 0$ and $M \otimes F \to \mathrm{Tor}^0(M, F)$ is onto. Hence, if $0 \to F \to A^0 \to A^1 \to \cdots$ is exact, then $M \otimes F \to M \otimes A^0 \to M \otimes A^1 \to M \otimes A^2$ is exact for any finitely presented right R-module M. By [5, Lemma 5.6] again, we get the desired exact sequence $0 \to F \to A^0 \to K \to 0$ with $K = \ker(A^1 \to A^2)$.

If $m = 0$, then $0 \to M \to F^0 \to F^1 \to \cdots$ being exact means $\mathrm{Tor}^k(M, F) = 0$ for $k > 0$ and $M \otimes F \to \mathrm{Tor}^0(M, F)$ is an isomorphism. This gives that $0 \to M \otimes F \to M \otimes A^0 \to M \otimes A^1$ is exact for all M which implies that F is a pure submodule of A^0, so $F \in \mathscr{WI}$. □

Corollary 3.6. *The following are equivalent for a ring R:*

(1) *every flat left R-module has weak injective dimension at most n;*
(2) *every injective right R-module has weak flat dimension at most n;*
(3) *${}_RR$ has weak injective dimension at most n;*
(4) *$(\mathscr{WI}, \mathscr{WI}^\perp)$ is a perfect cotorsion theory.*

Proof: (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows from Theorem 3.5.

(3) $\Rightarrow$ (4) is proved in [12, Proposition 4.17].

(4) $\Rightarrow$ (3). It follows from the fact that if $\mathscr{WI} = {}^\perp(\mathscr{WI}^\perp)$, then each projective left R-module is in $\mathscr{WI}$. □

Recall that a $\mathcal{C}$-envelope $\varphi\colon M \to C$ is said to have *unique mapping property*, see [1], if for any homomorphism $f\colon M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g\colon C \to C'$ such that $g\varphi = f$. Dually, we have the definition of $\mathcal{C}$-cover with unique mapping property.

We end this paper with the following result.

Theorem 3.7. *The following are equivalent for a ring R:*

(1) $\mathrm{l.sp.gldim}(R) \leq n$;
(2) $\mathrm{wid}_R(R) \leq n$ *and every left R-module has a monomorphic $\mathscr{WI}$-cover;*
(3) *every left R-module has an epimorphic $\mathscr{WI}$-cover with the unique mapping property;*

(4) *every left R-module has a $\mathscr{W}\mathscr{I}$-envelope with the unique mapping property.*

Proof: (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4). Let M be a left R-module. Then $M \in \mathscr{W}\mathscr{I}$ by (1). Then it is easy to verify that the identity homomorphism on M is a $\mathscr{W}\mathscr{I}$-cover with the unique mapping property. It is also a $\mathscr{W}\mathscr{I}$-envelope of M with the unique mapping property.

(2) $\Rightarrow$ (1). Let M be any left R-module. By (2), M has an epimorphic $\mathscr{W}\mathscr{I}$-cover $f\colon F \to M$. Since $\mathrm{wid}_R(R) \leq n$, it is easy to see that f is an epimorphism and hence $M \in \mathscr{W}\mathscr{I}$.

(3) $\Rightarrow$ (1). For any left R-module M, let $f\colon E \to M$ be a $\mathscr{W}\mathscr{I}$-cover of M with the unique mapping property, where $E \in \mathscr{W}\mathscr{I}$. By (3), $K = \ker(f)$ has an epimorphic $\mathscr{W}\mathscr{I}$-cover $g\colon E' \to K$. So we obtain the following row exact commutative diagram:

$$\begin{tikzcd}
& & E' \arrow[dl, "g"'] \arrow[d, "ig"] \arrow[dr, "0"] & & \\
0 \arrow[r] & K \arrow[r, "i"] & E \arrow[r, "f"] & M \arrow[r] & 0.
\end{tikzcd}$$

Since $f(ig) = 0$, we have $ig = 0$ by uniqueness. Note that g is an epimorphism. Hence $K = \ker(f) = \mathrm{im}(g) \subseteq \ker(i) = 0$. Hence $M \in \mathscr{W}\mathscr{I}$ and so (1) follows.

(4) $\Rightarrow$ (1). The proof is similar to that of (3) $\Rightarrow$ (1). $\square$

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