Weingarten hypersurfaces of the spherical type in Euclidean spaces

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Abstract. We generalize a parametrization obtained by A. V. Corro in (2006) in the three-dimensional Euclidean space. Using this parametrization we study a class of oriented hypersurfaces M^n, $n \geq 2$, in Euclidean space satisfying a relation $\sum_{r=1}^{n}(-1)^{r+1} r f^{r-1} \binom{n}{r} H_r = 0$, where H_r is the rth mean curvature and $f \in C^\infty(M^n;\mathbb{R})$, these hypersurfaces are called Weingarten hypersurfaces of the spherical type. This class of hypersurfaces includes the surfaces of the spherical type (Laguerré minimal surfaces). We characterize these hypersurfaces in terms of harmonic applications. Also, we classify the Weingarten hypersurfaces of the spherical type of rotation and we give explicit examples.

Keywords: Weingarten hypersurface; Laguerre minimal surface; rth mean curvature; Laplace–Beltrami operator

Classification: 53C42, 53A35

1. Introduction

The surfaces M^2 satisfying a functional relation of the form $W(H,K) = 0$, where H and K are the mean and Gaussian curvatures of the surface M^2, respectively, are called *Weingarten surfaces.* Examples of Weingarten surfaces are the surfaces of revolution and the surfaces of constant mean or Gaussian curvature. In [9], the authors study an important class of surfaces satisfying a linear relation of the form

$$aH + bK + c = 0,$$

where $a, b, c \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. These surfaces are called linear Weingarten surfaces.

W. K. Schief in [13] studied surfaces $M^2 \subset \mathbb{R}^3$ satisfying a Weingarten relation of the form $(\mu^2 \pm \varrho^2)K + 2\mu H + 1 = 0$, where $\mu, \varrho\colon M^2 \to \mathbb{R}$ are harmonic functions defined on the surface.

In [5], A. V. Corro presented a way of parameterizing surfaces as envelopes of a congruence of spheres in which an envelope is contained in a plane and with

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radius function h associated with a hydrodynamic type system. As an application, it studies the surfaces in hyperbolic space $\mathbb{H}^3$ satisfying the equality

$$2ach^{2(c-1)/c}(H-1) + (a+b-ach^{2(c-1)/c})K = 0,$$

where $a, b, c \in \mathbb{R}$, $a+b \neq 0$, $c \neq 0$, H is the mean curvature and K is the Gaussian curvature. This class of surfaces includes the Bryant surfaces and the flat surfaces of the hyperbolic space, and are called generalized Weingarten surfaces of Bryant type.

In [6] the authors study the surfaces M^2 in the hyperbolic space $\mathbb{H}^3$ satisfying the relation $2(H-1)e^{2\mu} + K(1-e^{2\mu}) = 0$, where μ is a harmonic function with respect to the quadratic form $\sigma = -KI + 2(H-1)II$, I and II are the first and second fundamental form of M^2. These surfaces are called generalized Weingarten surfaces of harmonic type.

D. G. Dias in [7] studied a class of oriented surfaces $M^2 \subset \mathbb{R}^3$ satisfying a relation of the form $2\Psi_v H + \Delta_v K = 0$, where $\Psi_v, \Delta_v \colon M^2 \to \mathbb{R}^3$ are given by $\Psi_v(p) = \langle p-v, N(p)\rangle$, $\Delta_v(p) = \langle p-v, p-v\rangle$ and $v \in \mathbb{R}^3$ is a fixed vector.

An oriented surface $\psi\colon M^2 \to \mathbb{R}^3$ with nonzero Gaussian curvature K and mean curvature H is called a *Laguerre minimal surface* if

$$\Delta_{III}\Big(\frac{H}{K}\Big) = 0,$$

where Δ_{III} is the Laplacian with respect to the third fundamental form III of ψ. The study of these surfaces was done by W. Blaschke in [1], [2], [3], [4], where such surfaces appear as critical points of the functional

$$L(\Psi) = \int \frac{H^2-K}{K}\,\mathrm{d}M,$$

where $\mathrm{d}M$ is the area element.

In [12], the authors study Laguerre's minimal surfaces as graphs of biharmonic functions in the isotropic model of Laguerre geometry. In particular, they study the surfaces of the spherical type (Laguerre minimal surfaces), namely the surfaces M^2 of $\mathbb{R}^3$ such that the set of spheres with center $p+(H(p)/K(p))N(p)$, $p \in M^2$, are tangent to a fixed oriented plane.

In [8], the authors study a class of oriented hypersurfaces M^n in $(n+1)$-dimensional hyperbolic space that satisfy a Weingarten relation in the form

$$\sum_{r=0}^{n}(c-n+2r)\binom{n}{r}H_r = 0,$$

where c is a real constant and H_r is the rth mean curvature of the hypersurface M^n. They show that this class of hypersurfaces is characterized by a harmonic application derived from the two hyperbolic Gauss map. Looking these hypersurfaces as orthogonal to a congruence of geodesics, they also show the relation of such hypersurfaces with solutions of the equation $\Delta u + ku^{(n+2)/(n-2)} = 0$, where $k \in \{-1, 0, 1\}$.

In this paper, motivated by the work of [5], we will present a way to parametrize hypersurfaces as congruence of spheres in which an envelope is contained in a hyperplane. Using this parametrization, we present a generalization of the surfaces of the spherical type (Laguerre minimal surfaces) studied in [12], namely the *Weingarten hypersurfaces of the spherical type*, i.e. the oriented hypersurfaces of the Euclidean space $M^n \subset \mathbb{R}^{n+1}$ satisfying a Weingarten relation of the form

$$\sum_{r=1}^{n} (-1)^{r+1} r f^{r-1} \binom{n}{r} H_r = 0,$$

where $f \in C^\infty(M^n; \mathbb{R})$ and H_r is the rth mean curvature of M^n. We characterize these hypersurfaces in terms of harmonic applications. Also, we classify the Weingarten hypersurfaces of the spherical type of rotation and we give explicit examples.

2. Preliminaries

Let Ω be an open subset of $\mathbb{R}^n$, $u = (u_1, u_2, \dots, u_n) \in \Omega$. Consider $M^n \subset \mathbb{R}^{n+1}$ an oriented hypersurface with n distinct principal curvatures k_i, $i = 1, \dots, n$.

If $X\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 2$, is a local parametrization of M^n and $N\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ is the Gauss map of M^n, then in local coordinates we get

$$N_{,i} = \sum_{j=1}^{n} W_{ij} X_{,j}, \qquad 1 \leq i \leq n,$$

where $W = (W_{ij})$ is called *Weingarten matrix of* M^n.

We note that, the subscript "$\,,i$" denotes the derivative with respect to u_i.

Moreover, when X is parametrized by lines of curvature we have

$$\langle X_{,i}, X_{,j} \rangle = \delta_{ij} g_{ii}, \qquad 1 \leq i, j \leq n, \tag{2.1}$$

$$N_{,i} = -k_i X_{,i}, \tag{2.2}$$

$$\Gamma^k_{ij} = 0, \qquad \Gamma^i_{ii} = \frac{g_{ii,i}}{2g_{ii}}, \qquad \Gamma^j_{ii} = -\frac{g_{ii,j}}{2g_{jj}}, \qquad \Gamma^i_{ij} = \frac{g_{ii,j}}{2g_{ii}}, \tag{2.3}$$

where i, j, k are distinct.

Definition 1. The *mean curvature* and the *Gauss–Kronecker curvature* of M^n are given by

$$H = \frac{1}{n}\sum_{i=1}^{n} k_i, \qquad K = \prod_{i=1}^{n} k_i.$$

Definition 2. The *rth-mean curvature* H_r of M^n is defined by

$$H_r = \frac{S_r(W)}{\binom{n}{r}},$$

where, for intergers $0 \le r \le n$, $S_r(W)$ is given by

$$S_0(W) = 1,$$
$$S_r(W) = \sum_{1 \le i_1 < \cdots < i_r \le n} k_{i_1} \dots k_{i_r}.$$

We observe that $H = H_1$ and $K = H_n$.

Definition 3. The restriction of the usual inner product of $\mathbb{R}^{n+1}$ to the tangent space T_pM^n of M^n induces a metric on M^n, called *first fundamental form* I of M^n and given by

$$I_p(w_1, w_2) = \langle w_1, w_2 \rangle, \qquad p \in M^n,\ w_1, w_2 \in T_pM^n.$$

The *second fundamental form* II and the *third fundamental form* III of M^n are given by

$$II_p(w_1, w_2) = \langle -\mathrm{d}N_p(w_1), w_2 \rangle,$$
$$III_p(w_1, w_2) = \langle -\mathrm{d}N_p(w_1), -\mathrm{d}N_p(w_2) \rangle,$$

where $p \in M^n$, $w_1, w_2 \in T_pM^n$ and $\mathrm{d}N_p$ is the differential of the Gauss map in p.

We observe that the third fundamental form III of M^n, according to M. Obata in [11], satisfies

$$III_p(w_1, w_2) = nH\langle \mathrm{d}N_p(w_1), w_2 \rangle - \mathrm{Ric}_p(w_1, w_2), \qquad w_1, w_2 \in T_pM^n.$$

where Ric stands for the Ricci tensor. In the case, i.e. in the two dimensional case of surfaces, the third fundamental form III is expressible entirely in terms of the first fundamental form I and second fundamental form II:

$$III = 2H\,II - KI,$$

where H and K are the mean curvature and the Gaussian curvature of the surface, respectively.

Definition 4. The *Laplace–Beltrami operator* is defined by

$$\Delta f = \operatorname{div}\operatorname{grad} f, \qquad f \in C^\infty(M^n, \mathbb{R}).$$

Let M^n be a Riemannian manifold with Riemannian metric $g = (g_{ij})$. Then in local coordinates the Laplace–Beltrami operator is given by

$$\Delta_g f = \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(g^{ij}\sqrt{|\det g|}\frac{\partial f}{\partial x_j}\Big),$$

where $g^{-1} = (g^{ij})$ and $f \in C^\infty(M^n, \mathbb{R})$. The proof can be seen in [10].

In particular, for $g = \mathit{III}$ the Laplace–Beltrami operator of a smooth function $f \in C^\infty(M^n, \mathbb{R})$ with respect to the third fundamental form III is given by

$$\Delta_{\mathit{III}} f = \frac{1}{\sqrt{|\det \mathit{III}|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(\mathit{III}^{-1}\sqrt{|\det \mathit{III}|}\frac{\partial f}{\partial x_j}\Big).$$

Definition 5. A surface $M^2 \subset \mathbb{R}^3$ is a *Laguerre minimal surface* if

$$\Delta_{\mathit{III}}\Big(\frac{H}{K}\Big) = 0,$$

where H and K are the mean and Gaussian curvature of M^2, respectively, and $\triangle_{\mathit{III}}$ is the Laplacian operator with respect to the third fundamental form III of M^2.

3. Sphere congruence in $\mathbb{R}^{n+1}$

Definition 6. A *sphere congruence* in $\mathbb{R}^{n+1}$ is a n-parameter family of spheres, with a differentiable *radius function*, whose centers lie on a hypersurface $M^n \subset \mathbb{R}^{n+1}$. An *envelope* of a sphere congruence is a hypersurface M^n such that each point of the hypersurface M^n is tangent to a sphere of the sphere congruence.

If there exist a diffeomorphism $\varphi\colon M^n \to \widetilde{M}^n$, a differentiable function $h\colon M^n \to \mathbb{R}$, unit normal vector fields N, $\widetilde{N}$ of M^n and $\widetilde{M}^n$, respectively, such that:

a) $q + h(q)N(q) = \varphi(q) + h(q)\widetilde{N}(\varphi(q))$, $\forall\, q \in M^n$;

b) the subset $q + h(q)N(q)$, $q \in M^n$, is a n-dimensional hypersurface;

we say that M^n and $\widetilde{M}^n$ are *locally associated by a sphere congruence.*

Definition 7. A surface $M^2 \subset \mathbb{R}^3$ is called *surface of the spherical type* if the spheres with center in $p + (H(p)/K(p))N(p)$ and radius $H(p)/K(p)$ are tangent to a fixed oriented plane, where N is the Gauss map of M^2.

In this paper the inner product $\langle\ ,\rangle\colon \mathbb{C}\times\mathbb{C}\to\mathbb{R}$ is defined by

$$\langle f,g\rangle = f_1g_1+f_2g_2, \qquad \text{where}\ \ f=f_1+\mathrm{i}f_2,\ g=g_1+\mathrm{i}g_2,$$

are holomorphic functions.

In the computation we use the following properties: If $f,g\colon\mathbb{C}\to\mathbb{C}$ are holomorphic functions of $z=u_1+\mathrm{i}u_2$, then

$$(3.1)\ \ \langle f,g\rangle_{,1}=\langle f',g\rangle+\langle f,g'\rangle,\quad \langle f,g\rangle_{,2}=\langle \mathrm{i}f',g\rangle+\langle f,\mathrm{i}g'\rangle,\quad \langle fg,h\rangle=\langle g,\bar{f}h\rangle.$$

Lemma 1. *Let $M^n\subset\mathbb{R}^{n+1}$ be an oriented hypersurface, N the Gauss map of M^n and Π a hyperplane with unit normal vector ϑ satisfying $N\neq\vartheta$, $\forall\, p\in M^n$. Then there exists a sphere congruence given by a function $h\colon M^n\to\mathbb{R}$ such that M^n and Π are envelopes of this sphere congruence and such an h is called* radius function*, in relation to the hyperplane Π.*

Proof: Given $p\in M^n$ and the Gauss map $N(p)$ there exist the radius function $h\colon M^n\longrightarrow\mathbb{R}$ given by

$$h(p)=\frac{\langle Y(p),\vartheta\rangle-\langle p,\vartheta\rangle}{\langle N(p),\vartheta\rangle-1},$$

where $Y(p)$ is a local parametrization of the hyperplane Π.

Note that $\langle Y(p),\vartheta\rangle$ measures the distance from the origin to the hyperplane Π and this distance does not depend on the point p. To see this, consider p_1, $p_2\in M^n$,

$$\langle Y(p_1),\vartheta\rangle-\langle Y(p_2),\vartheta\rangle=\langle Y(p_1)-Y(p_2),\vartheta\rangle=0.$$

Thus, the set of points $\{p+h(p)N(p)\colon p\in M^n\}$ is a hypersurface of $\mathbb{R}^{n+1}$ and the map $Y\colon U\to\Pi$, given by

$$Y(p)=p+h(p)(N(p)-\vartheta),\qquad p\in U,$$

is a diffeomorphism on $Y(U)$ satisfying $p+hN=Y+h\vartheta$. Therefore, M^n and Π are locally associated by a sphere congruence. □

Remark 1. The radius function h in relation to the hyperplane Π defined in Lemma 1, is a geometric invariant analogous to the rth mean curvature, in the sense that it does not depend on the parametrization of the hypersurface M^n.

The next theorem generalizes the result obtained in [5].

Theorem 1. *An oriented hypersuface M^n in $\mathbb{R}^{n+1}$, $n\geq 2$, is an envelope of sphere congruence, whose other envelope is contained in the hyperplane $\Pi=\{(x_1,x_2,\dots,x_{n+1})\in\mathbb{R}^{n+1}\colon x_{n+1}=0\}$ if and only if there exist an orthogonal local parametrization of Π, $Y\colon U\subset\mathbb{R}^n\to\Pi$ and a differentiable function h:*

$U \subset \mathbb{R}^n \to \mathbb{R}$, such that $X\colon U \subset \mathbb{R}^n \to M^n$, given by

$$X(u) = Y(u) - \frac{2h(u)}{S}\left[\sum_{j=1}^{n} \frac{h_{,j}}{L_{jj}} Y_{,j} - e_{n+1}\right] \tag{3.2}$$

is a parametrization of M^n with $e_{n+1} = (0, 0, \ldots, 0, 1)$, $L_{ij} = \langle Y_{,i}, Y_{,j}\rangle$ $1 \leq i, j \leq n$ and

$$S = \sum_{j=1}^{n} \frac{(h_{,j})^2}{L_{jj}} + 1. \tag{3.3}$$

Moreover, the Gauss map is given by

$$N(u) = e_{n+1} + \frac{2}{S}\left[\sum_{j=1}^{n} \frac{h_{,j}}{L_{jj}} Y_{,j} - e_{n+1}\right], \tag{3.4}$$

and the Weingarten matrix is given by

$$W = 2V(SI - 2hV)^{-1}, \tag{3.5}$$

where the matrix $V = (V_{ij})$ is given by

$$V_{ij} = \frac{1}{L_{jj}}\left(h_{,ij} - \sum_{l=1}^{n} \widetilde{\Gamma}_{ij}^{l} h_{,l}\right), \qquad 1 \leq i, j \leq n, \tag{3.6}$$

and $\widetilde{\Gamma}_{ki}^{l}$ are the Christoffel symbols of the metric L_{ij}.

The regularity condition of X is given by

$$P = \det\,(SI - 2hV) \neq 0. \tag{3.7}$$

Proof: A hypersurface M^n in $\mathbb{R}^{n+1}$ is an envelope of a sphere congruence, where the other envelope is contained in a hyperplane $\mathbb{R}^n \cong \Pi$ if and only if there exist a local orthogonal parametrization of Π, $Y\colon U \subset \mathbb{R}^n \to \Pi$ and a differentiable function $h\colon U \subset \mathbb{R}^n \to \mathbb{R}$, such that $X\colon U \subset \mathbb{R}^n \to M^n$ is a parametrization of M^n satisfying

$$X(u) = Y(u) + h(u)(e_{n+1} - N(u)), \tag{3.8}$$

where N is the Gauss map of X, such that

$$N = \sum_{j=1}^{n} B_j Y_{,j} + B_{n+1} e_{n+1} \tag{3.9}$$

and

$$\sum_{j=1}^{n}(B_j)^2 L_{jj} + (B_{n+1})^2 = 1. \tag{3.10}$$

Differentiating (3.8), we get

$$X_{,i} = Y_{,i} + h_{,i}(e_{n+1} - N) - hN_{,i}, \qquad 1 \leq i \leq n. \tag{3.11}$$

Using (3.9) and (3.11), we have that

$$0 = \langle N, X_{,i}\rangle = B_i L_{ii} + h_{,i}(B_{n+1} - 1).$$

Hence,

$$B_i = \frac{h_{,i}}{L_{ii}}(1 - B_{n+1}), \qquad 1 \leq i \leq n. \tag{3.12}$$

From (3.12) and (3.10) we get

$$\sum_{j=1}^{n} \frac{(h_{,j})^2}{L_{jj}^2}(1 - B_{n+1})^2 L_{jj} + (B_{n+1})^2 = 1,$$

which is equivalent to

$$B_{n+1}^2 S - 2B_{n+1}(S-1) + (S-1) = 1.$$

Hence, it follows that

$$B_{n+1} = 1 - \frac{2}{S} \quad \text{or} \quad 1.$$

If $B_{n+1} = 1$, then $N = e_{n+1}$, which is a contradiction. Therefore

$$B_{n+1} = 1 - \frac{2}{S}. \tag{3.13}$$

Thus, from (3.9), (3.12) and (3.13) we get (3.4). Also, using (3.4) in (3.8) we obtain (3.2).

Now we calculate the Weingarten matrix. To do this, consider the following vector valued functions

$$C = Y + he_{n+1}, \qquad D = \sum_{i=1}^{n} \frac{h_{,i}}{L_{ii}} Y_{,i} + \Big(\frac{S}{2} - 1\Big)e_{n+1}. \tag{3.14}$$

Differentiating (3.14) and (3.3), we have

$$S_{,i} = 2\sum_{k=1}^{n} V_{ik} h_{,k}, \tag{3.15}$$

$$C_{,i} = Y_{,i} + h_{,i} e_{n+1}, \tag{3.16}$$

$$D_{,i} = \sum_{k=1}^{n} V_{ik} C_{,k}. \tag{3.17}$$

Observe that

$$X = C - \frac{2h}{S} D \qquad \text{and} \qquad N = \frac{2}{S} D.$$

Taking the derivatives of these expressions and using (3.15)–(3.17), it follows that

$$X_{,i} = C_{,i} - h_{,i} N + \frac{2h}{S} \sum_{k=1}^{n} V_{ik} \Big(\frac{2Dh_{,k}}{S} - C_{,k} \Big), \tag{3.18}$$

$$N_{,i} = \frac{2}{S} \sum_{k=1}^{n} V_{ik} \Big(C_{,k} - \frac{2Dh_{,k}}{S} \Big). \tag{3.19}$$

Now, using (3.18) and (3.19)

$$\begin{aligned} \frac{2h}{S} \sum_{k=1}^{n} V_{ik} N_{,k} + \frac{2}{S} \sum_{j=1}^{n} V_{ij} X_{,j} &= \frac{2h}{S} \sum_{k=1}^{n} V_{ik} \Big(\frac{2}{S} \sum_{l=1}^{n} V_{kl} \Big(C_{,l} - \frac{2Dh_{,l}}{S} \Big) \Big) \\ &\quad + \frac{2}{S} \sum_{j=1}^{n} V_{ij} X_{,j} \\ &= \frac{2}{S} \sum_{j=1}^{n} V_{ij} (C_{,j} - h_{,j} N) \\ &= \frac{2}{S} \sum_{j=1}^{n} V_{ij} C_{,j} - \frac{2}{S} \sum_{j=1}^{n} V_{ij} h_{,j} N \\ &= \frac{2}{S} D_{,i} - \frac{N}{S} S_{,i} = \frac{2}{S} D_{,i} - \frac{2D}{S^2} S_{,i}. \end{aligned}$$

Thus, we have

$$\frac{2h}{S} \sum_{k=1}^{n} V_{ik} N_{,k} + \frac{2}{S} \sum_{j=1}^{n} V_{ij} X_{,j} = N_{,i}. \tag{3.20}$$

Multiplying (3.20) by S and using the fact that $N_{,i} = \sum_{j=1}^{n} W_{ij} X_{,j}$, we have

$$S\Big(\sum_{j=1}^{n} W_{ij} X_{,j}\Big) - 2h \sum_{k=1}^{n} V_{ik}\Big(\sum_{j=1}^{n} W_{kj} X_{,j}\Big) = 2\sum_{j=1}^{n} V_{ij} X_{,j},$$

$$SW_{ij} - 2h\sum_{k=1}^{n} V_{ik} W_{kj} = 2V_{ij}.$$

Therefore, we have

$$(SI - 2hV)W = 2V,$$

hence, we get (3.5), and we observe that X is regular if and only if $P = \det(SI - 2hV) \neq 0$. This completes the proof. $\square$

Corollary 1. *With the conditions of the Theorem 1, we have that the first, second and third fundamental forms are given by*

$$I = L - \frac{2h}{S}((VL)^T + VL) + \Big(\frac{2h}{S}\Big)^2 VLV^T, \tag{3.21}$$

$$II = -\frac{2}{S}(VL)^T + \frac{4h}{S^2} VLV^T, \tag{3.22}$$

$$III = \frac{4}{S^2} VLV^T, \tag{3.23}$$

where L is the matrix of the metric (L_{ij}) and T denotes the transpose.

Proof: To calculate the first fundamental form of X, observe that by (3.18), we have

$$\begin{aligned}
\langle X_{,i}, X_{,j}\rangle &= \langle C_{,i}, C_{,j}\rangle - h_{,j}\langle C_{,i}, N\rangle - h_{,i}\langle N, C_{,j}\rangle + h_{,i}h_{,j} \\
&\quad + \frac{2h}{S}\Big(\Big\langle C_{,i}, \sum_{l=1}^{n} V_{jl}\Big(\frac{2D}{S}h_{,l} - C_{,l}\Big)\Big\rangle \\
&\quad + \Big\langle C_{,j}, \sum_{k=1}^{n} V_{ik}\Big(\frac{2D}{S}h_{,k} - C_{,k}\Big)\Big\rangle\Big) \\
&\quad - \frac{2h}{S}\Big(h_{,i}\Big\langle N, \sum_{l=1}^{n} V_{jl}\Big(\frac{2D}{S}h_{,l} - C_{,l}\Big)\Big\rangle \\
&\quad + h_{,j}\Big\langle N, \sum_{k=1}^{n} V_{ik}\Big(\frac{2D}{S}h_{,k} - C_{,k}\Big)\Big\rangle\Big) \\
&\quad + \frac{4h^2}{S^2}\Big\langle \sum_{k=1}^{n} V_{ik}\Big(\frac{2D}{S}h_{,k} - C_{,k}\Big), \sum_{l=1}^{n} V_{jl}\Big(\frac{2D}{S}h_{,l} - C_{,l}\Big)\Big\rangle.
\end{aligned}$$

From (3.14) and (3.16), we get

$$\langle C_{,i}, C_{,j}\rangle = L_{ij} + h_{,i}h_{,j}, \qquad \langle D, C_{,i}\rangle = \frac{S}{2}h_{,i}, \qquad \langle C_{,i}, N\rangle = h_{,i},$$
$$\langle D, N\rangle = \frac{S}{2}, \qquad \langle D, D\rangle = \frac{S^2}{4}.$$

Therefore, using the above expressions we obtain

$$\begin{aligned}
\langle X_{,i}, X_{,j}\rangle &= L_{ij} + \frac{2h}{S}\sum_{k=1}^{n} V_{ik}\frac{2h_{,k}}{S}\Big(\frac{S}{2}h_{,i} - (L_{ik} + h_{,i}h_{,k})\frac{S}{2h_{,k}}\Big)\\
&\quad + \frac{2h}{S}\sum_{k=1}^{n} V_{jk}\frac{2h_{,k}}{S}\Big(\frac{S}{2}h_{,j} - (L_{jk} + h_{,j}h_{,k})\frac{S}{2h_{,k}}\Big)\\
&\quad + \frac{4h^2}{S^2}\sum_{k,l=1}^{n} V_{ik}V_{jl}L_{kl}\\
&= L_{ij} - \frac{2h}{S}\sum_{k=1}^{n} V_{ik}L_{ik} - \frac{2h}{S}\sum_{k=1}^{n} V_{jk}L_{jk} + \frac{4h^2}{S^2}\sum_{k,l=1}^{n} V_{ik}V_{jl}L_{kl}\\
&= L_{ij} - \frac{2h}{S}(V_{ji}L_{ii} + V_{ij}L_{jj}) + \Big(\frac{2h}{S}\Big)^2\sum_{k=1}^{n} V_{ik}V_{jk}L_{kk},
\end{aligned}$$

and this equation is equivalent to (3.21).

Also, the second fundamental form are given by

$$\begin{aligned}
-\langle X_{,i}, N_{,j}\rangle =& -\Big\langle C_{,i}, \frac{2}{S}\sum_{l=1}^{n} V_{jl}C_{,l}\Big\rangle + \Big\langle C_{,i}, \frac{2}{S}\sum_{l=1}^{n} V_{jl}\frac{2D}{S}h_{,l}\Big\rangle\\
& + \Big\langle h_{,i}N, \frac{2}{S}\sum_{l=1}^{n} V_{jl}C_{,l}\Big\rangle - \Big\langle h_{,i}N, \frac{2}{S}\sum_{l=1}^{n} V_{jl}\frac{2D}{S}h_{,l}\Big\rangle\\
& - \Big\langle \frac{2h}{S}\sum_{k=1}^{n} V_{ik}\Big(\frac{2D}{S}h_{,k} - C_{,k}\Big), \frac{2}{S}\sum_{l=1}^{n} V_{jl}\Big(C_{,l} - \frac{2D}{S}h_{,l}\Big)\Big\rangle\\
=& -\frac{2}{S}\sum_{l=1}^{n} V_{jl}(L_{il} + h_{,i}h_{,l}) + \frac{4}{S^2}\sum_{l=1}^{n} V_{jl}h_{,l}\frac{S}{2}h_{,i}\\
& + \frac{2h_{,i}}{S}\sum_{l=1}^{n} V_{jl}h_{,l} - \frac{4h_{,i}h_{,l}}{S^2}\sum_{l=1}^{n} V_{jl}\frac{S}{2} + \frac{4h}{S^2}\sum_{k,l=1}^{n} V_{ik}V_{jl}L_{kl}\\
=& -\frac{2}{S}V_{ji}L_{ii} + \frac{4h}{S^2}\sum_{k=1}^{n} V_{jk}V_{ik}L_{kk},
\end{aligned}$$

which is equivalent to (3.22).

Finally, the third fundamental form are given by

$$\begin{aligned}\langle N_{,i}, N_{,j}\rangle &= \left\langle \frac{2}{S}\sum_{k=1}^{n} V_{ik}\Big(C_{,k} - \frac{2Dh_{,k}}{S}\Big), \frac{2}{S}\sum_{l=1}^{n} V_{jl}\Big(C_{,l} - \frac{2Dh_{,l}}{S}\Big)\right\rangle \\ &= \frac{4}{S^2}\sum_{k,l=1}^{n} V_{ik}V_{jl}L_{kl} = \frac{4}{S^2}\sum_{k=1}^{n} V_{ik}V_{jk}L_{kk},\end{aligned}$$

and this equation is equivalent to (3.23). Thus, the proof is complete. □

Corollary 2. *Let $X\colon U \subset \mathbb{R}^n \to M^n \subset \mathbb{R}^{n+1}$ be a parametrization of the hypersurface M^n given by* (3.2). *Then the following conditions are equivalent*

- *parametrization X is parametrized by lines of curvature;*
- *$V_{ij} = 0$ for $i \neq j$;*
- *$N_{,i} = -k_i X_{,i}$;*

where k_i, $1 \leq i \leq n$, are the principal curvatures of X.

Proof: By equation (3.5), if V is diagonal, then the Weingarten matrix W also is diagonal. Therefore, V is diagonal if and only if X is parametrized by lines of curvature. By equation (3.20) we have

$$\begin{aligned}N_{,i} &= \frac{2h}{S}V_{ii}N_{,i} + \frac{2}{S}V_{ii}X_{,i}, \qquad 1 \leq i \leq n,\\ N_{,i} &= \frac{2V_{ii}}{S - 2hV_{ii}}X_{,i}.\end{aligned}$$

Since V is diagonal, V_{ii} are the eigenvalues of the matrix V. From (3.5), the principal curvatures are given by

$$k_i = \frac{2V_{ii}}{2hV_{ii} - S}, \qquad 1 \leq i \leq n, \tag{3.24}$$

which proves the last two equivalences. □

4. Weingarten hypersurfaces of the spherical type

In this section we will define and characterize the Weingarten hypersurfaces of the spherical type. We will give a characterization of these hypersurfaces using the trace of a matrix and also through harmonic functions.

Lemma 2. *Define P_i for $1 \leq i \leq n$ and $n \geq 2$, by*

$$P_i = (1 - hk_1)(1 - hk_2)\ldots(\widehat{1 - hk_i})\ldots(1 - hk_n), \tag{4.1}$$

where $h, k_i\colon U \to \mathbb{R}$ are functions defined in an open $U \subset \mathbb{R}^n$ and "$\widehat{(\ )}$" means that the factor is absent in the expression. Then

$$\begin{aligned}
P_i = 1 - h(S_1(W) - k_i) + h^2\Bigg(S_2(W) - \sum_{1\le j\le n} k_j k_i\Bigg) \\
- h^3\Bigg(S_3(W) - \sum_{1\le l<j\le n} k_l k_j k_i\Bigg) \\
+ \dots + (-1)^{n-1}h^{n-1}\Bigg(S_{n-1}(W) - \sum_{1\le j_1<j_2<\dots<j_{n-2}\le n} k_{j_1}k_{j_2}\dots k_{j_{n-2}}k_i\Bigg),
\end{aligned}$$

where

$$S_r(W) = \sum_{1\le i_1<\dots<i_r\le n} k_{i_1}\dots k_{i_r}, \qquad 1\le r\le n.$$

Proof: Clearly the statement is true for $n=2$. By induction, suppose that the equality is true for $n-1$ and we will prove it for n.

Denote by

$$\overline{S}_r(W) = \sum_{1\le i_1<\dots<i_r\le n-1} k_{i_1}\dots k_{i_r},$$

with $1\le r\le n-1$. Then, using the induction hypothesis we obtain

$$\begin{aligned}
P_i &= (1-hk_1)(1-hk_2)\dots(\widehat{1-hk_i})\dots(1-hk_{n-1})(1-hk_n) \\
&= (1-hk_n)\Bigg[1 - h(\overline{S}_1(W) - k_i) + h^2\Bigg(\overline{S}_2(W) - \sum_{1\le j\le n-1} k_j k_i\Bigg) \\
&\quad - h^3\Bigg(\overline{S}_3(W) - \sum_{1\le l<j\le n-1} k_l k_j k_i\Bigg) + \dots + (-1)^{n-2}h^{n-2} \\
&\quad \times\Bigg(\overline{S}_{n-2}(W) - \sum_{1\le j_1<j_2<\dots<j_{n-3}\le n-1} k_{j_1}k_{j_2}\dots k_{j_{n-3}}k_i\Bigg)\Bigg] \\
&= 1 - h(\overline{S}_1(W) - k_i) + h^2\Bigg(\overline{S}_2(W) - \sum_{1\le j\le n-1} k_j k_i\Bigg) \\
&\quad - h^3\Bigg(\overline{S}_3(W) - \sum_{1\le l<j\le n-1} k_l k_j k_i\Bigg) + \dots + (-1)^{n-2}h^{n-2} \\
&\quad \times\Bigg(\overline{S}_{n-2}(W) - \sum_{1\le j_1<j_2<\dots<j_{n-3}\le n-1} k_{j_1}k_{j_2}\dots k_{j_{n-3}}k_i\Bigg) \\
&\quad - hk_n + h^2k_n(\overline{S}_1(W) - k_i) - h^3k_n\Bigg(\overline{S}_2(W) - \sum_{1\le j\le n-1} k_j k_i\Bigg)
\end{aligned}$$

$$+ h^4 k_n\left(\overline{S}_3(W) - \sum_{1\le l<j\le n-1} k_l k_j k_i\right) + \cdots + (-1)^{n-1}h^{n-1}k_n$$
$$\times \left(\overline{S}_{n-2}(W) - \sum_{1\le j_1<j_2<\cdots<j_{n-3}\le n-1} k_{j_1}k_{j_2}\ldots k_{j_{n-3}}k_i\right).$$

Grouping terms with the same powers of h, follows the result. $\square$

Definition 8. An oriented hypersurface $M^n \subset \mathbb{R}^{n+1}$, $n \ge 2$, is called *Weingarten hypersurface of the spherical type*, if the rth mean curvature of M^n satisfy the relation

$$\sum_{r=1}^{n}(-1)^{r+1} r f^{r-1}\binom{n}{r}H_r = 0, \tag{4.2}$$

for some function $f \in C^\infty(M^n;\mathbb{R})$.

The following result characterizes the Weingarten hypersurfaces of the spherical type.

Theorem 2. *Let $M^n \subset \mathbb{R}^{n+1}$, $n \ge 2$, be a hypersurface as in Theorem 1. The surface M^n is a Weingarten hypersurface of the spherical type if and only if* $\mathrm{tr}(V) = 0$.

Proof: Let V_{ii} be the eigenvalues of the matrix V and k_i the principal curvatures of M^n, $1 \le i \le n$. By equation (3.24), we get

$$V_{ii} = \frac{Sk_i}{2hk_i - 2}.$$

Thus, say that $\mathrm{tr}(V) = 0$ is equivalent to

$$\frac{k_1}{1-hk_1} + \frac{k_2}{1-hk_2} + \cdots + \frac{k_n}{1-hk_n} = 0 \iff k_1P_1 + k_2P_2 + \cdots + k_nP_n = 0,$$

where P_i are defined by (4.1). By Lemma 2, the equality is true if and only if we have

$$k_1\Bigg[1 - h(S_1(W) - k_1) + h^2\left(S_2(W) - \sum_{1\le j\le n} k_j k_1\right)$$
$$+ \cdots + (-1)^{n-1}h^{n-1}\left(S_{n-1}(W) - \sum_{1\le j_1<j_2<\cdots<j_{n-2}\le n} k_{j_1}k_{j_2}\ldots k_{j_{n-2}}k_1\right)\Bigg]$$

$$+\cdots+k_n\Big[1-h(S_1(W)-k_n)+h^2\Big(S_2(W)-\sum_{1\le j\le n}k_jk_n\Big)$$
$$+\cdots+(-1)^{n-1}h^{n-1}\Big(S_{n-1}(W)-\sum_{1\le j_1<j_2<\cdots<j_{n-2}\le n}k_{j_1}k_{j_2}\ldots k_{j_{n-2}}k_n\Big)\Big]=0,$$

that is,

$$(k_1+\cdots+k_n)-h(k_1S_1(W)-k_1^2+k_2S_1(W)-k_2^2+\cdots+k_nS_1(W)-k_n^2)$$
$$+h^2\Big(k_1S_2(W)-k_1\sum_{1\le j\le n}k_jk_1+\cdots+k_nS_2(W)-k_n\sum_{1\le j\le n}k_jk_n\Big)$$
$$+\cdots+(-1)^{n-1}h^{n-1}\Big(k_1S_{n-1}(W)-k_1\sum_{1\le j_1<j_2<\cdots<j_{n-2}\le n}k_{j_1}\ldots k_{j_{n-2}}k_1$$
$$+\cdots+k_nS_{n-1}(W)-k_n\sum_{1\le j_1<j_2<\cdots<j_{n-2}\le n}k_{j_1}\ldots k_{j_{n-2}}k_n\Big)=0,$$

which is equivalent to

$$S_1(W)-2hS_2(W)+3h^2S_3(W)+\cdots+(-1)^{n-1}nh^{n-1}S_n(W)=0.$$

From Definition 2, we get the result. □

Remark 2. For $n=2$ and $K\neq 0$ the surface M^2 given by Theorem 2 is a surface of the spherical type. In fact, the equation (4.2) is reduced to $2H-2hK=0$, hence, the radius function of the sphere congruence is given by $h=H/K$. It is known that these surfaces are Laguerre minimal surfaces, i.e. the mean and Gauss curvature, satisfy $\Delta_{III}(H/K)=0$. We observe that, since h gives the radius of the sphere congruence then we have that $H\neq 0$.

Theorem 3. *Let $Y\colon U\subset\mathbb{R}^n\to\Pi\subset\mathbb{R}^{n+1}$, $n\ge 2$ be a parametrization of the hyperplane Π given by $Y(u)=(u,0)$, $u\in U$ and $h\colon U\to\mathbb{R}$ a differentiable function. Then $X\colon U\to\mathbb{R}^{n+1}$ given by (3.2) satisfies*

$$\Delta h=\mathrm{tr}(V).$$

In particular, $X(U)$ is a Weingarten hypersurface of the spherical type if and only if h is harmonic.

Proof: The metric L_{ij} of Y is given by $L_{ij}=\delta_{ij}$, hence, it follows that, $\widetilde{\Gamma}_{ij}^k=0$. Therefore, using (3.6) one has

$$V_{ij}=h_{,ij},\qquad 1\le i,j\le n, \tag{4.3}$$

thus, $\operatorname{tr}(V) = \Delta h$. From Theorem 2, $X(U)$ is a Weingarten hypersurface of the spherical type if and only if $\Delta h = 0$, i.e. h is harmonic. $\square$

Example 1. Consider the function $h\colon \mathbb{R}^n \to \mathbb{R}$, given by

$$h(u_1, \dots, u_n) = c + \sum_{i=1}^{n} b_i u_i + \sum_{i,j=1, i\neq j}^{n} a_{ij} u_i u_j,$$

where c, b_i and a_{ij}, $1 \le i, j \le n$, are real constans. Since h is a harmonic function in $\mathbb{R}^n$ and considering $Y(u) = (u, 0)$, we obtain from Theorem 3 and (3.2) that

$$\begin{aligned} X(u) = (u,0) - {} & \frac{2\big(c + \sum_{i=1}^{n} b_i u_i + \sum_{i,j=1,i\neq j}^{n} a_{ij} u_i u_j\big)}{\big(b_1 + \sum_{j=2}^{n} a_{1j} u_j\big)^2 + \cdots + \big(b_n + \sum_{j=1}^{n-1} a_{nj} u_j\big)^2 + 1} \\ & \times \left(b_1 + \sum_{j=2}^{n} a_{1j} u_j, \dots, b_n + \sum_{j=1}^{n-1} a_{nj} u_j, -1\right) \end{aligned}$$

is a Weingarten hypersurface of the spherical type.

For $n = 2$, we have the following generalization of Theorem 3.

Theorem 4. *Let $Y\colon U \subset \mathbb{R}^2 \to \Pi$ be a parametrization of the plane $\Pi = \{(u_1, u_2, u_3)\colon u_3 = 0\}$ given by $Y(u) = (g(u), 0)$ with $g\colon U \subset \mathbb{C} \to \mathbb{C}$ being a holomorphic function, $u = (u_1, u_2) \in U$ and $h\colon U \to \mathbb{R}$ a differentiable function. Then the immersion $X\colon U \to \mathbb{R}^3$ given by (3.2) satisfies*

$$\Delta h = \operatorname{tr}(V) \|g'\|^2. \tag{4.4}$$

In particular, $X(U)$ is a Weingarten surface of the spherical type if and only if h is harmonic.

Proof: From equations (2.3), (3.1) and (3.6) we get

$$\begin{aligned} V_{11} + V_{22} &= \frac{1}{\|g'\|^2}\Big(h_{,11} - \frac{\langle g'', g'\rangle}{\|g'\|^2} h_{,1} + \frac{\langle ig', g'\rangle}{\|g'\|^2} h_{,2}\Big) \\ &\quad + \frac{1}{\|g'\|^2}\Big(h_{,22} + \frac{\langle g'', g'\rangle}{\|g'\|^2} h_{,1} - \frac{\langle ig'', g'\rangle}{\|g'\|^2} h_{,2}\Big) \\ &= \frac{1}{\|g'\|^2}(h_{,11} + h_{,22}) = \frac{\Delta h}{\|g'\|^2}. \end{aligned}$$

This shows the equation (4.4). From this equation and Theorem 2, it follows that $\operatorname{tr}(V) = 0$ if and only if h is harmonic. $\square$

Remark 3. From Theorem 4 and (3.2), we obtain that the Weingarten surface of the spherical type is locally parametrized by

$$X(u) = (g, 0) - \frac{2h}{S}\Big(\frac{g' \cdot \nabla h}{|g'|^2}, -1\Big).$$

Moreover, from Remark 2, this representation provides examples of Laguerre minimal surfaces.

5. Weingarten hypersurfaces of the spherical type of rotation

The following theorem characterizes the hypersurfaces of rotation in $\mathbb{R}^{n+1}$ obtained by the parametrization (3.2).

Theorem 5. *Let $Y\colon U \to \Pi$ be a parametrization of the hyperplane Π given by $Y(u) = (u, 0)$, $u \in U$, $h\colon U \to \mathbb{R}$ a differentiable function and $X\colon U \to \mathbb{R}^{n+1}$ the immersion given by* (3.2) *with Gauss map N given by* (3.4). *Under these conditions, $X(U)$ is a hypersurface of rotation if and only if h is a radial function.*

Proof: Suppose that $X(U)$ is a hypersurface of rotation. Without loss of generality, we can suppose that the axis of rotation is the axis x_{n+1}. In this way, the sections orthogonal to the axis x_{n+1} determine in $X(U)$ $(n-1)$-dimensional spheres centered in axis x_{n+1}. Note also that on these spheres the angle between X and N is constant, that is, $\langle X, N\rangle = k$, where k is a constant.

On the other hand,

$$\begin{aligned}\langle X, N\rangle &= \langle Y, N\rangle + h\langle e_{n+1}, N\rangle - h\\ &= \Big\langle u, \frac{2}{|\nabla h|^2+1}\sum_{j=1}^{n} h_{,j} e_j\Big\rangle + h\Big(1 - \frac{2}{|\nabla h|^2+1}\Big) - h\\ &= \frac{2}{|\nabla h|^2+1}(\langle u, \nabla h\rangle - h),\end{aligned}$$

and therefore

$$k = \frac{2}{S}\big(\langle u, \nabla h\rangle - h\big).$$

Analogously, as $\langle X, X\rangle$ is constant on these spheres we get

$$\begin{aligned}c &= \langle X, X\rangle\\ &= |u|^2 - \frac{4h}{S}\langle u, \nabla h\rangle + \frac{4h^2}{S^2}\langle \nabla h - e_{n+1}, \nabla h - e_{n+1}\rangle,\end{aligned}$$

$$= |u|^2 - \frac{4h}{S}\big(\langle u, \nabla h\rangle - h\big) = |u|^2 - 2kh,$$

where c is a constant.

From the above expression we obtain that $h(u) = J(|u|^2)$, where $J(|u|^2) = (|u|^2 - c)/2k$, consequently, h is a radial function.

Conversely, suppose that $h\colon U \to \mathbb{R}$ is a radial function, that is, $h(u) = J(|u|^2)$, $u = (u_1, \dots, u_n) \in U$ for some differentiable function J. From (3.2) we have that

$$\begin{aligned} X &= \left(u - \frac{2J}{4(J')^2|u|^2}\sum_{j=1}^{n} 2u_j J' e_j, \frac{2J}{4(J')^2|u|^2}\right) \\ &= \left(u - \frac{2J}{4(J')^2|u|^2}2J'u, \frac{2J}{4(J')^2|u|^2}\right). \end{aligned}$$

Thus, if $(2J)/(4(J')^2|u|^2) = q$ is constant on $|u|^2$, we have

$$\left|\left(1 - \frac{2J}{4(J')^2|u|^2}2J'\right)u\right|^2 = |(1-2qJ')u|^2 = (1-2qJ')^2 t.$$

Therefore, the orthogonal sections to axis x_{n+1} determine in $X(U)$ $(n-1)$-dimensional spheres centered in axis x_{n+1}. $\square$

The following result classifies the Weingarten hypersurfaces of the spherical type of rotation.

Proposition 1. *Let $Y : U \subset \mathbb{R}^n \to \Pi$ be a parametrization of the hyperplane Π given by $Y(u) = (u, 0)$, $u \in U$, $h\colon U \to \mathbb{R}$ a differentiable function and $X : U \to \mathbb{R}^{n+1}$ an immersion given by* (3.2) *with Gauss map N given by* (3.4)*. Under these conditions, $X(U)$ is a Weingarten hypersurface of the spherical type of rotation if and only if $h(u)$ is given by*

$$h(u) = \begin{cases} C\ln(u_1^2 + u_2^2) + D, & \text{if } n = 2, \\ \dfrac{2C}{2-n}(u_1^2 + \cdots + u_n^2)^{(2-n)/2} + D, & \text{if } n \neq 2, \end{cases}$$

where C and D are constants, $C > 0$.

Proof: If $X(U)$ is a hypersurface of rotation, then from Theorem 5 $h(u)$ is a radial function, i.e. $h(u) = J(t)$, where $u = (u_1, \dots, u_n)$ and $t = |u|^2$. Therefore,

$$h_{,i} = 2J'(t)u_i, \qquad h_{,ii} = 4J''(t)u_i^2 + 2J'(t).$$

From Theorem 3, h is a harmonic function, thus, we obtain

$$4J''(t)t + 2J'(t)n = 0,$$

which can be rewritten as

$$\frac{J''(t)}{J'(t)} = -\frac{2n}{4t}, \qquad t > 0.$$

Integrating the above equation in the interval (t_o, t), $t_0 > 0$, we get

$$\ln J'(t) = -\frac{n}{2}\int_{t_0}^{t} \frac{1}{t}\,\mathrm{d}t + A = -\frac{n}{2}\ln t + A,$$

where A is a constant.

Hence,

$$J'(t) = Ct^{-n/2}, \qquad C = \mathrm{e}^A, \tag{5.1}$$

where $C > 0$ is a constant.

Now, if $n = 2$ integrating (5.1)

$$J(t) = C\ln t + D.$$

If $n \neq 2$, integrating (5.1) gives

$$J(t) = C\frac{t^{1-n/2}}{1-\frac{n}{2}} + D.$$

This completes the proof. □

Proposition 2. *Let $Y\colon U \subset \mathbb{R}^2 \to \Pi$ be a parametrization of the plane Π given by $Y(u) = (u, 0)$, $u \in U$, $h\colon U \to \mathbb{R}$ a differentiable function and $X\colon U \to \mathbb{R}^3$ an immersion given by (3.2) with Gauss map N given by (3.4). Under these conditions, if $X(U)$ is a Weingarten surface of the spherical type of rotation, then $X(U)$ can be locally parametrized by*

$$X(u) = (R(u_1)\cos u_2, R(u_1)\sin u_2, T(u_1)), \tag{5.2}$$

where

$$R(u_1) = \frac{(1-2u_1)4C^2\,\mathrm{e}^{u_1} - 4CD\,\mathrm{e}^{u_1} + \mathrm{e}^{3u_1}}{4C^2 + \mathrm{e}^{2u_1}}, \qquad T(u_1) = \frac{(4Cu_1 + 2D)\,\mathrm{e}^{2u_1}}{4C^2 + \mathrm{e}^{2u_1}},$$

C and D are constants and $C > 0$. Moreover, $X(U)$ has always a circle of singularities and at most two isolated singularities.

Proof: If we choose $Y(w) = (w, 0)$ and $h(w) = C\ln(|w|^2) + D$, $w = (w_1, w_2) \in \mathbb{C}$ from Proposition 1, then (3.2) is a parametrization of a Weingarten surface of the spherical type of rotation. Making the change of parameters

$$w = \mathrm{e}^u, \qquad u = u_1 + \mathrm{i}u_2 \in \mathbb{C},$$

we have that $Y = (\mathrm{e}^{u_1}\cos u_2, \mathrm{e}^{u_1}\sin u_2, 0)$ and $h(u) = 2Cu_1 + D$, because $|\,\mathrm{e}^{u}| = \mathrm{e}^{u_1}$. Consequently, $Y = Y_{,1}$, $L_{11} = \mathrm{e}^{2u_1}$, $h_{,2} = 0$. Substituting these expressions into (3.2), we get

$$X(u) = \Big(\frac{4(1-2u_1)C^2 - 4CD + \mathrm{e}^{2u_1}}{4C^2 + \mathrm{e}^{2u_1}}\,\mathrm{e}^{u}, \frac{4Cu_1 + 2D}{4C^2 + \mathrm{e}^{2u_1}}\,\mathrm{e}^{2u_1}\Big),$$

which is equivalent to (5.2).

On the other hand, from (3.6)

$$V_{11} = -\frac{2C}{\mathrm{e}^{2u_1}}, \qquad V_{22} = \frac{2C}{\mathrm{e}^{2u_1}}, \qquad V_{12} = V_{21} = 0.$$

Using these expressions in (3.7) we obtain

$$P = \frac{1}{\mathrm{e}^{4u_1}}(4(1+2u_1)C^2 + 4CD + \mathrm{e}^{2u_1})(4(1-2u_1)C^2 - 4CD + \mathrm{e}^{2u_1}).$$

Note that $X(U)$ is regular only at points where $P \neq 0$. Thus, $P = 0$ if and only if

$$4(1+2u_1)C^2 + 4CD + \mathrm{e}^{2u_1} = 0, \tag{5.3}$$

or

$$4(1-2u_1)C^2 - 4CD + \mathrm{e}^{2u_1} = 0. \tag{5.4}$$

From expression (5.3) it follows that $\mathrm{e}^{2u_1} = -8C^2u_1 - 4C^2 - 4CD$ and as the straight line $-8C^2u_1 - 4C^2 - 4CD$ has negative angular coefficient we have that it always intersects the exponential curve e^{2u_1} at a point, say $u_1 = u_1^0$. We affirm that the point u_1^0 always generates a circle of singularities on the surface $X(U)$. In fact, from (5.2), the profile curve is given by

$$\alpha(u_1) = \Big(\frac{(1-2u_1)4C^2\,\mathrm{e}^{u_1} - 4CD\,\mathrm{e}^{u_1} + \mathrm{e}^{3u_1}}{4C^2 + \mathrm{e}^{2u_1}}, 0, \frac{(4Cu_1 + 2D)\,\mathrm{e}^{2u_1}}{4C^2 + \mathrm{e}^{2u_1}}\Big)$$

and its tangent vector is given by

$$\begin{aligned}\alpha'(u_1) = \Big(&\frac{(\,\mathrm{e}^{2u_1} - 4C^2)(4(1+2u_1)C^2 + 4CD + \mathrm{e}^{2u_1})}{(4C^2 + \mathrm{e}^{2u_1})^2}, 0,\\ &\frac{4C\,\mathrm{e}^{2u_1}(4(1+2u_1)C^2 + 4CD + \mathrm{e}^{2u_1})}{(4C^2 + \mathrm{e}^{2u_1})^2}\Big).\end{aligned}$$

Therefore, in the point u_1^0 the tangent vector vanishes, this proves that the point u_1^0 generates a circle of singularities.

Similarly, from equation (5.4) one has $\mathrm{e}^{2u_1} = 8C^2u_1 - 4C^2 + 4CD$ and as the straight line $8C^2u_1 - 4C^2 + 4CD$ has positive angular coefficient, we conclude that this straight line and the exponential curve e^{2u_1} can have at most two intersections. These points of intersection are the points where the function $R(u_1) = 0$, and therefore these points generate isolated singularities on the surface. □

Remark 4. In order to obtain Weingarten surfaces of the spherical type of rotation with one isolated singularity and a circle of singularities, it follows from (5.4) that the constants C and D must satisfy

$$D = 2C + 2C \ln(2C). \tag{5.5}$$

Example 2. Considering $C = 1$ and $D = 1$ in (5.2) we obtain

$$X(u_1, u_2) = (R(u_1)\cos u_2, R(u_1)\sin u_2, T(u_1)),$$

and the profile curve is given by

$$\alpha(u_1) = (R(u_1), 0, T(u_1)),$$

where

$$R(u_1) = \frac{\mathrm{e}^{u_1}(\mathrm{e}^{2u_1} - 8u_1)}{4 + \mathrm{e}^{2u_1}}, \qquad T(u_1) = \frac{\mathrm{e}^{2u_1}(2 + 4u_1)}{4 + \mathrm{e}^{2u_1}}.$$

Here $R(u_1) = 0$ only in two points and the profile curve is not regular only in one point, therefore, the Weingarten surface of spherical type of rotation has two isolated singularities and a circle of singularities (see Figures 1 and 2).

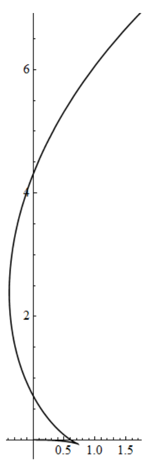

FIGURE 1

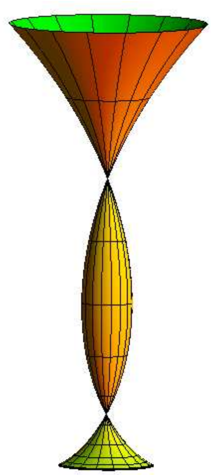

FIGURE 2

Example 3. Considering $C = 1$ and $D = 0$ in (5.2) we obtain

$$X(u_1, u_2) = (R(u_1)\cos u_2, R(u_1)\sin u_2, T(u_1)),$$

and the profile curve is given by

$$\alpha(u_1) = (R(u_1), 0, T(u_1)),$$

where

$$R(u_1) = \frac{e^{u_1}(4 + e^{2u_1} - 8u_1)}{4 + e^{2u_1}}, \qquad T(u_1) = \frac{4\, e^{2u_1} u_1}{4 + e^{2u_1}}.$$

Here $R(u_1) \neq 0$, $\forall\, u_1 \in \mathbb{R}$ and the profile curve is not regular only in one point, therefore, the Weingarten surface of spherical type of rotation has a circle of singularities (see Figures 3 and 4).

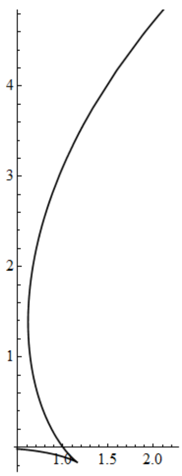

FIGURE 3

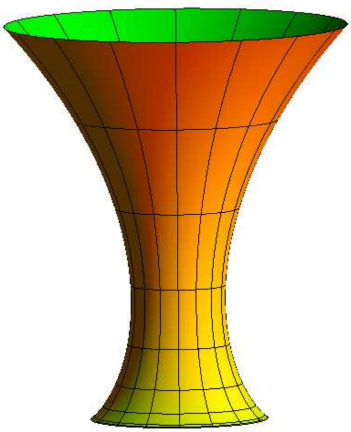

FIGURE 4

Example 4. From (5.5), considering $C = 1$ and $D = 2 - \ln 2$ in (5.2) we obtain

$$X(u_1, u_2) = (R(u_1) \cos u_2, R(u_1) \sin u_2, T(u_1))$$

and the profile curve is given by

$$\alpha(u_1) = (R(u_1), 0, T(u_1)),$$

where

$$R(u_1) = \frac{e^{u_1}(-4 + e^{2u_1} - 8u_1 + 8 \log 2)}{4 + e^{2u_1}}, \qquad T(u_1) = \frac{4\, e^{2u_1}(1 + u_1 - \log 2)}{4 + e^{2u_1}}.$$

Here $R(u_1) = 0$ only in one point and the profile curve is not regular only in one point, therefore, the Weingarten surface of spherical type of rotation has one isolated singularity and a circle of singularities (see Figures 5 and 6).

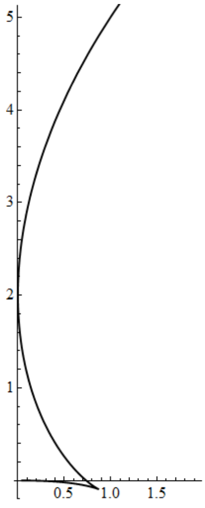

Figure 5

Figure 6

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