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Approximate biflatness and Johnson pseudo-contractibility of some Banach algebras

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Abstract. We study the structure of Lipschitz algebras under the notions of approximate biflatness and Johnson pseudo-contractibility. We show that for a compact metric space X, the Lipschitz algebras $\mathrm{Lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(X)$ are approximately biflat if and only if X is finite, provided that $0 < \alpha < 1$. We give a necessary and sufficient condition that a vector-valued Lipschitz algebras is Johnson pseudo-contractible. We also show that some triangular Banach algebras are not approximately biflat.

Keywords: approximate biflatness; Johnson pseudo-contractibility; Lipschitz algebra; triangular Banach algebra

Classification: 46M10, 46H20, 46H05

1. Introduction and preliminaries

A Banach algebra A is called amenable if there exists a bounded net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$ for every $a \in A$, where $\pi_A\colon A \otimes_p A \to A$ is the product morphism given by $\pi_A(a \otimes b) = ab$. Johnson showed that for a locally compact group G, $L^1(G)$ is amenable if and only if G is amenable. For more information about the history of amenability, the reader refers to [12].

An important notion of homological theory related to amenability is biflatness. In fact a Banach algebra A is called biflat if there exists a bounded A-bimodule morphism $\varrho\colon (A \otimes_p A)^* \to A^*$ such that $\varrho \circ \pi_A^* = \mathrm{id}_{A^*}$. It is well-known that a Banach algebra A is amenable if and only if A is biflat and A has a bounded approximate identity.

Motivated by these considerations, E. Samei et al. introduced in [15] the approximate version of biflatness. Indeed a Banach algebra A is approximately biflat if there exists a net of A-bimodule morphisms (ϱ_α) from $(A \otimes_p A)^*$ into A^* such that $\varrho_\alpha \circ \pi_A^* \xrightarrow{\mathrm{W^*OT}} \mathrm{id}_{A^*}$, where $\mathrm{W^*OT}$ stands for the weak star operator

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topology. Indeed for the Banach spaces E and F, the weak star operator topology on $B(E, F^*)$ (the set of all bounded linear operators from E into F^*) is the locally convex topology given by the seminorms $\{\|\cdot\|_{e,f} \colon e \in E, \ f \in F\}$, where $\|T\|_{e,f} = |\langle f, T(e)\rangle|$ and $T \in B(E, F^*)$. E. Samei et al. also studied approximate biflatness of the Segal algebras and the Fourier algebras.

The Lipschitz algebras are concrete Banach algebras, see [16]. These algebras rely upon the metric spaces. In this paper, we characterize approximate biflatness of Lipschitz algebras and we show that for a compact metric space X, the Lipschitz algebras $\mathrm{Lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(X)$ are approximately biflat if and only if X is finite, provided that $0 < \alpha < 1$. We also study the Johnson pseudo-contractibility of vector-valued Lipschitz algebras and we investigate the approximate biflatness of some triangular Banach algebras.

We present some standard notations and definitions that we shall need in this paper. Let A be a Banach algebra. Throughout this work, the character space of A is denoted by $\Delta(A)$, that is, the set of all nonzero multiplicative linear functionals on A. For each $\varphi \in \Delta(A)$ there exists a unique extension $\widetilde{\varphi}$ to A^{**} which is defined by $\widetilde{\varphi}(F) = F(\varphi)$. It is easy to see that $\widetilde{\varphi} \in \Delta(A^{**})$. The projective tensor product $A \otimes_p A$ is a Banach A-bimodule via the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca, \qquad a, b, c \in A.$$

Let X and Y be Banach A-bimodules. The linear map $T \colon X \to Y$ is called A-bimodule morphism if

$$T(a \cdot x) = a \cdot T(x), \quad T(x \cdot a) = T(x) \cdot a, \qquad a \in A, \ x \in X.$$

2. Johnson pseudo-contractibility and approximate biflatness

We recall that the Banach algebra A is Johnson pseudo-contractible if there exists a not necessarily bounded net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$ for each $a \in A$, see [13] and [14].

We should remind that the Banach algebra A is called pseudo-contractible if there exists a not necessarily bounded net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A(m_\alpha)a \to a$ for each $a \in A$, for more details see [6]. In fact Johnson pseudo-contractibility is an extended version of pseudo-contractibility in $(A \otimes_p A)^{**}$, for the relations and differences of these two concepts see [14].

Theorem 2.1. *Let A be a Johnson pseudo-contractible Banach algebra. Then A is approximately biflat.*

Proof: Suppose that A is a Johnson pseudo-contractible Banach algebra. Then there exists a net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \to a$

for each $a \in A$. Define $\theta_\alpha(a) = a \cdot m_\alpha$. Clearly $(\theta_\alpha)_\alpha$ is a net of A-bimodule morphisms from A into $(A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \theta_\alpha(a) \to a$ for each $a \in A$. Put $\varrho_\alpha = \theta_\alpha^*|_{(A\otimes_p A)^*} \colon (A \otimes_p A)^* \to A^*$. It is easy to see that $(\varrho_\alpha)_\alpha$ is a net of A-bimodule morphisms. We claim that

$$\varrho_\alpha \circ \pi_A^* \xrightarrow{\mathrm{W^*OT}} \mathrm{id}_{A^*}.$$

To see this, let $a \in A$ and $f \in A^*$.

$$\begin{aligned}
\langle \varrho_\alpha \circ \pi_A^*(f), a\rangle - \langle a, f\rangle &= \langle \theta_\alpha^*|_{(A\otimes_p A)^*} \circ \pi_A^*(f), a\rangle - \langle a, f\rangle \\
&= \langle \theta_\alpha^{***} \circ \pi_A^*(f), a\rangle - \langle a, f\rangle \\
&= \langle \pi_A^*(f), \theta_\alpha^{**}(a)\rangle - \langle a, f\rangle \\
&= \langle \pi_A^*(f), \theta_\alpha(a)\rangle - \langle a, f\rangle \\
&= \langle \theta_\alpha(a), \pi_A^*(f)\rangle - \langle a, f\rangle \\
&= \langle \pi_A^{**} \circ \theta_\alpha(a), f\rangle - \langle a, f\rangle \to 0.
\end{aligned}$$

It follows that A is approximately biflat. □

Remark 2.2. The converse of above theorem is not always true. To see this, suppose that S is the left zero semigroup with $|S| \geq 2$, that is, a semigroup with product $st = s$ for all $s, t \in S$. Then the related semigroup algebra $l^1(S)$ has the following product

$$fg = \varphi_S(f)g, \qquad f, g \in l^1(S),$$

where φ_S is denoted for the augmentation character on $l^1(S)$. Define $\varrho \colon l^1(S) \to (l^1(S) \otimes_p l^1(S))^{**}$ by $\varrho(f) = f_0 \otimes f$. Clearly ϱ is a bounded $l^1(S)$-bimodule morphism which $\pi_{l^1(S)}^{**} \circ \varrho(f) = f$ for each $f \in l^1(S)$. Applying [12, Lemma 4.3.22], $l^1(S)$ becomes biflat. So $l^1(S)$ is approximately biflat. We claim that $l^1(S)$ is not Johnson pseudo-contractible. We assume conversely that $l^1(S)$ is Johnson pseudo-contractible. It is easy to see that $l^1(S)$ has an approximate identity, say (e_α). Consider

$$\varphi_S(e_\alpha) \to 1, \quad e_\alpha f - f e_\alpha = \varphi_S(e_\alpha)f - \varphi_S(f)e_\alpha \to 0, \qquad f \in l^1(S).$$

It follows that $f - \varphi_S(f)e_\alpha \to 0$ for each $f \in l^1(S)$. Since there exist at least two different elements s_1 and s_2 in S, replace two distinct elements δ_{s_1} and δ_{s_2} of $l^1(S)$ with f in $f - \varphi_S(f)e_\alpha \to 0$. It follows that $\delta_{s_1} = \delta_{s_2}$, so $s_1 = s_2$ which is a contradiction.

It is still open, whether the approximately biflatness of A implies the Johnson pseudo-contractibility of A.

Lemma 2.3. *Let A be an approximately biflat Banach algebra with a central approximate identity. Then there is a net (m_γ) in $(A\otimes_p A)^{**}$ such that*

$$a\cdot m_\gamma = m_\gamma\cdot a,\quad \pi_A^{**}(m_\gamma)a \xrightarrow{w^*} a,\qquad a\in A.$$

Proof: Suppose that A is an approximately biflat Banach algebra with a central approximate identity, say $(e_\beta)_{\beta\in J}$. Then there exists a net of A-bimodule morphism $(\varrho_\alpha)_{\alpha\in I}$ from $(A\otimes_p A)^*$ into A^* such that $\varrho_\alpha\circ\pi_A^* \xrightarrow{\mathrm{W^*OT}} \mathrm{id}_{A^*}$. Set $m_\alpha^\beta=\varrho_\alpha^*(e_\beta)$. Since (ϱ_α^*) is a net of A-bimodule morphism, we have

$$a\cdot m_\alpha^\beta = a\cdot\varrho_\alpha^*(e_\beta)=\varrho_\alpha^*(ae_\beta)=\varrho_\alpha^*(e_\beta a)=\varrho_\alpha^*(e_\beta)\cdot a = m_\alpha^\beta\cdot a$$

for each $\alpha\in I$, $\beta\in J$ and $a\in A$. Also for each $a\in A$ and $\varphi\in A^*$, we have

$$\begin{aligned}\lim_\beta\lim_\alpha\langle\varphi,\pi_A^{**}(m_\alpha^\beta)\cdot a\rangle &= \lim_\beta\lim_\alpha\langle\varphi\cdot a,\pi_A^{**}(m_\alpha^\beta)\rangle\\ &=\lim_\beta\lim_\alpha\langle\varphi\cdot a,\pi_A^{**}(\varrho_\alpha^*(e_\beta))\rangle\\ &=\lim_\beta\lim_\alpha\langle\varrho_\alpha\circ\pi_A^*(\varphi\cdot a),e_\beta\rangle\\ &=\lim_\beta\langle\varphi\cdot a,e_\beta\rangle\\ &=\lim_\beta\langle\varphi,ae_\beta\rangle=\langle a,\varphi\rangle.\end{aligned}\tag{2.1}$$

Set $E=J\times I^J$, where I^J is the set of all functions from J into I. Consider the product ordering on E as follow

$$(\beta,\alpha)\leq_E(\beta',\alpha')\Leftrightarrow\beta\leq_J\beta',\ \alpha\leq_{I^J}\alpha'\qquad\beta,\beta'\in J,\ \alpha,\alpha'\in I^J,$$

here $\alpha\leq_{I^J}\alpha'$ means that $\alpha(d)\leq_I\alpha'(d)$ for each $d\in J$. Suppose that $\gamma=(\beta,\alpha_\beta)\in E$ and $m_\gamma=\varrho_{\alpha_\beta}^*(e_\beta)\in(A\otimes_p A)^{**}$. Now using iterated limit theorem [11, page 69] and the equation (2.1), we have

$$a\cdot m_\gamma = m_\gamma\cdot a,\quad \pi_A^{**}(m_\gamma)a \xrightarrow{w^*} a,\qquad a\in A.$$

□

Let A be a Banach algebra and $\varphi\in\Delta(A)$. An element $m\in A^{**}$ that satisfies $am=\varphi(a)m$ and $\widetilde{\varphi}(m)=1$ is called φ-mean. Suppose that $m\in A^{**}$ is a φ-mean for A. Since $\|\varphi\|=1$, we have $\|m\|\geq 1$. So for $C\geq 1$, A is called C-φ-amenable if A has a φ-mean m which $\|m\|\leq C$. Also A is called C-character amenable if A has a bounded right approximate identity and has φ-mean m which $\|m\|\leq C$ for every $\varphi\in\Delta(A)$, see [10] and [9].

Proposition 2.4. *Let A be a Johnson pseudo-contractible Banach algebra and $\Delta(A)\neq\emptyset$. If A has a right identity, then A is C-character amenable.*

PROOF: Similar to the proof of [14, Lemma 3.5]. □

Lemma 2.5. *Let A be an approximately biflat Banach algebra with an identity and $\Delta(A) \neq \emptyset$. Then A is φ-amenable for every $\varphi \in \Delta(A)$.*

PROOF: Suppose that A is an approximately biflat with an identity e. Then by Lemma 2.3, there exists a net (m_α) in $(A \otimes_p A)^{**}$ such that $a \cdot m_\alpha = m_\alpha \cdot a$ and $\pi_A^{**}(m_\alpha)a \xrightarrow{w^*} a$ for every $a \in A$. So for every $\varepsilon > 0$ there exists $\alpha_\varepsilon^\varphi$ such that

$$|\widetilde{\varphi} \circ \pi_A^{**}(m_{\alpha_\varepsilon^\varphi}) - 1| = |\widetilde{\varphi} \circ \pi_A^{**}(m_{\alpha_\varepsilon^\varphi}) - \widetilde{\varphi}(e)| = |\pi_A^{**}(m_{\alpha_\varepsilon^\varphi})e(\varphi) - e(\varphi)| < \varepsilon$$

and $a \cdot m_{\alpha_\varepsilon^\varphi} = m_{\alpha_\varepsilon^\varphi} \cdot a$. Let $T\colon A \otimes_p A \to A$ be a map defined by $T(a \otimes b) = \varphi(b)a$ for every $a, b \in A$. Since $\widetilde{\varphi} \circ T^{**} = \widetilde{\varphi} \circ \pi_A^{**}$, it follows that

$$|\widetilde{\varphi} \circ T^{**}(m_{\alpha_\varepsilon^\varphi}) - 1| = |\widetilde{\varphi}(\pi_A^{**}(m_{\alpha_\varepsilon^\varphi})) - 1| < \varepsilon. \tag{2.2}$$

As we know that T^{**} is a w^*-continuous map, thus

$$T^{**}(a \cdot F) = a \cdot T^{**}(F), \quad \varphi(a)T^{**}(F) = T^{**}(F \cdot a), \qquad a \in A,\ F \in (A \otimes_p A)^{**}.$$

Then

$$a \cdot T^{**}(m_{\alpha_\varepsilon^\varphi}) = T^{**}(a \cdot m_{\alpha_\varepsilon^\varphi}) = T^{**}(m_{\alpha_\varepsilon^\varphi} \cdot a) = \varphi(a)T^{**}(m_{\alpha_\varepsilon^\varphi})$$

for every $a \in A$. Replacing $T^{**}(m_{\alpha_\varepsilon^\varphi})$ by $T^{**}(m_{\alpha_\varepsilon^\varphi})/(\widetilde{\varphi} \circ T^{**}(m_{\alpha_\varepsilon^\varphi}))$, we may suppose that

$$aT^{**}(m_{\alpha_\varepsilon^\varphi}) = \varphi(a)T^{**}(m_{\alpha_\varepsilon^\varphi}), \qquad \widetilde{\varphi} \circ T^{**}(m_{\alpha_\varepsilon^\varphi}) = 1,$$

for every $a \in A$. It shows that A is left φ-amenable. □

3. Applications to Lipschitz algebras

Let X be a metric space and $\alpha > 0$. Also let $(E, \|\cdot\|)$ be a Banach space. Set

$$\mathrm{Lip}_\alpha(X, E) = \{f\colon X \to E\colon f \text{ is bounded and } p_{\alpha,E}(f) < \infty\},$$

where

$$p_{\alpha,E}(f) = \sup\Big\{\frac{\|f(x) - f(y)\|}{d(x,y)^\alpha}\colon\ x, y \in X,\ x \neq y\Big\}.$$

Also

$$\mathrm{lip}_\alpha(X, E) = \Big\{f \in \mathrm{Lip}_\alpha(X, E)\colon\ \frac{\|f(x) - f(y)\|}{d(x,y)^\alpha} \to 0 \ \text{ as } \ d(x,y) \to 0\Big\},$$

provided that $0 < \alpha < 1$. Define

$$\|f\|_{\alpha,E} = \|f\|_{\infty,E} + p_{\alpha,E}(f),$$

where $\|f\|_{\infty,E} = \sup_{x\in X} \|f(x)\|$. For each Banach algebra E, with the pointwise multiplication and norm $\|\cdot\|_{\alpha,E}$, $\mathrm{Lip}_\alpha(X,E)$ and $\mathrm{lip}_\alpha(X,E)$ become Banach algebras. Also we denote $\mathrm{Lip}_\alpha(X)$ for $\mathrm{Lip}_\alpha(X,\mathbb{C})$ and $\mathrm{lip}_\alpha(X)$ for $\mathrm{lip}_\alpha(X,\mathbb{C})$, respectively. The amenability of $\mathrm{Lip}_\alpha(X,E)$ and $\mathrm{lip}_\alpha(X,E)$ was investigated by F. Gourdeau, see [8].

If X is a compact metric space, it is well-known that each nonzero multiplicative linear functional on $\mathrm{Lip}_\alpha(X)$ (also on $\mathrm{lip}_\alpha(X)$) has a form φ_x for some $x \in X$, where $\varphi_x(f) = f(x)$ for every $x \in X$. For further information about the Lipschitz algebras see [2], [16], [8] and [5]. A metric space (X,d) is called uniformly discrete if there exists $\varepsilon > 0$ such that $d(x,y) > \varepsilon$ for every $x,y \in X$ with $x \neq y$.

Note that $\mathrm{Lip}_\alpha(X,E)$ always separates the elements of X (even for $\alpha = 1$), while $\mathrm{lip}_\alpha(X,E)$ always does so when $\alpha < 1$.

Theorem 3.1. *Let (X,d) be a metric space and $\alpha > 0$ and let E be a Banach algebra with a right identity with $\Delta(E) \neq \emptyset$. If $\mathrm{Lip}_\alpha(X,E)$ or $\mathrm{lip}_\alpha(X,E)$ (in this case $0 < \alpha < 1$) is Johnson pseudo-contractible, then X is uniformly discrete and E is Johnson pseudo-contractible.*

Proof: Let A be $\mathrm{lip}_\alpha(X,E)$ or $\mathrm{Lip}_\alpha(X,E)$. Since E has a right identity, A has a right identity. Using Johnson pseudo-contractibility of A and Proposition 2.4, we have A is C-character amenable. By [3, Lemma 3.1] X is uniformly discrete. Let $x_0 \in X$. Define $\varphi_{x_0} \colon A \to E$ by $\varphi_{x_0}(f) = f(x_0)$. Clearly φ_{x_0} is a homomorphism and onto bounded linear map. Since A is Johnson pseudo-contractible by [14, Proposition 2.9], E is Johnson pseudo-contractible. $\square$

Proposition 3.2. *Let (X,d) be a metric space and $\alpha > 0$ and let E be a Banach algebra with an identity which $\Delta(E) \neq \emptyset$. If A is $\mathrm{Lip}_\alpha(X,E)$ or $\mathrm{lip}_\alpha(X,E)$ (in this case $0 < \alpha < 1$) , then the following statements are equivalent:*

(i) *Banach algebra A is Johnson pseudo-contractible.*
(ii) *Metric space X is uniformly discrete.*
(iii) *Banach algebra A is amenable.*

Proof: (i) $\Leftrightarrow$ (ii) Since E is unital, by [1, Theorem 1.1] E is amenable. Clearly A is a unital Banach algebra. Also by [1, Theorem 1.1], Johnson pseudo-contractibility of A implies that A is amenable. Applying [3, Theorem 3.4] finishes the proof.

(ii) $\Leftrightarrow$ (iii) It is clear by [3, Theorem 3.4]. $\square$

Theorem 3.3. *Let X be a compact metric space and let A be $\mathrm{Lip}_\alpha(X)$ or $\mathrm{lip}_\alpha(X)$ with $0 < \alpha < 1$. Then the following statements are equivalent:*

(i) *Banach algebra A is approximately biflat.*

(ii) *Metric space X is finite.*

(iii) *Banach algebra A is amenable.*

Proof: (i) $\Rightarrow$ (ii) Let A be an approximately biflat Banach algebra. Since A has an identity, by Lemma 2.5, A is φ-amenable for every $\varphi \in \Delta(A)$. On the other hand the existence of an identity for A implies that A is character amenable. Suppose, towards a contradiction, that X is infinite and $x_0 \in X$ is not isolated point of X. Since by [4, Theorem 4.4.30 (iv)], $\ker \varphi_{x_0}$ does not have a bounded approximate identity, [9, Lemma 3.3] implies that A is not character amenable, which is impossible. It implies that X is discrete, so X is finite.

(ii) $\Rightarrow$ (iii) See [7, Theorem 3].

(iii) $\Rightarrow$ (i) Suppose that A is amenable. Then there exists an element $M \in (A \otimes_p A)^{**}$ such that $a \cdot M = M \cdot a$ such that $\pi_A^{**}(M)a = a$ for each $a \in A$. Define $\varrho\colon A \to (A \otimes_p A)^{**}$ by $\varrho(a) = a \cdot M$. It is easy to see that ϱ is a bounded A-bimodule morphism and A is biflat, see [12, Lemma 4.3.22]. It follows that A is approximately biflat. $\square$

4. Applications to triangular Banach algebras

Let A be a Banach algebra and $\varphi \in \Delta(A)$. Suppose that X is a Banach left A-module. A nonzero linear functional $\eta \in X^*$ is called left φ-character if $\eta(a \cdot x) = \varphi(a)\eta(x)$ and it is called right φ-character if $\eta(x \cdot a) = \varphi(a)\eta(x)$. A left and a right φ-character is called φ-character. Note that if A is a Banach algebra and $\varphi \in \Delta(A)$, then $\varphi \otimes \varphi$ on $A \otimes_p A$ (defined by $\varphi \otimes \varphi(a \otimes b) = \varphi(a)\varphi(b)$) and $\widetilde{\varphi}$ on A^{**} are φ-characters.

Let A and B be Banach algebras and let X be a Banach (A, B)-module. That is, X is a Banach left A-module and a Banach right B-module that satisfy $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ and $\|a \cdot x \cdot b\| \le \|a\| \|x\| \|b\|$ for every $a \in A$, $b \in B$ and $x \in X$. Consider

$$T = \mathrm{Tri}(A, B, X) = \left\{ \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \colon\ a \in A,\ x \in X,\ b \in B \right\},$$

with the usual matrix operations and

$$\left\| \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \right\| = \|a\| + \|x\| + \|b\|, \qquad a \in A,\ x \in X,\ b \in B,$$

T becomes a Banach algebra which is called triangular Banach algebra. Let $\varphi \in \Delta(B)$. We define a character $\psi_\varphi \in \Delta(T)$ via $\psi_\varphi\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) = \varphi(b)$ for every $a \in A$, $b \in B$ and $x \in X$.

Theorem 4.1. *Let $T = \mathrm{Tri}(A, B, X)$ be a triangular Banach algebra such that A and B have a central approximate identity ($\Delta(B) \neq \emptyset$). Let one of the followings hold:*

(i) *Banach algebra B is not left φ-amenable.*
(ii) *Metric space X has a right φ-character.*

Then T is not approximately biflat.

Proof: Suppose, in contradiction, that T is approximately biflat. Since T has a central approximate identity, by similar argument as in Lemma 2.5, T is left ψ_φ-amenable. Clearly $I = \left(\begin{smallmatrix} 0 & X \\ 0 & B \end{smallmatrix}\right)$ is a closed ideal of T and $\psi_\varphi|_I \neq 0$, then by [10, Lemma 3.1] I is left ψ_φ-amenable. Thus by [10, Theorem 1.4] there exists a net (m_α) in I such that $am_\alpha - \psi_\varphi|_I(a)m_\alpha \to 0$ and $\psi_\varphi|_I(m_\alpha) = 1$, where $a \in I$. Let $x_\alpha \in X$ and $b_\alpha \in B$ be such that $m_\alpha = \left(\begin{smallmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{smallmatrix}\right)$. Then we have $\psi_\varphi\left(\begin{smallmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{smallmatrix}\right) = \varphi(b_\alpha) = 1$ and

$$\begin{pmatrix} 0 & x_0 \\ 0 & b_0 \end{pmatrix}\begin{pmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{pmatrix} - \psi_\varphi\begin{pmatrix} 0 & x_0 \\ 0 & b_0 \end{pmatrix}\begin{pmatrix} 0 & x_\alpha \\ 0 & b_\alpha \end{pmatrix} \to 0 \tag{4.1}$$

for each $x_0 \in X$ and $b_0 \in B$. Using (4.1) we obtain $b_\alpha b_0 - \varphi(b_0)b_\alpha \to 0$ and since $\varphi(b_\alpha) = 1$, we see that B is left φ-amenable, which contradicts (i).

Now suppose that the statement (ii) holds. Then from (4.1) we have $x_0 b_\alpha - \varphi(b)x_\alpha \to 0$. By hypothesis from (ii) there exists a right φ-character $\eta \in X^*$. Applying η on $x_0 b_\alpha - \varphi(b)x_\alpha \to 0$, we have $\eta(x_0 b_\alpha) - \varphi(b)\eta(x_\alpha) \to 0$ for every $b \in B$ and $x \in X$, which is impossible (take $b \in \ker\varphi$, it implies that η is zero), that is, (ii) does not hold. □

It is well-known that if X is a compact metric space, then $\mathrm{Lip}_\alpha(X)$ is unital and the character space $\mathrm{Lip}_\alpha(X)$ is nonempty, so we have the following corollary.

Corollary 4.2. *Suppose that X is a compact metric space. Then*

$$T = \mathrm{Tri}(\mathrm{Lip}_\alpha(X), \mathrm{Lip}_\alpha(X), \mathrm{Lip}_\alpha(X))$$

is not approximately biflat.

Theorem 4.3. *Let $T = \mathrm{Tri}(A, B, X)$ be a triangular Banach algebra with $\Delta(B) \neq \emptyset$. Let one of the followings hold:*

(i) *Banach algebra B is not left φ-amenable.*
(ii) *Metric space X has a right φ-character.*

Then T is not Johnson pseudo-contractible.

Proof: Let T be Johnson pseudo-contractible. Using a similar argument as in the proof of Lemma 2.5, we can see that T is left ψ_φ-amenable. Following the proof of Theorem 4.1 finishes the proof. □

Corollary 4.4. *Suppose that S is the left zero semigroup. Then*

$$T = \mathrm{Tri}(l^1(S), l^1(S), l^1(S))$$

is not Johnson pseudo-contractible.

Proof: It is known that every semigroup algebra $l^1(S)$ has a character (for instance, the augmentation character φ_S). So the Banach $(l^1(S), l^1(S))$-module $l^1(S)$ (with natural action) has a right φ_S-character. Thus by previous theorem T is not Johnson pseudo-contractible. $\square$

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