Roughness in G-graphs

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Abstract. G-graphs are a type of graphs associated to groups, which were proposed by A. Bretto and A. Faisant (2005). In this paper, we first give some theorems regarding G-graphs. Then we introduce the notion of rough G-graphs and investigate some important properties of these graphs.

Keywords: coset; G-graph; rough set; group; normal subgroup; lower approximation; upper approximation

Classification: 05C25, 03E75, 03E99

1. Introduction

In [12] Z. Pawlak proposed rough set theory as an extension of set theory in 1982. Also, N. Kuroki and P. P. Wang in [11] introduced the notion of rough subgroups with respect to a normal subgroup of a group and investigated some properties of the lower and the upper approximations in a group.

The Cayley graphs are the popular representations of groups by graphs, first studied by A. Cayley in [8] and [9]. Another type of graphs associated to groups are G-graphs. A. Bretto and A. Faisant introduced these graphs to study the graph isomorphism problem [2]. For more information on the properties of G-graphs, we refer to [1]–[7].

In [13], the notions of rough edge Cayley graphs, pseudo-Cayley graphs, rough vertex pseudo-Cayley graphs and rough pseudo-Cayley graphs have been introduced and their properties have been investigated.

In this paper, we first give some theorems regarding G-graphs. We then introduce the notion of rough G-graphs and investigate their important properties.

2. Preliminaries

In the following, we first briefly review some definitions and terminologies related to groups, rough sets, and graphs. For rough set and graph-theoretic concepts not defined here, we refer to [11] and [14], respectively. In this paper, all groups and graphs are finite.

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2.1 Group definitions. Let G be a group and $g \in G$. Denote by $o(G)$ and $o(g)$ the order of G and g, respectively. Let S be a nonempty subset of a group G such that every $g \in G$ can be written as form $g = s_{i_1} \dots s_{i_k}$, where $s_{i_1}, \dots, s_{i_k} \in S$. Then we say that G is generated by S and write $G = \langle S \rangle$. Throughout this paper, let $D_{2n} = \langle r, s \colon o(r) = n,\ o(s) = 2,\ srs = r^{-1} \rangle$ be the dihedral group of order $2n$, $n \geq 2$.

Let H be a subgroup of a group G. Then G can be partitioned in the disjoint union of all the right cosets of H. A right transversal for H in G is a set $T_H^G = \{t_\alpha\}_{\alpha \in I} \subseteq G$ such that for each right coset Hg, there is precisely one $\alpha \in I$ such that $Ht_\alpha = Hg$. If $H = \langle t \rangle$ then we use T_t^G instead of $T_{\langle t \rangle}^G$.

2.2 The lower and upper approximations in a group. Let G be a group, N be a normal subgroup of G and A be a nonempty subset of G. Then the sets $N_-(A) := \{x \in G \colon Nx \subseteq A\}$ and $N^\wedge(A) := \{x \in G \colon Nx \cap A \neq \emptyset\}$ are called the lower and upper approximations of A with respect to N, respectively, and $\big(N_-(A), N^\wedge(A)\big)$ is called the rough set of A in G.

Proposition 2.1 ([10], [11])**.** *Let H and N be two normal subgroups of a group G. Let A and B be two nonempty subsets of G. Then:*

(i) $N_-(A) \subseteq A \subseteq N^\wedge(A)$;
(ii) $N_-(A \cup B) \supseteq N_-(A) \cup N_-(B)$;
(iii) $N^\wedge(A \cup B) = N^\wedge(A) \cup N^\wedge(B)$;
(iv) $N_-(A \cap B) = N_-(A) \cap N_-(B)$;
(v) $N^\wedge(A \cap B) \subseteq N^\wedge(A) \cap N^\wedge(B)$;
(vi) $A \subseteq B \implies N_-(A) \subseteq N_-(B)$;
(vii) $A \subseteq B \implies N^\wedge(A) \subseteq N^\wedge(B)$;
(viii) $N \subseteq H \implies N_-(A) \supseteq H_-(A)$;
(ix) $N \subseteq H \implies N^\wedge(A) \subseteq H^\wedge(A)$.

The following proposition is a modified version of Propositions 2.4 and 2.5 in [11].

Proposition 2.2 ([10])**.** *Let H and N be two normal subgroups of a group G. Let A be a nonempty subset of G. Then:*

(i) $(H \cap N)_-(A) \supseteq H_-(A) \cup N_-(A) \supseteq H_-(A) \cap N_-(A)$;
(ii) $(H \cap N)^\wedge(A) \subseteq H^\wedge(A) \cap N^\wedge(A) \subseteq H^\wedge(A) \cup N^\wedge(A)$.

2.3 Graph definitions. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a graph. Denote by $\|\Gamma\|$ the number of edges in Γ. A graph Γ is called an empty graph if its edge set is empty. A graph Γ' is a subgraph of Γ (written $\Gamma' \subseteq \Gamma$) if $V_{\Gamma'} \subseteq V_\Gamma$ and $E_{\Gamma'} \subseteq E_\Gamma$. The union $\Gamma_1 \cup \Gamma_2$ of two graphs Γ_1 and Γ_2 is a graph with vertex set $V_{\Gamma_1} \cup V_{\Gamma_2}$ and edge set $E_{\Gamma_1} \cup E_{\Gamma_2}$. The intersection $\Gamma_1 \cap \Gamma_2$ of Γ_1 and Γ_2 is defined analogously.

Let $r \geq 2$ be an integer. A graph Γ is called r-partite if V_Γ can be partitioned into r subsets, or parts, in such a way that no edge has both ends in the same part.

Let S be a nonempty subset of a group G. For any $s \in S$, we have $G = \bigcup_{x \in T_s} \langle s \rangle x$, where $T_s := T_s^G$ is a right transversal for $\langle s \rangle$ in G. Consider the cycles

$$(s)x := (x, sx, s^2x, \ldots, s^{o(s)-1}x)$$

of the permutation $g_s \colon x \longmapsto sx$ on G. The set $\langle s \rangle x$ is called the support of the cycle $(s)x$. A G-graph $\varphi(G,S)$ is a graph with vertex set $V := \bigcup_{s \in S} V_s$, where $V_s = \{(s)x \colon x \in T_s\}$ are such that for each $(s)x, (t)y \in V$, if $|\langle s \rangle x \cap \langle t \rangle y| := l \geq 1$ then the vertices $(s)x$ and $(t)y$ are linked by l edges. We consider $\varphi(G, \emptyset)$ as null graph $(\emptyset, \emptyset)$. One can see that for any $s \in S$ and $x \in T_s$, the vertex $(s)x$ has $o(s)$ loops. We denote by $\widetilde{\varphi}(G,S)$ the graph constructed by deleting all loops from $\varphi(G,S)$. The graph $\widetilde{\varphi}(G,S)$ is also called G-graph.

Hereafter, we just deal with G-graph $\widetilde{\varphi}(G,S)$.

Proposition 2.3 ([2], [3])**.** *Let $\Gamma := \widetilde{\varphi}(G,S)$ be a G-graph. Then:*

(i) *Graph Γ is connected if and only if $G = \langle S \rangle$.*
(ii) *Graph Γ is a simple graph if and only if for all distinct $s, t \in S$, $\langle s \rangle \cap \langle t \rangle = 1_G$.*

3. More facts on G-graphs

In this section, we give some basic facts regarding G-graphs.

Proposition 3.1. *Let $\Gamma := \widetilde{\varphi}(G,S)$ be a G-graph. Then Γ is an r-partite graph, where $r \leq |S|$.*

Proof: If there exist $s, t \in S$ such that $\langle s \rangle = \langle t \rangle$, then for every $x \in G$, $\langle s \rangle x = \langle t \rangle x$ and so $(s)x = (t)x$. Moreover, $T_s = T_t$ and then $V_s = V_t$. Set $r := |\{V_s \colon s \in S\}|$. Obviously $r \leq |S|$. One can easily see that Γ is r-partite. □

Example 3.2. Let $G = \mathbb{Z}_6$ and $S = \{1,2,3,4,5\}$. Obviously, $V_1 = V_5$ and $V_2 = V_4$. So, the G-graph $\widetilde{\varphi}(G,S)$ is 3-partite (see Figure 1).

A modified version of Proposition 2 in [2] for G-graph $\widetilde{\varphi}(G,S)$ is as follows:

Proposition 3.3. *Let $\Gamma := \widetilde{\varphi}(G,S)$ be a G-graph. Then, for every $v \in V_s$, $\deg(v) = o(s)(r-1)$ and $\|\Gamma\| = (r(r-1)/2)o(G)$, where $r = |\{V_s \colon s \in S\}|$.*

Theorem 3.4. *Let $\widetilde{\varphi}(G,S_1)$ and $\widetilde{\varphi}(G,S_2)$ be two G-graphs such that $S_1 \subseteq S_2$. Then $\widetilde{\varphi}(G,S_1) \subseteq \widetilde{\varphi}(G,S_2)$.*

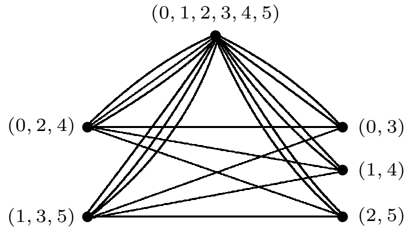

Figure 1. $\widetilde{\varphi}(\mathbb{Z}_6, \{1,2,3,4,5\})$.

Proof: Let $S_1 \subseteq S_2$. Then

$$V_{\widetilde{\varphi}(G,S_1)} = \bigcup_{s\in S_1} V_s \subseteq \left(\bigcup_{s\in S_1} V_s\right) \cup \left(\bigcup_{s\in S_2-S_1} V_s\right) = V_{\widetilde{\varphi}(G,S_2)}.$$

Thus $V_{\widetilde{\varphi}(G,S_1)} \subseteq V_{\widetilde{\varphi}(G,S_2)}$.

Now, suppose that there exist $p \geq 1$ edges between two distinct vertices $(s)x$ and $(t)y$ in $\widetilde{\varphi}(G,S_1)$. Since $(s)x \in V_s$ and $(t)y \in V_t$, there are p edges between every vertex in V_s and every vertex in V_t. This implies that $|\langle s\rangle \cap \langle t\rangle| = p$. Hence there exist p edges between $(s)x$ and $(t)y$ in $\widetilde{\varphi}(G,S_2)$. So $\widetilde{\varphi}(G,S_1) \subseteq \widetilde{\varphi}(G,S_2)$. □

Remark 3.5. The converse of Theorem 3.4 is not necessarily true. For example, $\widetilde{\varphi}(\mathbb{Z}_6,\{1,2,3\}) \subseteq \widetilde{\varphi}(\mathbb{Z}_6,\{3,4,5\})$ but $\{1,2,3\} \nsubseteq \{3,4,5\}$.

Corollary 3.6. *Let $\Gamma_1 := \widetilde{\varphi}(G,S_1)$ and $\Gamma_2 := \widetilde{\varphi}(G,S_2)$ be two G-graphs. Then:*

(i) $\Gamma_1 \cup \Gamma_2 \subseteq \widetilde{\varphi}(G, S_1 \cup S_2)$;
(ii) $\Gamma_1 \cap \Gamma_2 \supseteq \widetilde{\varphi}(G, S_1 \cap S_2)$.

Proof: (i) Since $S_1, S_2 \subseteq S_1 \cup S_2$, by Theorem 3.4, we have $\Gamma_1, \Gamma_2 \subseteq \widetilde{\varphi}(G, S_1 \cup S_2)$. Therefore $\Gamma_1 \cup \Gamma_2 \subseteq \widetilde{\varphi}(G, S_1 \cup S_2)$.

(ii) Similarly, since $S_1 \cap S_2 \subseteq S_1, S_2$, it follows that $\widetilde{\varphi}(G, S_1 \cap S_2) \subseteq \Gamma_1, \Gamma_2$. So $\widetilde{\varphi}(G, S_1 \cap S_2) \subseteq \Gamma_1 \cap \Gamma_2$. □

Remark 3.7. The converse of Corollary 3.6 is not necessarily true. For example:

(i) Let $\Gamma_1 := \widetilde{\varphi}(\mathbb{Z}_6,\{1\})$ and $\Gamma_2 := \widetilde{\varphi}(\mathbb{Z}_6,\{4\})$. Then $\Gamma_1 \cup \Gamma_2 \nsupseteq \widetilde{\varphi}(\mathbb{Z}_6,\{1,4\})$.
(ii) Let $\Gamma_1 := \widetilde{\varphi}(\mathbb{Z}_6,\{1,4\})$ and $\Gamma_2 := \widetilde{\varphi}(\mathbb{Z}_6,\{2,4,5\})$. Then $\Gamma_1 \cap \Gamma_2 \nsubseteq \widetilde{\varphi}(\mathbb{Z}_6,\{4\})$.

Theorem 3.8. *Let $\widetilde{\varphi}(G_1,S)$ and $\widetilde{\varphi}(G_2,S)$ be two G-graphs. Then $\widetilde{\varphi}(G_1,S) \subseteq \widetilde{\varphi}(G_2,S)$ if and only if $G_1 \subseteq G_2$.*

PROOF: Let $G_1 \subseteq G_2$ and $(s)x \in V_{\widetilde{\varphi}(G_1,S)}$. Then $s \in S$ and $x \in G_1$. Suppose that $(s)x \notin V_{\widetilde{\varphi}(G_2,S)}$. Since $x \in G_2 = \bigcup_{y \in T_s^{G_2}} \langle s \rangle y$, there exists $y \in T_s^{G_2}$ such that $x \in \langle s \rangle y$. On the other hand, $x \in \langle s \rangle x$. Hence $\langle s \rangle x = \langle s \rangle y$. So $(s)x = (s)y$, a contradiction. Therefore $(s)x \in V_{\widetilde{\varphi}(G_2,S)}$ and then $V_{\widetilde{\varphi}(G_1,S)} \subseteq V_{\widetilde{\varphi}(G_2,S)}$. By similar argument as in the proof of Theorem 3.4, one can show that $E_{\widetilde{\varphi}(G_1,S)} \subseteq E_{\widetilde{\varphi}(G_2,S)}$. Thus $\widetilde{\varphi}(G_1,S) \subseteq \widetilde{\varphi}(G_2,S)$.

Conversely, let $\widetilde{\varphi}(G_1,S) \subseteq \widetilde{\varphi}(G_2,S)$ and $g \in G_1$. Let s be an arbitrary fixed element of S. Since $g \in G_1 = \bigcup_{x \in T_s^{G_1}} \langle s \rangle x$, there exists $x \in T_s^{G_1}$ such that $g \in \langle s \rangle x$. Note that $(s)x \in V_{\widetilde{\varphi}(G_1,S)}$. Hence $(s)x \in V_{\widetilde{\varphi}(G_2,S)}$. Therefore $\langle s \rangle x \subseteq G_2$ and then $g \in G_2$. Thus $G_1 \subseteq G_2$. $\square$

Theorem 3.9. *Let $\Gamma_1 := \widetilde{\varphi}(H_1,S_1)$ and $\Gamma_2 := \widetilde{\varphi}(H_2,S_2)$ be two G-graphs, where H_1 and H_2 are two subgroups of a group G. Then $\Gamma_1 \cap \Gamma_2 \supseteq \widetilde{\varphi}(H_1 \cap H_2, S_1 \cap S_2)$.*

PROOF: Since $H_1 \cap H_2 \subseteq H_1, H_2$, by Theorem 3.8, it follows that

$$\widetilde{\varphi}(H_1 \cap H_2, S_1 \cap S_2) \subseteq \widetilde{\varphi}(H_1, S_1 \cap S_2), \widetilde{\varphi}(H_2, S_1 \cap S_2).$$

Now, since $S_1 \cap S_2 \subseteq S_1, S_2$, by Theorem 3.4, we have $\widetilde{\varphi}(H_1, S_1 \cap S_2) \subseteq \Gamma_1$ and $\widetilde{\varphi}(H_2, S_1 \cap S_2) \subseteq \Gamma_2$, respectively. Therefore $\widetilde{\varphi}(H_1 \cap H_2, S_1 \cap S_2) \subseteq \Gamma_1, \Gamma_2$ and then $\widetilde{\varphi}(H_1 \cap H_2, S_1 \cap S_2) \subseteq \Gamma_1 \cap \Gamma_2$. $\square$

Remark 3.10. The converse of Theorem 3.9 is not necessarily true. For example, if $\Gamma_1 := \widetilde{\varphi}(\mathbb{Z}_6, \{1,4\})$ and $\Gamma_2 := \widetilde{\varphi}(\mathbb{Z}_6, \{2,4,5\})$ then $\Gamma_1 \cap \Gamma_2 \nsubseteq \widetilde{\varphi}(\mathbb{Z}_6, \{4\})$.

4. Rough G-graphs

In this section, the notions of the lower and upper approximations of a G-graph with respect to a normal subgroup are introduced and their properties are investigated.

Definition 4.1. Let G be a group, N be a normal subgroup of G and $\Gamma := \widetilde{\varphi}(G,S)$ be a G-graph. Then the graphs $\underline{\Gamma} := \widetilde{\varphi}\big(G, N_{-}(S)\big)$ and $\overline{\Gamma} := \widetilde{\varphi}\big(G, N^{\wedge}(S)\big)$ are called the lower and upper approximations of Γ with respect to N, respectively and $(\underline{\Gamma}, \overline{\Gamma})$ is called the rough G-graph of Γ with respect to N.

Example 4.2. Let $G = \mathbb{Z}_8$, $S = \{1,2,3,5,7\}$, $N = \{0,2,4,6\}$ and $\Gamma := \widetilde{\varphi}(G,S)$. Note that $N_{-}(S) = \{1,3,5,7\}$ and $N^{\wedge}(S) = \{0,1,2,3,4,5,6,7\}$. Then $\underline{\Gamma} = \widetilde{\varphi}(\mathbb{Z}_8, \{1,3,5,7\})$ and $\overline{\Gamma} = \widetilde{\varphi}(\mathbb{Z}_8, \{0,1,2,3,4,5,6,7\})$ (see Figure 2).

Theorem 4.3. *Let N be a normal subgroup of a group G and $\Gamma := \widetilde{\varphi}(G,S)$ be a G-graph. Then $\underline{\Gamma} \subseteq \Gamma \subseteq \overline{\Gamma}$.*

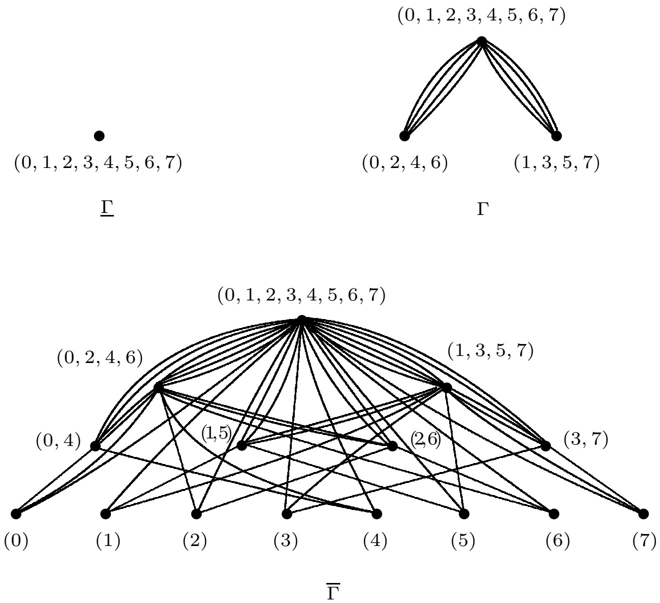

FIGURE 2. Rough G-graph $\widetilde{\varphi}(\mathbb{Z}_8, \{1, 2, 3, 5, 7\})$ with respect to $N = \{0, 2, 4, 6\}$.

PROOF: By Proposition 2.1 (i), we have $N_-(S) \subseteq S \subseteq N^\wedge(S)$. Now, Theorem 3.4 implies that $\underline{\Gamma} \subseteq \Gamma \subseteq \overline{\Gamma}$. $\square$

Theorem 4.4. *Let N be a normal subgroup of a group G. Let $\widetilde{\varphi}(G, S_1)$ and $\widetilde{\varphi}(G, S_2)$ be two G-graphs. Then:*

(i) $\widetilde{\varphi}\big(G, N_-\big(S_1 \cup S_2\big)\big) \supseteq \widetilde{\varphi}\big(G, N_-(S_1)\big) \cup \widetilde{\varphi}\big(G, N_-(S_2)\big)$;
(ii) $\widetilde{\varphi}\big(G, N^\wedge(S_1 \cup S_2)\big) \supseteq \widetilde{\varphi}\big(G, N^\wedge(S_1)\big) \cup \widetilde{\varphi}\big(G, N^\wedge(S_2)\big)$;
(iii) $\widetilde{\varphi}\big(G, N_-(S_1 \cap S_2)\big) \subseteq \widetilde{\varphi}\big(G, N_-(S_1)\big) \cap \widetilde{\varphi}\big(G, N_-(S_2)\big)$;
(iv) $\widetilde{\varphi}\big(G, N^\wedge(S_1 \cap S_2)\big) \subseteq \widetilde{\varphi}\big(G, N^\wedge(S_1)\big) \cap \widetilde{\varphi}\big(G, N^\wedge(S_2)\big)$.

PROOF: (i) By Proposition 2.1 (ii), $N_-(S_1 \cup S_2) \supseteq N_-(S_1) \cup N_-(S_2)$. On the other hand, $N_-(S_1) \cup N_-(S_2) \supseteq N_-(S_1), N_-(S_2)$. So $N_-(S_1 \cup S_2) \supseteq N_-(S_1), N_-(S_2)$. Now, by Theorem 3.4, it follows that $\widetilde{\varphi}\big(G, N_-(S_1 \cup S_2)\big) \supseteq \widetilde{\varphi}\big(G, N_-(S_1)\big)$, $\widetilde{\varphi}\big(G, N_-(S_2)\big)$. Therefore $\widetilde{\varphi}\big(G, N_-(S_1 \cup S_2)\big) \supseteq \widetilde{\varphi}\big(G, N_-(S_1)\big) \cup \widetilde{\varphi}\big(G, N_-(S_2)\big)$.

(ii) By Proposition 2.1 (iii) , $N^\wedge(S_1 \cup S_2) = N^\wedge(S_1) \cup N^\wedge(S_2)$. Now, Corollary 3.6 (i) implies that $\widetilde{\varphi}\big(G, N^\wedge(S_1 \cup S_2)\big) \supseteq \widetilde{\varphi}\big(G, N^\wedge(S_1)\big) \cup \widetilde{\varphi}\big(G, N^\wedge(S_2)\big)$.

(iii) By Proposition 2.1 (iv), $N_-(S_1 \cap S_2) = N_-(S_1) \cap N_-(S_2)$. Now, Corollary 3.6 (ii) yields $\widetilde{\varphi}\big(G, N_-(S_1 \cap S_2)\big) \subseteq \widetilde{\varphi}\big(G, N_-(S_1)\big) \cap \widetilde{\varphi}\big(G, N_-(S_2)\big)$.

(iv) By Proposition 2.1 (v), $N^\wedge(S_1 \cap S_2) \subseteq N^\wedge(S_1) \cap N^\wedge(S_2)$. On the other hand, $N^\wedge(S_1) \cap N^\wedge(S_2) \subseteq N^\wedge(S_1)$, $N^\wedge(S_2)$. Then $N^\wedge(S_1 \cap S_2) \subseteq N^\wedge(S_1)$,

$N^\wedge(S_2)$. Now, by using Theorem 3.4, we have $\widetilde{\varphi}\big(G, N^\wedge(S_1 \cap S_2)\big) \subseteq \widetilde{\varphi}\big(G, N^\wedge(S_1)\big)$, $\widetilde{\varphi}\big(G, N^\wedge(S_2)\big)$. Therefore $\widetilde{\varphi}\big(G, N^\wedge(S_1 \cap S_2)\big) \subseteq \widetilde{\varphi}\big(G, N^\wedge(S_1)\big) \cap \widetilde{\varphi}\big(G, N^\wedge(S_2)\big)$. $\square$

Remark 4.5. The converse of Theorem 4.4 is not necessarily true. For example:

(i) Let $G = D_6$, $S_1 = \{s, r^2s\}$, $S_2 = \{s, rs\}$, $N = \{1, r, r^2\}$, $\Gamma_1 := \widetilde{\varphi}(G, S_1)$ and $\Gamma_2 := \widetilde{\varphi}(G, S_2)$. Note that $N_-(S_1) = N_-(S_2) = \emptyset$ and $N_-(S_1 \cup S_2) = \{s, rs, r^2s\}$. Then $\widetilde{\varphi}\big(G, N_-(S_1 \cup S_2)\big) \nsubseteq \widetilde{\varphi}\big(G, N_-(S_1)\big) \cup \widetilde{\varphi}\big(G, N_-(S_2)\big)$.

(ii) Let $G = D_8$, $S_1 = \{r, s\}$, $S_2 = \{r^2, s\}$, $N = \{1, r^2\}$, $\Gamma_1 := \widetilde{\varphi}(G, S_1)$ and $\Gamma_2 := \widetilde{\varphi}(G, S_2)$. Note that $N^\wedge(S_1) = \{r, r^3, s, r^2s\}$, $N^\wedge(S_2) = \{1, r^2, s, r^2s\}$ and $N^\wedge(S_1 \cup S_2) = \{1, r, r^2, r^3, s, r^2s\}$. Then $\widetilde{\varphi}\big(G, N^\wedge(S_1 \cup S_2)\big) \nsubseteq \widetilde{\varphi}\big(G, N^\wedge(S_1)\big) \cup \widetilde{\varphi}\big(G, N^\wedge(S_2)\big)$.

(iii) Let $G = \mathbb{Z}_6$, $S_1 = \{1, 4\}$, $S_2 = \{2, 4, 5\}$, $N = \{0\}$, $\Gamma_1 := \widetilde{\varphi}(G, S_1)$ and $\Gamma_2 := \widetilde{\varphi}(G, S_2)$. Note that $N_-(S_1) = \{1, 4\}$, $N_-(S_2) = \{2, 4, 5\}$ and $N_-(S_1 \cap S_2) = \{4\}$. Then $\widetilde{\varphi}\big(G, N_-(S_1 \cap S_2)\big) \nsupseteq \widetilde{\varphi}\big(G, N_-(S_1)\big) \cap \widetilde{\varphi}\big(G, N_-(S_2)\big)$.

(iv) Let $G = D_6$, $S_1 = \{r, s\}$, $S_2 = \{r, rs\}$, $N = \{1, r, r^2\}$, $\Gamma_1 := \widetilde{\varphi}(G, S_1)$ and $\Gamma_2 := \widetilde{\varphi}(G, S_2)$. Note that $N^\wedge(S_1) = N^\wedge(S_2) = D_6$ and $N^\wedge(S_1 \cap S_2) = \{1, r, r^2\}$. Then $\widetilde{\varphi}\big(G, N^\wedge(S_1 \cap S_2)\big) \nsupseteq \widetilde{\varphi}\big(G, N^\wedge(S_1)\big) \cap \widetilde{\varphi}\big(G, N^\wedge(S_2)\big)$.

Theorem 4.6. *Let N and H be two normal subgroups of a group G such that $N \subseteq H$. Let $\Gamma := \widetilde{\varphi}(G, S)$ be a G-graph. Then:*

(i) $\widetilde{\varphi}\big(G, N_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big)$;

(ii) $\widetilde{\varphi}\big(G, N^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\big)$.

Proof: (i) By Proposition 2.1 (viii), $N_-(S) \supseteq H_-(S)$. So, Theorem 3.4 yields $\widetilde{\varphi}\big(G, N_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big)$.

(ii) By Proposition 2.1 (ix) and Theorem 3.4, the proof is similar to (i). $\square$

Theorem 4.7. *Let N and H be two normal subgroups of a group G. Let $\Gamma := \widetilde{\varphi}(G, S)$ be a G-graph. Then:*

(i) $\widetilde{\varphi}\big(G, (H \cap N)_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big) \cup \widetilde{\varphi}\big(G, N_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big) \cap \widetilde{\varphi}\big(G, N_-(S)\big)$;

(ii) $\widetilde{\varphi}\big(G, (H \cap N)^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\big) \cap \widetilde{\varphi}\big(G, N^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\big) \cup \widetilde{\varphi}\big(G, N^\wedge(S)\big)$.

Proof: (i) By Proposition 2.2 (i), $(H \cap N)_-(S) \supseteq H_-(S) \cup N_-(S)$. Now, Theorem 3.4 implies that $\widetilde{\varphi}\big(G, (H \cap N)_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S) \cup N_-(S)\big)$. On the other hand, by Corollary 3.6 (i), we have $\widetilde{\varphi}\big(G, H_-(S) \cup N_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big) \cup \widetilde{\varphi}\big(G, N_-(S)\big)$. Obviously $\widetilde{\varphi}\big(G, H_-(S)\big) \cup \widetilde{\varphi}\big(G, N_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big) \cap \widetilde{\varphi}\big(G, N_-(S)\big)$. Therefore $\widetilde{\varphi}\big(G, (H \cap N)_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big) \cup \widetilde{\varphi}\big(G, N_-(S)\big) \supseteq \widetilde{\varphi}\big(G, H_-(S)\big) \cap \widetilde{\varphi}\big(G, N_-(S)\big)$.

(ii) By Proposition 2.2 (ii) , $(H\cap N)^\wedge(S) \subseteq H^\wedge(S)\cap N^\wedge(S)$. Now, Theorem 3.4 implies that $\widetilde{\varphi}\big(G, H\cap N)^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\cap N^\wedge(S)\big)$. On the other hand, by Corollary 3.6 (ii), we have $\widetilde{\varphi}\big(G, H^\wedge(S)\cap N^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\big)\cap\widetilde{\varphi}\big(G, N^\wedge(S)\big)$. Obviously $\widetilde{\varphi}\big(G, H^\wedge(S)\big)\cap\widetilde{\varphi}\big(G, N^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\big)\cup\widetilde{\varphi}\big(G, N^\wedge(S)\big)$. Therefore $\widetilde{\varphi}\big(G, (H\cap N)^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\big)\cap\widetilde{\varphi}\big(G, N^\wedge(S)\big) \subseteq \widetilde{\varphi}\big(G, H^\wedge(S)\big)\cup \widetilde{\varphi}\big(G, N^\wedge(S)\big)$. □

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