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Lipschitz approximable Banach spaces

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This work is dedicated to the memory of Eve Oja (1948–2019)

Abstract. We show the existence of Lipschitz approximable separable spaces which fail Grothendieck's approximation property. This follows from the observation that any separable space with the metric compact approximation property is Lipschitz approximable. Some related results are spelled out.

Keywords: compact approximation property; Lipschitz map; Lipschitz-free Banach space

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1. Introduction

N. Kalton proved a score of fundamental results on nonlinear geometry of Banach spaces during the last decade of his life, see [7]. He investigated in particular nonlinear versions of Grothendieck's approximation properties which may be valid for every Banach space, see [10], [11], although we know since P. Enflo's work that linear approximation fails in general. We recall that a gauge ω is a subadditive map from $[0, \infty)$ to itself such that $\lim_{t\to 0} \omega(t) = 0$. If X and Y are Banach spaces, a map $f\colon X \to Y$ is uniformly continuous if the map ω_f defined by

$$\omega_f(t) = \sup\{\|f(x) - f(y)\|_Y : \|x - y\|_X \le t\}$$

is a gauge, which amounts to request that $\lim_{t\to 0} \omega_f(t) = 0$. If there is $C > 0$ such that $\omega_f(t) \le Ct$ for all $t \ge 0$, we say that f is Lipschitz. We now recall definitions which are given in Section 4 of [11], in the larger frame of metric spaces.

Definition 1.1. Let X be a separable Banach space. We say that X is approximable if there exists a gauge ω and a sequence ψ_n of maps from X to itself, such that $\psi_n(X)$ is relatively compact in X equipped with the norm topology, with $\omega_{\psi_n}(t) \le \omega(t)$ for all $t > 0$ and $\lim_{n\to\infty} \|x - \psi_n(x)\| = 0$ for all $x \in X$. We say that X is Lipschitz-approximable if the above holds for some $C > 0$ with the gauge $\omega(t) = Ct$.

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In other words, a separable Banach space is approximable if there exists an equiuniformly continuous sequence of maps from X to itself with relatively compact range which converges to Id_X uniformly on all compact subsets of X. Theorem 4.6 in [11] states that if X^* is separable, then X and X^* are both approximable. However, Lipschitz-approximable Banach spaces are not considered further in [11] since it is claimed in the introduction of Section 4 of this article that Lipschitz-approximable Banach spaces actually enjoy the usual (linear) bounded approximation property (BAP). For supporting this claim, reference is made to [6]. However, Definition 5.2 in [6] includes a specific linear restriction on finite-dimensionality of the ranges of the approximating Lipschitz maps which is not part of the definition of Lipschitz-approximable spaces. The present work shows that this restriction is actually necessary since we exhibit Lipschitz approximable spaces which fail BAP, and actually fail the approximation property. This answers a question which has been around for some time, see Question 4 in [8] or Problem 3 in [4].

Linear approximation properties are surveyed in [2]. We use the usual pieces of notation (AP, BAP, MAP) for the classical linear approximation properties. We recall that a Banach space X has the compact approximation property (CAP) (or the λ-bounded CAP) if Id_X belongs to the closure of the space $K(X)$ of compact operators (or of the ball of radius λ in $K(X)$, respectively) for the topology of uniform convergence on compact subsets of X. The 1-bounded CAP is called the metric CAP. Reflexive spaces with CAP actually have the metric CAP, see [3] for a stronger result. The existence of separable reflexive spaces which have CAP but fail the approximation property has been shown by G. A. Willis [15]. We refer to [6] for definitions and some properties of Lipschitz-free Banach spaces.

2. The use of compact approximation property

Our main result is the following simple observation.

Theorem 2.1. *Let X be a separable Banach space with the bounded compact approximation property. Then X is Lipschitz-approximable.*

Proof: By assumption, there exist $\lambda \in \mathbb{R}$ and a sequence (K_n) of compact operators on X such that $\|K_n\| \leq \lambda$ for all n, and $\lim \|K_n(x) - x\| = 0$ for all $x \in X$. For any integer $k \geq 1$, the radial projection $r_k \colon X \to kB_X$ is 2-Lipschitz, see [14] for a more precise result. We define $f_{n,k} = K_n \circ r_k$. The maps $(f_{n,k})$ are (2λ)-Lipschitz, their range $f_{n,k}(X) = K_n(kB_X)$ is relatively compact in X. If L is an arbitrary compact subset of X, there is $k_0 \geq 1$ such that $L \subset k_0 B_X$ and then the restriction of (f_{n,k_0}) to L converges uniformly on L to the canonical injection from L to X when n increases to ∞. Re-indexation shows that X is Lipschitz-approximable, with a constant $C = 2\lambda$. $\square$

The following corollary is the motivation for this result.

Corollary 2.2. *There exist Lipschitz-approximable reflexive separable spaces which fail the approximation property.*

Indeed G. A. Willis in [15] showed the existence of reflexive spaces with the CAP (and thus the metric CAP) which fail the AP, and then Theorem 1 concludes the proof. Let us note more precisely that it is shown in [15] that such spaces X exist which are subspaces of quotients of L_p, $1 < p < \infty$, $p \neq 2$, and thus are super-reflexive.

3. Some related results

We first recall that the λ-Lipschitz BAP is defined in [6, Definition 5.2] as Lipschitz-approximable with the additional condition that the relatively compact sets $\psi_n(X)$ are contained in finite-dimensional linear subspaces of X. Theorem 5.3 in [6] states that a Banach space X has the λ-Lipschitz BAP if and only if it has λ-BAP if and only if its free space $\mathcal{F}(X)$ has λ-BAP. Let us outline proofs of these results for separable spaces which are slightly different from the proofs in [6]. For instance we use spaces of convergent sequences in the spirit of Section 2 in [1]. The following proposition roughly states that "good limits" of metric spaces (E_n) which are such that $\mathcal{F}(E_n)$ has BAP (uniformly in n) still enjoy this property.

Proposition 3.1. *Let M be a metric space, and $\lambda \in \mathbb{R}$. Assume that there exists a sequence (ψ_n) of λ-Lipschitz maps from M to M such that $\lim \psi_n(x) = x$ for all $x \in M$. Assume moreover that $\psi_n(M) \subset E_n$, where (E_n) is an increasing sequence of subsets of M such that the Banach spaces $\mathcal{F}(E_n)$ have ν-bounded approximation property for some $\nu \in \mathbb{R}$ and all $n \geq 1$. Then the free space $\mathcal{F}(M)$ has the $(\lambda\nu)$-bounded approximation property.*

Proof: We denote by $S = c(\mathcal{F}(E_n))$ the Banach space of all sequences (μ_n) with $\mu_n \in \mathcal{F}(E_n)$ for all n which converge in the space $\mathcal{F}(M)$, equipped with the canonical supremum norm. It follows from our assumptions that the set $D = \bigcup_n E_n$ is dense in M, and thus $Q((\mu_n)) = \lim(\mu_n)$ defines a quotient map from S onto $\mathcal{F}(M)$ with $\|Q\| = 1$.

We claim that the space S has ν-BAP. Indeed the projections defined by $P_n((\mu_k)) = (\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n, \mu_n, \mu_n, \dots)$ have norm 1 and composing them with the approximation operators on the spaces $\mathcal{F}(E_n)$ shows our claim. Moreover there exist linear maps, see [6],

$$\widehat{\psi}_n \colon \mathcal{F}(M) \to \mathcal{F}(E_n)$$

with $\|\widehat{\psi}_n\| \le \lambda$ extending the maps (ψ_n) and thus such that $\mu = \lim \widehat{\psi}_n(\mu)$ for all $\mu \in \mathcal{F}(M)$. We define $\Psi \colon \mathcal{F}(M) \to S$ by $\Psi(\mu) = (\widehat{\psi}_n(\mu))_n$. We have $\|\Psi\| \le \lambda$ and $Q\Psi = \mathrm{Id}_{\mathcal{F}(M)}$. The conclusion follows since S has the ν-BAP. $\square$

The special case when M is a separable Banach space provides the hard implication of Theorem 5.3 in [6].

Corollary 3.2. *Let X be a separable Banach space with the λ-Lipschitz bounded approximation property. Then the space $\mathcal{F}(X)$ has the λ-bounded approximation property.*

Proof: By Proposition 5.1 in [6], the space $\mathcal{F}(E)$ has the MAP for every finite dimensional Banach space E. We may therefore apply Proposition 3.1 above with $M = X$ and an increasing sequence (E_n) of finite dimensional subspaces of X. $\square$

Note that Proposition 3.1 leads also to applications to nonlinear metric spaces. We spell out for instance the following:

Corollary 3.3. *Let M be a separable metric space, which is the closure of an increasing union of subsets (E_n) which satisfy the following properties:*

- (i) *There is $\nu > 0$ such that for every set E_n, there exist a compact convex subset K_n of a finite-dimensional normed space and a bi-Lipschitz bijective map F_n from E_n onto K_n with $Lip(F_n)$, $Lip(F_n^{-1}) \le \nu$.*
- (ii) *There exists a sequence (ψ_n) of λ-Lipschitz maps from M to E_n such that $\lim \psi_n(x) = x$ for all $x \in M$.*

Then $\mathcal{F}(M)$ has the $(\lambda\nu)$-Banach approximation property.

Indeed this follows from Proposition 3.1 since it is shown in [13] that $\mathcal{F}(K)$ has MAP for every compact convex subset of a finite-dimensional Banach space. In fact convexity is not fully necessary and it suffices by [13] to request that K is starlike with respect to all points from an open subset of K. It is not known if $\mathcal{F}(E)$ has MAP for every subset E of a finite-dimensional Banach space.

We now gather in a single proposition some properties which *fail* to imply that a given space X has the approximation property. The gist of this statement is that the λ-Lipschitz BAP as defined in [6] is optimal. An approximating sequence is a sequence of maps which converge to Id_X for the topology of uniform convergence on compact subsets of X. We say that a (possibly nonlinear) map F has finite rank if $F(X)$ is contained in a finite-dimensional subspace of X. We refer to [9] for the investigation of compact-valued Lipschitz maps and the corresponding linear operators on free spaces.

Proposition 3.4. *We denote by X a separable Banach space. Then each of the following properties fail to imply that X has the approximation property.*

(i) *There exists an approximating sequence (φ_n) of 2-Lipschitz maps from X to X such that for all n, the set $\varphi_n(X)$ is a compact convex subset of X.*

(ii) *There exist a gauge ω and an approximating sequence (φ_n) of equiuniformly continuous finite rank maps with relatively compact range such that for all n, we have $\omega_{\varphi_n} \leq \omega$ and moreover $\omega_{\varphi_n}(t) \leq 2t + 1/n$ for all $t > 0$ and all n.*

(iii) *For any $\varepsilon > 0$ and any compact subset $K \subset X$, there exists a finite rank Lipschitz map φ such that $\varphi(X)$ is relatively compact and $\|\varphi(x) - x\| < \varepsilon$ for all $x \in K$.*

PROOF: Let X be a reflexive space with CAP and failing AP. Then the proof of Theorem 2.1 provides an approximating sequence of 2-Lipschitz maps whose ranges are convex and compact (since X is reflexive) and this shows (i). By Proposition 4.3 in [11], any space with (i) satisfies condition (ii) as well, hence the same counterexample works.

Let us now check the easy fact that any Banach space satisfies condition (iii). Pick $\varepsilon > 0$, and let $(x_i)_{i \leq N}$ be a finite ε-net in K. For any $i \leq N$ and $x \in K$, we define $g_i(x) = \sup(\varepsilon - \|x_i - x\|, 0)$ and $h_i = g_i / \big(\sum_{k \leq N} g_k\big)$. Since K is compact, there is $\eta > 0$ such that $\big(\sum_{k \leq N} g_k\big) \geq \eta$ on K and thus the collection (h_i) is a partition of unity subordinated to the cover $(B(x_i, \varepsilon))_{i \leq N}$ which consists of Lipschitz functions. Mac Shane's formula and truncation shows that for every i, there exists Lipschitz extensions $\tilde{h}_i \colon X \to [0,1]$ to h_i. We now define

$$\varphi(x) = \sum_{i=1}^{N} \tilde{h}_i(x) x_i$$

and it is easy to check that φ satisfies our requirements. □

We conclude this note with some remarks and problems.

Remarks. 1) The existence of a compact convex set K such that $\mathcal{F}(K)$ fails AP was shown in [8], as a consequence of the existence of Banach spaces failing AP and of the lifting property as shown in [6]. Theorem 2.1 combined with Proposition 3.1 provides a slightly alternative approach for the construction of compact convex sets L such that $\mathcal{F}(L)$ fails the BAP, where cube measures are not needed but only the lifting property which can be shown e.g. through convolution with a Gaussian measure. Let us outline the argument: by the proof of Theorem 2.1 and Proposition 3.1, a separable reflexive space X with compact BAP but failing AP contains an increasing sequence of compact convex sets (L_n) whose union is dense, such that the corresponding free spaces $\mathcal{F}(L_n)$ do not have BAP uniformly in n. We equip the Cartesian product $L = \prod_n L_n$ with the distance $d((x_k),(y_k)) = \sup_k\{2^{-k}d_k^{-1}\|x_k - y_k\|\}$, where d_k denotes the diameter of L_k. For each n, the compact set L contains a 1-Lipschitz retract subset $\widetilde{L}_n$ whose free space is isometric to $\mathcal{F}(L_n)$ and our claim follows.

2) Extensions of Lipschitz functions defined on subsets of compact sets K such that $\mathcal{F}(K)$ fail BAP are investigated in [5], where it is observed that the Lipschitz norms of such extensions cannot be controlled. Along these lines, we observe the following: Let K be a compact convex subset of a Banach space such that $\mathcal{F}(K)$ fails the BAP. Let (C_n) be an increasing sequence of finite-dimensional compact convex sets whose union is dense in K. Then there is a sequence (γ_n) of real numbers increasing to ∞ such that any Lipschitz retraction r_n from K onto C_n satisfies $\mathrm{Lip}(r_n) \geq \gamma_n$. This follows indeed from Corollary 3.3.

3) Willis' examples are super-reflexive with CAP and thus we may and do assume that they have uniformly convex norms and the metric CAP. It follows in this case that in the proof of Theorem 2.1 we have $\lambda = 1$ and by [14] the radial projections (r_k) are α-Lipschitz for some $\alpha < 2$. Hence such a space X is Lipschitz-approximable with some constant $\alpha < 2$.

4) The class of Lipschitz approximable Banach spaces strictly contains the class of spaces with the BAP, and is stable under Lipschitz isomorphism. Is every separable Banach space Lipschitz approximable?

5) It is not known either if every separable Banach space is approximable in the sense of Definition 1.1. This important problem goes back to N. Kalton (Problem 1 in [11]). If the answer is negative, the existence of an equivalent norm on l_1 which fails MAP follows, and thus by dualizing the existence of a dual space which has BAP but fails MAP. This would solve a problem which goes back to A. Grothendieck. We refer to [12] for a useful survey on this problem.

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