On congruence permutable G-sets

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Abstract. An algebraic structure is said to be congruence permutable if its arbitrary congruences α and β satisfy the equation $\alpha\circ\beta=\beta\circ\alpha$, where $\circ$ denotes the usual composition of binary relations. To an arbitrary G-set X satisfying $G\cap X=\emptyset$, we assign a semigroup $(G,X,0)$ on the base set $G\cup X\cup\{0\}$ containing a zero element $0\notin G\cup X$, and examine the connection between the congruence permutability of the G-set X and the semigroup $(G,X,0)$.

Keywords: G-set; congruence permutable algebras; semigroup

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1. Introduction and motivation

An algebraic structure A is said to be congruence permutable if for every congruences α and β on A, the equation $\alpha\circ\beta=\beta\circ\alpha$ is satisfied, where '$\circ$' denotes the usual composition of binary relations. Recall that for arbitrary binary relations α and β on a set X, $\alpha\circ\beta=\{(a,b)\in X\times X\colon (\exists x\in X)\ (a,x)\in\alpha,\ (x,b)\in\beta\}$. Every group is congruence permutable, but this cannot be said about the G-sets and the semigroups. In [2], finite congruence permutable transitive right G-sets play an important role in the description of a special type of finite congruence permutable semigroups. To a finite group G and a finite congruence permutable transitive right G-set $N^*=G/G_a$ (G_a is a subgroup of G and G/G_a is the right coset space of G modulo G_a), the authors assign a semigroup (in [2, Construction 1]), and prove (in [2, Theorem 2]) that a finite semigroup S is a congruence permutable semigroup which is a semilattice of a group G and a nil semigroup such that the identity element of G is a right identity element of S and $SN=\{0\}$ if and only if S is isomorphic to a semigroup defined in Construction 1 of [2].

It is easy to see that Construction 1 of [2] also gives a semigroup when the group G is arbitrary and an arbitrary right G-set is considered instead of the special right G-set N^*. This fact and the result of Theorem 2 of [2] inspire us

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to find connection between the congruence permutability of right G-sets and the semigroups assigned to them.

In our present paper, to an arbitrary group G and an arbitrary right G-set X (satisfying $G \cap X = \emptyset$), we shall assign a semigroup $(G, X, 0)$ (containing a zero $0 \notin G \cup X$), and examine the connection between the congruence permutability of the right G-set X and the semigroup $(G, X, 0)$. First we characterize the congruence permutable semigroup $(G; X; 0)$ by the help of the right G-set X. We prove that the semigroup $S = (G, X, 0)$ is congruence permutable if and only if the right G-set X is transitive and congruence permutable, see Theorem 1. We define the notion of the orbit subsemigroup of the semigroup $(G, X, 0)$ and characterize arbitrary congruence permutable right G-sets by the help of the semigroup $(G, X, 0)$ and the orbit subsemigroups of $(G, X, 0)$. We prove that a right G-set X is congruence permutable if and only if the semigroup $(G, X, 0)$ is segregated (which means that every congruence α on $(G, X, 0)$ satisfies the following condition: if A and B are different orbits of X such that $(a_0, b_0) \in \alpha$ for some $a_0 \in A$ and $b_0 \in B$ then $(a, b) \in \alpha$ for all $a, b \in A \cup B$) such that it has at most two orbit subsemigroups, and every orbit subsemigroup of $(G, X, 0)$ is congruence permutable, see Theorem 2.

2. Preliminaries

Let G be a group with the identity element e. By a G-set we shall mean a right G-set, that is, a nonempty set X together with a mapping

$$X \times G \mapsto X; \qquad (x, g) \mapsto x^g \in X,$$

satisfying the equations $x^e = x$ and $(x^g)^h = x^{(gh)}$ for every $x \in X$ and every $g, h \in G$.

A G-set X is said to be transitive if for every $x, y \in X$ there is a $g \in G$ such that $x^g = y$. A transitive G-subset of a G-set X is called an orbit of X. Clearly, any G-set is a disjoint union of its orbits.

Every G-set X can be considered as a unary algebra $(X; G)$ with the set G of operations where the operation $g \in G$ is defined by the role $g(x) = x^g$ for every $x \in X$.

By a congruence of a G-set X we mean an equivalence relation σ of X which satisfies the following condition: for every $a, b \in X$, the assumption $(a, b) \in \sigma$ implies $(a^g, b^g) \in \sigma$ for every $g \in G$ (that is, σ is a congruence of the unary algebra (X, G)).

The next lemma is about the congruence lattice of a transitive G-set X, see [6, Lemma 3] and [4, Lemma 4.20].

Lemma 1. *Let X be a transitive G-set. Then the congruence lattice* $\mathrm{Con}(X)$ *of the G-set X is isomorphic to the interval* $[\mathrm{Stab}_G(x), G]$ *of the subgroup lattice of the group G, where x is an arbitrary element of X and*

$$\mathrm{Stab}_G(x) = \{g \in G \colon x^g = x\}.$$

The corresponding isomorphisms are

$$\varphi \colon \alpha \mapsto H_\alpha = \{g \in G \colon (x^g, x) \in \alpha\}$$

and

$$\psi \colon H \mapsto \alpha_H = \{(x^g, x^h) \in A \times A \colon Hg = Hh\}$$

$(\alpha \in \mathrm{Con}(X),\ H \in [\mathrm{Stab}_G(x), G])$ *which are inverses of each other.*

By [7, Lemma 1], $\alpha \circ \beta = \beta \circ \alpha$ is satisfied for congruences α and β of a transitive G-set X if and only if $H_\alpha H_\beta = H_\beta H_\alpha$ is satisfied. Thus the following lemma is a characterization of the congruence permutable transitive G-sets.

Lemma 2. *A transitive G-set X is congruence permutable if and only if $HK = KH$ is satisfied for every subgroups H and K of G belonging to the interval* $[\mathrm{Stab}_G(x), G]$*, where x is an arbitrary element of X.*

Arbitrary congruence permutable G-sets are characterized in [8]. A G-set X is called segregated if every congruence α of the G-set X satisfies the following condition: if A and B are different orbits of X such that $(a_0, b_0) \in \alpha$ for some $a_0 \in A$ and $b_0 \in B$ then $(a, b) \in \alpha$ for all $a, b \in A \cup B$. By [8, Theorem 3.4] the following lemma is true.

Lemma 3. *A G-set X is congruence permutable if and only if X is a segregated G-set such that X has at most two orbits and every orbit of X is congruence permutable.*

In the next section, to an arbitrary group G and an arbitrary G-set X satisfying $G \cap X = \emptyset$, we shall assign a semigroup $(G, X, 0)$ containing a zero 0 $(0 \notin G \cup X)$, and examine the connection between the congruence permutability of the G-set X and the semigroup $(G, X, 0)$.

For semigroup theoretical terminologies used in our investigation, we refer to the paper [3] and the books [1], [5].

3. Results

It is clear that every G-set is isomorphic to a G-set X with $G \cap X = \emptyset$. In the next we suppose that the considered G-sets X satisfy this condition.

Construction. Let X be a right G-set (with condition $G \cap X = \emptyset$). Let 0 be symbol with $0 \notin G \cup X$. On the set $S = G \cup X \cup \{0\}$, define an operation '$*$' as follows. For arbitrary $g, h \in G$, let $g * h = gh$, where gh is the original product of g and h in G. For arbitrary $x \in X$ and arbitrary $g \in G$, let $x * g = x^g$. Let $0 * g = 0$ for every $g \in G$. If $a \in X \cup \{0\}$ then for arbitrary $s \in S$, let $s * a = 0$. It is easy to check that S is a semigroup in which 0 is the zero element, G is a subgroup of S, and $X \cup \{0\}$ is a zero subsemigroup of S (that is, $a * b = 0$ for all $a, b \in X \cup \{0\}$). The semigroup S will be denoted by $(G, X, 0)$.

The next example shows that the congruence permutability of a G-set X and the congruence permutability of the semigroup $(G, X, 0)$ are not equivalent conditions, in general.

Example. Let $X = \{a, b\}$ be a two-element set and G be an arbitrary group. Assume $a^g = a$ and $b^g = b$ for every $g \in G$. Then the orbits of the G-set X are $\{a\}$ and $\{b\}$. It is clear that X is a congruence permutable G-set. Let α and β be equivalence relations on the semigroup $S = (G, X, 0)$ whose classes are α: $\{a; 0\}, \{b\}, G$ and β: $\{b; 0\}, \{a\}, G$. It is easy to see that α and β are congruences on the semigroup $(G, X, 0)$. Since $(a, 0) \in \alpha$ and $(0, b) \in \beta$, then we have $(a, b) \in \alpha \circ \beta$. If the semigroup $(G, X, 0)$ was congruence permutable then we would have $(a, b) \in \beta \circ \alpha$ from which we would get $(a, t) \in \beta$ and $(t, b) \in \alpha$ for some $t \in (G, X, 0)$. Since $[a]_\beta = \{a\}$ and $[b]_\alpha = \{b\}$, we would get $a = b$ which is a contradiction. Consequently the semigroup $(G, X, 0)$ is not congruence permutable.

The next theorem characterizes the congruence permutable semigroup $(G, X, 0)$ by the help of the G-set X.

Theorem 1. *The semigroup $S = (G, X, 0)$ is congruence permutable if and only if the G-set X is transitive and congruence permutable.*

Proof: Assume that the semigroup $S = (G, X, 0)$ is congruence permutable. Let α, β be arbitrary congruences of the G-set X. Let α' be the equivalence relation on the semigroup $S = (G, X, 0)$ defined by $\alpha' = \alpha \cup \iota_S$, where ι_S denotes the identity relation on S. We show that α' is a congruence relation on S. Assume $(a, b) \in \alpha'$ for some $a, b \in S$. We can suppose that $a \neq b$. Then $a, b \in X$ and $(a, b) \in \alpha$. Let $s \in S$ be an arbitrary element. Since $s * a = 0 = s * b$, then $(s * a, s * b) \in \alpha'$, and so α' is a left congruence on the semigroup S. If $s \in G$, then $a * s = a^s$ and $b * s = b^s$ and so $(a * s, b * s) \in \alpha \subseteq \alpha'$. If $s \in X \cup \{0\}$, then $a * s = 0 = b * s$ and $(a * s, b * s) \in \alpha'$. Hence α' is a right congruence on S. Consequently α' is a congruence on S. Similarly, β' defined by $\beta' = \beta \cup \iota_S$ is a congruence on the semigroup $S = (G, X, 0)$. We show that $\alpha \circ \beta = \beta \circ \alpha$. Let $a, b \in X$ be arbitrary elements. Assume $(a, b) \in \alpha \circ \beta$. Then there is an element $x \in X$ such that $(a, x) \in \alpha$ and $(x, b) \in \beta$. As $\alpha \subseteq \alpha'$ and $\beta \subseteq \beta'$, we have

$(a, b) \in \alpha' \circ \beta'$. Since $S = (G, X, 0)$ is a congruence permutable semigroup, then $(a, b) \in \beta' \circ \alpha'$ and so there is an element $t \in S = (G, X, 0)$ such that $(a, t) \in \beta'$ and $(t, b) \in \alpha'$. As X is saturated by α' and β', we have $t \in X$ and so $(a, t) \in \beta$ and $(t, b) \in \alpha$. Hence $(a, b) \in \beta \circ \alpha$. Consequently $\alpha \circ \beta \subseteq \beta \circ \alpha$, and by symmetry $\alpha \circ \beta = \beta \circ \alpha$. Hence X is a congruence permutable G-set.

Assume that X has at least two orbits. Let A and B be different orbits of X. It is clear that $A \cup \{0\}$ and $B \cup \{0\}$ are ideals of the semigroup $(G, X, 0)$. By [3, Theorem 4], the ideals of a congruence permutable semigroup form a chain with respect to inclusion. Then $A \subseteq B$ or $B \subseteq A$ which contradicts $A \cap B = \emptyset$. Consequently X has one orbit. Thus X is a transitive congruence permutable G-set.

To prove the converse, assume that X is a transitive congruence permutable G-set. Let N denote the set $X \cup \{0\}$. First we show that for an arbitrary non-universal congruence α on the semigroup $S = (G, X, 0)$, we have $[g]_\alpha \subseteq G$ for every $g \in G$, and $[0]_\alpha = \{0\}$ or $[0]_\alpha = N$. Let α be a non-universal congruence on the semigroup $S = (G, X, 0)$. Assume $(a, g) \in \alpha$ for some $a \in N, g \in G$. Then $(e * a, g) \in \alpha$, where e is the identity element of G. As $e * a = 0$, we get $g \in [0]_\alpha$ from which it follows that $G \subseteq [0]_\alpha$. Let $a \in X$ be an arbitrary element. Then $X = a * G \subseteq [0]_\alpha$ and so $[0]_\alpha = S$. This contradicts the assumption that α is a non-universal congruence on S. Consequently $[a]_\alpha \subseteq N$ and $[g]_\alpha \subseteq G$ for every $a \in N$ and every $g \in G$. Consider the case when $[0]_\alpha \neq \{0\}$. Then there is an element $a \in X$ such that $a \in [0]_\alpha$ and so $X = a * G \subseteq [0]_\alpha$. Hence $[0]_\alpha = N$.

Let α and β be arbitrary congruences on the semigroup $S = (G, X, 0)$. We show that $\alpha \circ \beta = \beta \circ \alpha$. We can suppose that α and β are not the universal relations of S. Let $b, c \in S$ be arbitrary elements. Assume $(b, c) \in \alpha \circ \beta$. Then there is an element $x \in S$ such that $(b, x) \in \alpha$ and $(x, c) \in \beta$. We have two cases.

Case 1: $x \in G$. In this case $b, c \in G$. As G is congruence permutable, there is an element $y \in G$ with $(b, y) \in \beta$ and $(y, c) \in \alpha$. Hence $(b, c) \in \beta \circ \alpha$.

Case 2: $x \in N = X \cup \{0\}$. In this case $b, c \in N$. We have two subcases. If $[0]_\beta = N$ or $[0]_\alpha = N$, then $(b, c) \in \beta \cup \alpha \subseteq \beta \circ \alpha$. Consider the case $[0]_\beta = [0]_\alpha = \{0\}$. In this case X is saturated by both α and β. If $x = 0$, then $b = c = 0$ and so $(b, c) \in \beta \circ \alpha$. If $x \in X$, then $b, c \in X$. Let α^+ and β^+ denote the restriction of α and β to X. Then α^+ and β^+ are congruences on the G-set X. Moreover $(b, c) \in \alpha^+ \circ \beta^+$. Since X is a congruence permutable G-set, we get $(b, c) \in \beta^+ \circ \alpha^+$. Then there is an element $y \in X$ such that $(b, y) \in \beta^+$ and $(y, c) \in \alpha^+$ from which we get $(b, y) \in \beta$ and $(y, c) \in \alpha$, that is, $(b, c) \in \beta \circ \alpha$.

Thus we have $(b, c) \in \beta \circ \alpha$ in both cases. Hence $\alpha \circ \beta \subseteq \beta \circ \alpha$, and by symmetry $\alpha \circ \beta = \beta \circ \alpha$. Thus $S = (G, X, 0)$ is congruence permutable. $\square$

Let X be a G-set. We say that the semigroup $(G, X, 0)$ is segregated if the G-set X is segregated.

Lemma 4. *Let X be a G-set. Then the semigroup $(G, X, 0)$ is segregated if and only if every congruence α on $(G, X, 0)$ satisfies the following condition: if A and B are different orbits of X such that $(a_0, b_0) \in \alpha$ for some $a_0 \in A$ and $b_0 \in B$ then $(a, b) \in \alpha$ for all $a, b \in A \cup B$.*

Proof: It is clear that if α is a congruence on the semigroup $(G, X, 0)$, then the restriction of α to X is a congruence of the G-set X. Moreover, if α is a congruence of the G-set X, then $\alpha' = \alpha \cup \iota_S$ is a congruence on the semigroup $S = (G, X, 0)$, where ι_S denotes the identity relation on $S = (G, X, 0)$. Thus the assertion of the lemma is obvious. □

Let A be an orbit of a G-set X. The subsemigroup $(G, A, 0)$ is called an orbit subsemigroup of the semigroup $(G, X, 0)$. The next theorem characterizes arbitrary congruence permutable G-sets by the help of the semigroup $(G, X, 0)$ and the orbit subsemigroups of $(G, X, 0)$.

Theorem 2. *A G-set X is congruence permutable if and only if the semigroup $(G, X, 0)$ is segregated such that it has at most two orbit subsemigroups, and every orbit subsemigroup of $(G, X, 0)$ is congruence permutable.*

Proof: Let a G-set X be congruence permutable. By Lemma 3, X is a segregated G-set such that X has at most two orbits and every orbit of X is a congruence permutable transitive G-set. Then the semigroup $(G, X, 0)$ is segregated by definition, and it contains at most two orbit subsemigroups. By Theorem 1, every orbit subsemigroup of $(G, X, 0)$ is congruence permutable.

Conversely, assume that the semigroup $(G, X, 0)$ is segregated such that it has at most two orbit subsemigroups, and every orbit subsemigroup of $(G, X, 0)$ is congruence permutable. Then the G-set X is segregated by definition, and it has at most two orbits. Every orbit of X is a congruence permutable G-set by Theorem 1. Consequently X is a congruence permutable G-set by Lemma 3. □

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