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Gr-$(2,n)$-ideals in graded commutative rings

Khaldoun Al-Zoubi, Shatha Alghueiri, Ece Y. Celikel

Abstract. Let G be a group with identity e and let R be a G-graded ring. In this paper, we introduce and study the concept of graded $(2,n)$-ideals of R. A proper graded ideal I of R is called a graded $(2,n)$-ideal of R if whenever $rst \in I$ where $r,s,t \in h(R)$, then either $rt \in I$ or $rs \in Gr(0)$ or $st \in Gr(0)$. We introduce several results concerning gr-$(2,n)$-ideals. For example, we give a characterization of graded $(2,n)$-ideals and their homogeneous components. Also, the relations between graded $(2,n)$-ideals and others that already exist, namely, the graded prime ideals, the graded 2-absorbing primary ideals, and the graded n-ideals are studied.

Keywords: gr-$(2,n)$-ideals; gr-2-absorbing primary ideals; gr-prime ideal

Classification: 13A02, 16W50

1. Introduction and preliminaries

Throughout this article, rings are assumed to be commutative with $1 \neq 0$. Let R be a ring, I be a proper ideal of R. By $\sqrt{I}$, we mean the radical of I which is $\{r \in R\colon r^n \in I$ for some positive integer $n\}$. In particular, $\sqrt{0}$ is the set of nilpotent elements in R. Recall from [13] that a proper ideal I of R is said to be an $(2,n)$-*ideal* if whenever $a,b,c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in \sqrt{0}$ or $bc \in \sqrt{0}$.

The scope of this paper is devoted to the theory of graded commutative rings. One use of rings with gradings is in describing certain topics in algebraic geometry. Here, in particular, we are dealing with gr-$(2,n)$-ideals in a G-graded commutative ring. First, we recall some basic properties of graded rings which will be used in the sequel. We refer to [8]–[10] for these basic properties and more information on graded rings. Let G be a group with identity e. A ring R is called graded (or more precisely, G-graded) if there exists a family of subgroups $\{R_g\}$ of R such that $R = \bigoplus_{g\in G} R_g$ (as abelian groups) indexed by the elements $g \in G$, and $R_gR_h \subseteq R_{gh}$ for all $g,h \in G$. The summands R_g are called homogeneous components and elements of these summands are called homogeneous elements. If $a \in R$, then a can be written uniquely $a = \sum_{g\in G} a_g$ where a_g is the component of a in R_g. Also, we write $h(R) = \bigcup_{g\in G} R_g$. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded

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ring. An ideal I of R is said to be a graded ideal if $I = \bigoplus_{g\in G}(I \cap R_g) := \bigoplus_{g\in G} I_g$. An ideal of a graded ring need not be graded. If I is a graded ideal of R, then the quotient ring R/I is a G-graded ring. Indeed, $R/I = \bigoplus_{g\in G}(R/I)_g$ where $(R/I)_g = \{x + I : x \in R_g\}$. Let R be a G-graded ring and $S \subseteq h(R)$ a multiplicatively closed subset of R. Then graded ring of fractions is denoted by $S^{-1}R$ which is defined by $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \bigl\{\frac{a}{s} : a \in R, s \in S, g = (\deg s)^{-1}(\deg a)\bigr\}$.

The *graded radical* of a graded ideal I, denoted by $Gr(I)$, is the set of all $r = \sum_{g\in G} r_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $r_g^{n_g} \in I$. Note that, if x is a homogeneous element, then $x \in Gr(I)$ if and only if $x^n \in I$ for some $n \in \mathbb{N}$, see [12].

Let R be a G-graded ring. A proper graded ideal M of R is said to be *graded maximal ideal of* R (gr-maximal) if J is a graded ideal of R such that $M \subseteq J \subseteq R$, then $M = J$ or $J = R$. A proper graded ideal I of R is said to be a *graded prime (gr-prime)* if whenever $r_g, s_h \in h(R)$ with $r_g s_h \in I$, then either $r_g \in I$ or $s_h \in I$, see [3], [12].

The concepts of graded primary ideals and graded weakly primary ideals of a graded ring have been introduced in [11] and [5], respectively. A proper graded ideal I of a G-graded ring R is called a *graded primary (gr-primary) (*or *graded weakly primary) ideal* if whenever $r_g, s_h \in h(R)$ and $r_g s_h \in I$ (or $0 \neq r_g s_h \in I$, respectively), then either $r_g \in I$ or $s_h \in Gr(I)$.

Graded 2-absorbing ideals of commutative graded rings have been introduced in [1]. According to that paper, I is said to be a *graded* 2*-absorbing (gr-*2*-absorbing) ideal of a G-graded ring* R if whenever $r_g, s_h, t_i \in h(R)$ with $r_g s_h t_i \in I$, then $r_g s_h \in I$ or $r_g t_i \in I$ or $s_h t_i \in I$.

In [4] K. Al-Zoubi and N. Sharafat introduced a generalization of graded primary ideals called graded 2-absorbing primary ideals. A graded ideal I of a G-graded ring R is said to be a *graded* 2*-absorbing primary (gr-*2*-absorbing primary) ideal of* R if whenever $r_g, s_h, t_i \in h(R)$ with $r_g s_h t_i \in I$, then $r_g s_h \in I$ or $r_g t_i \in Gr(I)$ or $s_h t_i \in Gr(I)$.

Recently, K. Al-Zoubi, F. Al-Turman and S. Çeken in [2] introduced and studied the concepts of graded n-ideals in commutative graded rings. A proper graded ideal I of a G-graded ring R is called a graded n-ideal (gr-n-ideal) of R if whenever $r_g, s_h \in h(R)$ with $r_g s_h \in I$ and $r_g \notin Gr(0)$, then $s_h \in I$.

In this paper, we introduce the concept of graded $(2, n)$-ideals (gr-$(2, n)$-ideals) and investigate the basic properties and facts concerning gr-$(2, n)$-ideals.

2. Results

Definition 2.1. Let R be a G-graded ring. A proper graded ideal I of R is called a *graded* $(2,n)$*-ideal of* R if whenever $rst \in I$ where $r,s,t \in h(R)$, then either $rt \in I$ or $rs \in Gr(0)$ or $st \in Gr(0)$. In short, we call it a gr-$(2,n)$-ideal.

Lemma 2.2. *Let R be a G-graded ring and I a proper graded ideal of R. If I is a gr-$(2,n)$-ideal, then $Gr(I) = Gr(0)$.*

Proof: Suppose that I is gr-$(2,n)$-ideal. Clearly, $Gr(0) \subseteq Gr(I)$. Now, let $r_g \in Gr(I) \cap h(R)$, then $r_g^n \in I$ for some $n \in \mathbb{Z}^+$. It follows that $1 \cdot 1 \cdot r_g^n \in I$. Since I is a gr-$(2,n)$-ideal of R and $1 \in R_e - I$, we get $r_g^n \in Gr(0)$ and so $r_g \in Gr(0)$. Hence $Gr(I) \subseteq Gr(0)$. Therefore $Gr(I) = Gr(0)$. □

It is clear that every gr-$(2,n)$-ideal is a gr-2-absorbing primary ideal. However, the converse is not true in general. The example of this is given below.

Example 2.3. Let $R = \mathbb{Z}[\mathrm{i}]$ and $G = \mathbb{Z}_2$. Then R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \mathrm{i}\mathbb{Z}$. Let $I = 6R$. Then I is not gr-$(2,n)$-ideal since we have $Gr(I) \neq Gr(0)$. However an easy computation shows that I is a gr-2-absorbing primary ideal of R.

It is clear that gr-n-ideal is gr-$(2,n)$-ideal. However, the converse is not true in general. The example of this is given below.

Example 2.4. Let $G = \mathbb{Z}_2$. Then $R = \mathbb{Z}_6$ is a G-graded ring with $R_0 = R$ and $R_1 = \{0\}$. Consider the graded ideal $I = \langle 0 \rangle$ of R. It is clear that I is gr-$(2,n)$-ideal of R. However, I is not gr-n-ideal since $2,3 \in h(R) = \mathbb{Z}_6$ with $2 \cdot 3 \in I$, $2 \notin I$ and $3 \notin Gr(0) = 0$.

Theorem 2.5. *Let R be a G-graded ring and I a proper graded ideal of R. Then the following statements are equivalent:*

(i) *Ideal I is a gr-$(2,n)$-ideal of R.*

(ii) *Ideal I is a gr-2-absorbing primary ideal of R and $Gr(I) = Gr(0)$.*

Proof: (i) $\Rightarrow$ (ii) Suppose that I is a gr-$(2,n)$-ideal of R, clearly I is a gr-2-absorbing primary ideal of R. By Lemma 2.1, $Gr(0) = Gr(I)$.

(ii) $\Rightarrow$ (i) Let $r,s,t \in h(R)$ with $rst \in I$. Then $rs \in I$ or $rt \in Gr(I)$ or $st \in Gr(I)$ as I is a gr-2-absorbing primary ideal of R. This implies that $rs \in I$ or $rt \in Gr(0)$ or $st \in Gr(0)$ since $Gr(I) = Gr(0)$. Thus I is a gr-$(2,n)$-ideal of R. □

Note that a gr-prime ideal is not necessarily a gr-$(2,n)$-ideal. For example, let us take a G-graded ring R as in Example 2.3. Then $I = 2R$ be a graded prime ideal. However, I is not gr-$(2,n)$-ideal since we have $Gr(I) \neq Gr(0)$.

Conversely, a gr-$(2,n)$-ideal is not a gr-prime in general. For instance take a G-graded ring R as in Example 2.4. The graded ideal $I = \langle 0\rangle$ is gr-$(2,n)$-ideal but I is not gr-prime ideal since $2\cdot 3 \in I$ but $2 \notin I$ and $3 \notin I$.

Theorem 2.6. *Let R be a G-graded ring and P a gr-prime ideal of R, then the following statements are equivalent:*

(i) *Ideal P is a gr-$(2,n)$-ideal of R.*
(ii) *$P = Gr(0)$.*
(iii) *Ideal P is a gr-n-ideal of R.*

Proof: Since P is gr-prime, then $Gr(P) = P$ by [11, Proposition 1.2 (5)]. It is clear that every gr-prime ideal is gr-2-absorbing primary so the equivalence (i) $\Leftrightarrow$ (ii) follows from Theorem 2.5. Now, the equivalence (ii) $\Leftrightarrow$ (iii) is just [2, Theorem 2.4]. $\square$

The proof of the following result is an analogue of the proof of [6, Lemma 2.18].

Theorem 2.7. *Let R be a G-graded ring, I a gr-$(2,n)$-ideal of R and $J = \bigoplus_{g\in G} J_g$ be a graded ideal of R. If $r,s \in h(R)$ and $g \in G$ such that $rsJ_g \subseteq I$ and $rs \notin I$, then either $rJ_g \subseteq Gr(0)$ or $sJ_g \subseteq Gr(0)$.*

Proof: Let $r,s \in h(R)$ and $g \in G$ such that $rsJ_g \subseteq I$ and $rs \notin I$. Assume that $rJ_g \nsubseteq Gr(0)$ and $sJ_g \nsubseteq Gr(0)$. Then there exist $j_{g_1}, j_{g_2} \in J_g$ such that $rj_{g_1} \notin Gr(0)$ and $sj_{g_2} \notin Gr(0)$. Since $rsj_{g_1} \in I$, $rs \notin I$ and $rj_{g_1} \notin Gr(0)$, we get $sj_{g_1} \in Gr(0)$ as I is gr-$(2,n)$-ideal of R. Similarly, by $rsj_{g_2} \in I$, $rs \notin I$ and $sj_{g_2} \notin Gr(0)$, we conclude that $rj_{g_2} \in Gr(0)$. By $j_{g_1} + j_{g_2} \in J_g$, we get $rs(j_{g_1} + j_{g_2}) \in I$. Then either $r(j_{g_1} + j_{g_2}) \in Gr(0)$ or $s(j_{g_1} + j_{g_2}) \in Gr(0)$ as I is gr-$(2,n)$-ideal of R. This implies that $rj_{g_1} \in Gr(0)$ or $sj_{g_2} \in Gr(0)$ since $rj_{g_2} \in Gr(0)$ and $sj_{g_1} \in Gr(0)$ which is a contradiction. $\square$

Theorem 2.8. *Let R be a G-graded ring, I be a gr-$(2,n)$-ideal of R. Let $J = \bigoplus_{g\in G} J_g$ and $K = \bigoplus_{g\in G} K_g$ be two graded ideals of R. If $r \in h(R)$ and $g,h \in G$ with $rJ_gK_h \subseteq I$, then either $J_gK_h \subseteq I$ or $rJ_g \subseteq Gr(0)$ or $rK_h \subseteq Gr(0)$.*

Proof: Let $r \in h(R)$ and $g,h \in G$ with $rJ_gK_h \subseteq I$ and $J_gK_h \nsubseteq I$. We show that $rJ_g \subseteq Gr(0)$ or $rK_h \subseteq Gr(0)$. Suppose that neither $rJ_g \subseteq Gr(0)$ nor $rK_h \subseteq Gr(0)$. Then there are $j_g \in J_g$ and $k_h \in K_h$ such that $rj_g \notin Gr(0)$ and $rk_h \notin Gr(0)$, but $rj_gk_h \in I$ so we have $j_gk_h \in I$ since I is a gr-$(2,n)$-ideal of R. Now, since $J_gK_h \nsubseteq I$, there exist $j'_g \in J_g$ and $k'_h \in K_h$ such that $j'_gk'_h \notin I$. Since $rj'_gk'_h \in I$ and $j'_gk'_h \notin I$, we get either $rj'_g \in Gr(0)$ or $rk'_h \in Gr(0)$ as I is a gr-$(2,n)$-ideal of R. We consider three cases.

Case 1: Suppose that $rj'_g \in Gr(0)$ but $rk'_h \notin Gr(0)$. Since $rj_gk'_h \in I$, $rj_g \notin Gr(0)$ and $rk'_h \notin Gr(0)$, we get $j_gk'_h \in I$. Since $rj'_g \in Gr(0)$ but $rj_g \notin Gr(0)$, we have $r(j_g+j'_g) \notin Gr(0)$. By $r(j_g+j'_g)k'_h \in I$ and $rk'_h \notin Gr(0)$, we have $(j_g+j'_g)k'_h = j_gk'_h + j'_gk'_h \in I$ as I is gr-$(2,n)$-ideal of R. It follows that $j'_gk'_h \in I$ since $j_gk'_h \in I$, a contradiction.

Case 2: Suppose that $rk'_h \in Gr(0)$ but $rj'_g \notin Gr(0)$, similar to Case 1.

Case 3: Suppose that $rj'_g \in Gr(0)$ and $rk'_h \in Gr(0)$. By $rk'_h \in Gr(0)$ and $rk_h \notin Gr(0)$, we get $r(k_h+k'_h) \notin Gr(0)$. Since $rj_g(k_h+k'_h) \in I$, $rj_g \notin Gr(0)$ and $r(k_h+k'_h) \notin Gr(0)$, we get $j_g(k_h+k'_h) = j_gk_h + j_gk'_h \in I$ as I is gr-$(2,n)$-ideal of R. It follows that $j_gk'_h \in I$ since $j_gk_h \in I$. By $rj'_g \in Gr(0)$ and $rj_g \notin Gr(0)$, we get $r(j_g+j'_g) \notin Gr(0)$. Since $r(j_g+j'_g)k_h \in I$, $rk_h \notin Gr(0)$ and $r(j_g+j'_g) \notin Gr(0)$, we have $(j_g+j'_g)k_h = j_gk_h + j'_gk_h \in I$ as I is gr-$(2,n)$-ideal of R. This yields that $j'_gk_h \in I$ since $j_gk_h \in I$. Now since $r(j_g+j'_g)(k_h+k'_h) \in I$, $r(j_g+j'_g) \notin Gr(0)$ and $r(k_h+k'_h) \notin Gr(0)$, we get $(j_g+j'_g)(k_h+k'_h) = j_gk_h + j'_gk_h + j_gk'_h + j'_gk'_h \in I$. It follows that $j'_gk'_h \in I$, a contradiction. □

Theorem 2.9. *Let R be a G-graded ring, I a proper graded ideal of R. Let $J = \bigoplus_{g\in G} J_g$, $K = \bigoplus_{g\in G} K_g$ and $L = \bigoplus_{g\in G} L_g$ be graded ideals of R. Then the following statements are equivalent:*

(i) *Ideal I is a gr-$(2,n)$-ideal of R.*
(ii) *For every $g,h,\lambda \in G$ with $K_hJ_gL_\lambda \subseteq I$, either $J_gL_\lambda \subseteq I$ or $K_hL_\lambda \subseteq Gr(0)$ or $K_hJ_g \subseteq Gr(0)$.*

Proof: (i) $\Rightarrow$ (ii) Assume that I is a gr-$(2,n)$-ideal of R. Let $g,h,\lambda \in G$ with $K_hJ_gL_\lambda \subseteq I$ and $J_gL_\lambda \nsubseteq I$. Then for all $k_h \in K_h$ either $k_hL_\lambda \subseteq Gr(0)$ or $k_hJ_g \subseteq Gr(0)$ by Theorem 2.8. If $k_hJ_g \subseteq Gr(0)$ for all $k_h \in K_h$, then $K_hJ_g \subseteq Gr(0)$, we are done. Similarly, if $k_hL_\lambda \subseteq Gr(0)$ for all $k_h \in K_h$, then $K_hL_\lambda \subseteq Gr(0)$, we are done. Suppose that $k_{h_1}, k_{h_2} \in K_h$ are such that $k_{h_1}J_g \nsubseteq Gr(0)$ and $k_{h_2}L_\lambda \nsubseteq Gr(0)$. Since $k_{h_1}J_gL_\lambda \subseteq I$, $J_gL_\lambda \nsubseteq I$ and $k_{h_1}J_g \nsubseteq Gr(0)$, by Theorem 2.8, we get $k_{h_1}L_\lambda \subseteq Gr(0)$. Similarly we have $k_{h_2}J_g \subseteq Gr(0)$. By $(k_{h_1}+k_{h_2}) \in K_h$, we get $(k_{h_1}+k_{h_2})J_gL_\lambda \subseteq I$. Then either $(k_{h_1}+k_{h_2})J_g \subseteq Gr(0)$ or $(k_{h_1}+k_{h_2})L_\lambda \subseteq Gr(0)$ by Theorem 2.8. By $(k_{h_1}+k_{h_2})J_g \subseteq Gr(0)$ it follows that $k_{h_1}J_g \subseteq Gr(0)$, which is a contradiction. Similarly by $(k_{h_1}+k_{h_2})L_\lambda \subseteq Gr(0)$ we get a contradiction. Therefore either $K_hL_\lambda \subseteq Gr(0)$ or $K_hJ_g \subseteq Gr(0)$.

(ii) $\Rightarrow$ (i) Assume that (ii) holds. Let $r_g, s_h, t_\lambda \in h(R)$ with $r_gs_ht_\lambda \in I$. Let $J = (r_g)$, $K = (s_h)$ and $L = (t_\lambda)$, be graded ideals of R generated by r_g, s_h, t_λ, respectively. Hence $K_hJ_gL_\lambda \subseteq I$, by our assumption we have $J_gL_\lambda \subseteq I$ or $K_hL_\lambda \subseteq Gr(0)$ or $K_hJ_g \subseteq Gr(0)$. It follows that $r_gt_\lambda \in I$ or $s_ht_\lambda \in Gr(0)$ or $s_hr_g \in Gr(0)$. Thus I is a gr-$(2,n)$-ideal of R. □

Theorem 2.10. *Let R be a G-graded ring and I and J be two proper graded ideals of R, if I and J are gr-$(2,n)$-ideals, then so is $I \cap J$.*

Proof: Let $r, s, t \in h(R)$ such that $rst \in I \cap J$ with $rs \notin I \cap J$, then $rs \notin I$ or $rs \notin J$, suppose for example $rs \notin I$. Then $rt \in Gr(0)$ or $st \in Gr(0)$ since I is a gr-$(2, n)$-ideal, hence $I \cap J$ is a gr-$(2, n)$-ideal of R. $\square$

Note that a gr-primary ideal is not necessarily a gr-$(2, n)$-ideal. For example, let us take a G-graded ring R as in Example 2.3. Let $I = 2R$ be a gr-prime ideal (and so gr-primary). But I is not gr-$(2, n)$-ideal since we have $Gr(I) \neq Gr(0)$. Next we characterize the rings over which every gr-primary ideal (every graded 2-absorbing primary ideal, respectively) is gr-$(2, n)$-ideal.

Recall that a graded principal ideal of a G-graded ring R is a graded ideal of R generated by a single homogeneous element, see [10].

Theorem 2.11. *Let R be a G-graded ring, then the following statements are equivalent:*

(i) *For each $r \in h(R)$, either r is unit or $r \in Gr(0)$.*
(ii) *Every proper graded principal ideal is a gr-n-ideal.*
(iii) *Every proper graded ideal is gr-n-ideal.*
(iv) *Every gr-2-absorbing primary ideal is gr-$(2, n)$-ideal.*
(v) *Every gr-primary ideal is gr-$(2, n)$-ideal.*
(vi) *Every gr-prime ideal is gr-$(2, n)$-ideal.*
(vii) *Every gr-maximal ideal is gr-$(2, n)$-ideal.*
(viii) *Ideal $Gr(0)$ is a gr-maximal ideal of R.*

Proof: (i) $\Rightarrow$ (ii) Let $I = \langle r \rangle$ be a proper graded principal ideal of R where $r \in h(R)$ and let $s, t \in h(R)$ such that $st \in I$ and $s \notin Gr(0)$, so we have s is unit in R, hence $t \in I$, so I is a gr-n-ideal.

(ii) $\Rightarrow$ (iii) Let I be a proper graded ideal of R, and $r, s \in h(R)$ with $rs \in I$ and $r \notin Gr(0)$, but $rs \in \langle rs \rangle \subseteq I$ which is a gr-n-ideal of R, so we conclude $s \in \langle rs \rangle \subseteq I$, hence I is a gr-n-ideal.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) Trivial.

(vii) $\Rightarrow$ (viii) Let M be a gr-maximal ideal of R. By (vii) and Lemma 2.2, we get $Gr(M) = Gr(0)$. Since M is gr-maximal ideal, by [11, Proposition 1.2 (5)], $M = Gr(M) = Gr(0)$. Hence $Gr(0)$ is the unique gr-maximal ideal of R.

(viii) $\Rightarrow$ (i) Let $r \in h(R)$ which is not unit, so $\langle r \rangle$ is a proper graded ideal of R, but $Gr(0)$ is gr-maximal ideal of R, so $Gr(\langle r \rangle) = Gr(0)$ and so $r \in Gr(0)$. $\square$

For G-graded rings R and R', a G-graded ring homomorphism $f\colon R \to R'$ is a ring homomorphism such that $f(R_g) \subseteq R'_g$ for every $g \in G$. The following result studies the behavior of gr-$(2, n)$-ideals under graded homomorphism.

Theorem 2.12. *Let R and R' be two G-graded rings and $\varphi\colon R \to R'$ a graded ring homomorphism. Then the following statements hold:*

(i) *If φ is a graded onto homomorphism and I is a gr-$(2,n)$-ideal of R containing* $\ker\varphi$*, then $\varphi(I)$ is a gr-$(2,n)$-ideal of R'.*

(ii) *If φ is a graded monomorphism and I' is a gr-$(2,n)$-ideal of R', then $\varphi^{-1}(I')$ is a gr-$(2,n)$-ideal of R.*

PROOF: (i) Suppose that I is a gr-$(2,n)$-ideal of R with $\ker\varphi \subseteq I$. Let $r', s', t' \in h(R')$ such that $r's't' \in \varphi(I)$. Since φ is a graded onto homomorphism, there exist $r, s, t \in h(R)$ such that $\varphi(r) = r', \varphi(s) = s'$, $\varphi(t) = t'$. Hence $\varphi(rst) = r's't' \in \varphi(I)$, it follows that $\varphi(rst) = \varphi(i)$ for some $i \in I \cap h(R)$. Then $rst \in I$ since $rst - i \in \ker(\varphi) \subseteq I$. This yields that either $rs \in I$ or $st \in Gr(0_R)$ or $rt \in Gr(0_R)$ as I is a gr-$(2,n)$-ideal of R. Hence either $r's' \in \varphi(I)$ or $s't' \in \varphi(Gr(0_R)) \subseteq Gr(0_{R'})$ or $r't' \in \varphi(Gr(0_R)) \subseteq Gr(0_{R'})$. Therefore $\varphi(I)$ is a gr-$(2,n)$-ideal of R'.

(ii) Suppose that I' is a gr-$(2,n)$-ideal of R'. Let $r, s, t \in h(R)$ such that $rst \in \varphi^{-1}(I')$. Then $\varphi(rst) = \varphi(r)\varphi(s)\varphi(t) \in I'$. Since $\varphi(r)$, $\varphi(s)$, $\varphi(t) \in h(R')$ and I' is a gr-$(2,n)$-ideal of R', we get either $\varphi(r)\varphi(s) = \varphi(rs) \in I'$ or $\varphi(s)\varphi(t) = \varphi(st) \in Gr(0_{R'})$ or $\varphi(r)\varphi(t) = \varphi(rt) \in Gr(0_{R'})$. But φ is a graded monomorphism, so we have either $rs \in \varphi^{-1}(I')$ or $st \in Gr(0_R)$ or $rt \in Gr(0_R)$. Therefore $\varphi^{-1}(I')$ is a gr-$(2,n)$-ideal of R. □

Corollary 2.13. *Let R be a G-graded ring.*

(i) *Let I and J be graded ideals of R with $J \subseteq I$. Then I is a gr-$(2,n)$-ideal of R if and only if I/J is a gr-$(2,n)$-ideal of R/J and $J \subseteq Gr(0)$.*

(ii) *If R' is a graded subring of R and I is a gr-$(2,n)$-ideal of R, then $I \cap R'$ is a gr-$(2,n)$-ideal of R'.*

PROOF: (i) Consider the natural graded ring epimorphism map $\pi\colon R \to R/J$, defined by $\pi(r) = r + J$. The result is clear by Theorem 2.12 (i). Furthermore $J \subseteq I \subseteq Gr(I) = Gr(0)$.

Conversely, suppose that I/J is a gr-$(2,n)$-ideal of R/J and $J \subseteq Gr(0)$. Then I/J is a 2-absorbing primary ideal of R/J and $Gr(I/J) = Gr(0_{R/J})$. Thus $Gr(J)/J = Gr(0_{R/J}) = Gr(I/J) = Gr(I)/J$ implies that $Gr(I) = Gr(J) = Gr(0)$ from our assumption. On the other hand, we conclude from [4, Theorem 2.6 (i)] that I is a 2-absorbing primary ideal of R. Consequently, I is a gr-$(2,n)$-ideal of R by Theorem 2.5.

(ii) Considering the natural injection $i\colon R' \to R$, we conclude the result by Theorem 2.12 (ii) as $i^{-1}(I) = I \cap R'$. □

Let I be a proper graded ideal of G-graded ring R. Then $G - Z_I(R) = \{r \in h(R)\colon rs \in I \text{ for some } s \in h(R) - I\}$.

Theorem 2.14. *Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R.*

(i) *If I is a gr-$(2,n)$-ideal of R, then $S^{-1}I$ is a gr-$(2,n)$-ideal of $S^{-1}R$.*

(ii) *If $S^{-1}I$ is a gr-$(2,n)$-ideal of $S^{-1}R$, $S \cap G\text{-}Z_0(R) = \emptyset$, and $S \cap G\text{-}Z_I(R) = \emptyset$, then I is a gr-$(2,n)$-ideal of R.*

Proof: (i) Assume that I is a gr-$(2,n)$-ideal of R. Let $s \in S \cap I \subseteq h(R)$, hence $s \in I \subseteq Gr(I) = Gr(0)$ by Lemma 2.2, it follows that $s^n = 0$ for some $n \in \mathbb{Z}^+$, so $0 \in S$, a contradiction. Thus $S \cap I = \emptyset$ and $S^{-1}I$ is a proper graded ideal of $S^{-1}R$. Now, let $\frac{a}{s}\frac{b}{t}\frac{c}{k} \in S^{-1}I$ for some $\frac{a}{s}, \frac{b}{t}, \frac{c}{k} \in h(S^{-1}R)$. So there exists $u \in S$ such that $uabc \in I$. Then either $uab \in I$ or $bc \in Gr(0)$ or $uac \in Gr(0)$ as I is a gr-$(2,n)$-ideal of R. If $uab \in I$, then $\frac{a}{s}\frac{b}{t} = \frac{uab}{ust} \in S^{-1}I$, and if $bc \in Gr(0)$, then $\frac{b}{t}\frac{c}{k} \in S^{-1}Gr(0_R) = Gr(S^{-1}0_R) = Gr(0_{S^{-1}R})$. And if $uac \in Gr(0)$ then $\frac{a}{s}\frac{c}{k} = \frac{uac}{usk} \in S^{-1}Gr(0_R) = Gr(S^{-1}0_R) = Gr(0_{S^{-1}R})$. Therefore $S^{-1}I$ is a gr-$(2,n)$-ideal of $S^{-1}R$.

(ii) Suppose that $abc \in I$ for some $a, b, c \in h(R)$. Then $\frac{abc}{1} = \frac{a}{1}\frac{b}{1}\frac{c}{1} \in S^{-1}I$. Since $S^{-1}I$ is a gr-$(2,n)$-ideal of $S^{-1}R$, we conclude that either $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ or $\frac{b}{1}\frac{c}{1} \in Gr(0_{S^{-1}R})$ or $\frac{a}{1}\frac{c}{1} \in Gr(0_{S^{-1}R})$. If $\frac{a}{1}\frac{b}{1} = \frac{ab}{1} \in S^{-1}I$, then $vab \in I$ for some $v \in S$. Since $v \in S$ and $S \cap G\text{-}Z_I(R) = \emptyset$, we have $ab \in I$. If $\frac{b}{1}\frac{c}{1} = \frac{bc}{1} \in Gr(0_{S^{-1}R}) = S^{-1}(Gr(0_R))$, then there exists $t \in S$ and $n \in \mathbb{Z}^+$ such that $(tbc)^n = t^n\, b^n c^n = 0$. Since $t \in S$, we have $t^n \notin G\text{-}Z_0(R)$. Thus $b^n c^n = 0$, and so $bc \in Gr(0_R)$. With a same argument, we can show that if $\frac{a}{1}\frac{c}{1} \in Gr(0_{S^{-1}R})$, then $ac \in Gr(0_R)$. Therefore I is a gr-$(2,n)$-ideal of R. □

Lemma 2.15. *Let R be a G-graded ring. If P_1 and P_2 are two gr-prime ideals of R, then $P_1 \cap P_2$ is gr-2-absorbing ideal of R.*

Proof: Straightforward. □

The set of all minimal gr-prime ideals is denoted by $\mathrm{Min}_g(R)$.

Lemma 2.16. *Let R be a G-graded ring. If R has at most two minimal gr-prime ideals, then there exists a gr-$(2,n)$-ideal. In this case, $Gr(0)$ immediately is a gr-$(2,n)$-ideal.*

Proof: Suppose that R has at most two minimal gr-prime ideals. Assume first that R has only one minimal gr-prime ideal say I, then $Gr(0) = I$, hence by Theorem 2.6, $Gr(0)$ is a gr-$(2,n)$-ideal. Assume that R has exactly two minimal gr-prime ideals say I_1 and I_2, then $Gr(0) = I_1 \cap I_2$, so by Lemma 2.15, $Gr(0)$ is a gr-2-absorbing ideal, then $Gr(0)$ is a gr-2-absorbing primary ideal of R and $Gr(Gr(0)) = Gr(0)$ by [12, Proposition 2.4]. So by Theorem 2.5, we have $Gr(0)$ is a gr-$(2,n)$-ideal of R. □

Theorem 2.17. *Let R be a graded ring with $|\mathrm{Min}_g(R)| \leq 2$. If $Gr(0)$ is not a gr-maximal ideal of R, then the following statements are equivalent:*

(i) *Every* gr-$(2,n)$-*ideal is* gr-*primary.*
(ii) *Ideal* $Gr(0)$ *is a* gr-*prime ideal and it is the only* gr-$(2,n)$-*ideal of* R.
(iii) *Every* gr-$(2,n)$-*ideal is a* gr-n-*ideal.*

Proof: (i) $\Rightarrow$ (ii) Since $|\operatorname{Min}_g(R)| \leq 2$, by Lemma 2.16, gr-$(2,n)$-ideals exist in R, and in particular $Gr(0)$ is a gr-$(2,n)$-ideal of R, so $Gr(0)$ is gr-primary and hence $Gr(0)$ is a gr-prime by [11, Lemma 1.8]. Now let I be a gr-$(2,n)$-ideal of R. Then $Gr(I) = Gr(0)$ by Lemma 2.2, so we have $I \subseteq Gr(0)$. Let L be any gr-maximal ideal of R, consider $l \in (L - (Gr(0)) \cap h(R))$ and $r \in Gr(0) \cap h(R)$, and set $J = I + \langle lr \rangle$, it follows that $J \subseteq Gr(0)$ and this implies $Gr(J) = Gr(0)$, it follows that $Gr(J)$ is a gr-prime ideal of R. By [4, Theorem 2.4], we have J is gr-2-absorbing primary ideal. Thus by Theorem 2.5, J is a gr-$(2,n)$-ideal, so J is gr-primary. Since $rl \in J$ and $l \notin Gr(J) = Gr(0)$, $r \in J$. Then there exist $i \in I \cap h(R)$ and $t \in h(R)$ such that $r = i + tlr$, so $r(1-tl) \in I$. Since $(1-tl) \notin L$, $(1-tl) \notin Gr(I) = Gr(0)$. So there exists $g \in G$ such that $(1-tl)_g \notin Gr(I) = Gr(0)$. By $r(1-tl) \in I$ we get $r(1-tl)_g \in I$ and since I is gr-primary, we get $r \in I$. Therefore $I = Gr(0)$, so $Gr(0)$ is the only gr-$(2,n)$-ideal of R.

(ii) $\Rightarrow$ (iii) It is clear by [2, Theorem 2.4].

(iii) $\Rightarrow$ (i) Since every gr-n-ideal is gr-primary, the result is clear. $\square$

Recall that a G-graded ring R is said to be graded field if $0 \neq a \in h(R)$, then $ab = 1$ for some $b \in h(R)$, see [10].

Theorem 2.18. *Let* R *be a graded ring with* $|\operatorname{Min}_g(R)| \leq 2$. *Then every* gr-$(2,n)$-*ideal is a* gr-*prime ideal of* R *if and only if* $Gr(0)$ *is a* gr-*prime ideal and it is the only* gr-$(2,n)$-*ideal of* R.

Proof: Suppose that every gr-$(2,n)$-ideal is gr-prime. If $Gr(0)$ is not a gr-maximal ideal of R, then the result follows from Theorem 2.8. Now suppose that $Gr(0)$ is a gr-maximal ideal of R. Then every proper graded ideal is gr-n-ideal; so gr-$(2,n)$-ideal of R by Theorem 2.11. Thus every proper graded ideal is gr-prime by our assumption. Therefore R is a graded field by [7, Lemma 2.15], and we are done by [7, Lemma 2.3 (iv)]. The converse part is obvious. $\square$

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K. Al-Zoubi, S. Alghueiri:
Department of Mathematics and Statistics,
Jordan University of Science and Technology, P. O. Box 3030, Irbid 22110, Jordan

E-mail: kfzoubi@just.edu.jo

E-mail: ghweiri64@gmail.com

E. Y. Celikel:
Department of Electrical Electronics Engineering, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, TR-27410, Turkey

E-mail: yetkinece@gmail.com

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