On the nontrivial solvability of systems of homogeneous linear equations over $\mathbb{Z}$ in ZFC

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Abstract. Motivated by the paper by H. Herrlich, E. Tachtsis (2017) we investigate in ZFC the following compactness question: for which uncountable cardinals κ, an arbitrary nonempty system S of homogeneous $\mathbb{Z}$-linear equations is nontrivially solvable in $\mathbb{Z}$ provided that each of its subsystems of cardinality less than κ is nontrivially solvable in $\mathbb{Z}$?

Keywords: homogeneous $\mathbb{Z}$-linear equation; κ-free group; $\mathcal{L}_{\omega_1\omega}$-compact cardinal

Classification: 08A45, 13C10, 20K30, 03E35, 03E55

1. Introduction and preliminaries

Throughout the paper, group means always an abelian group, i.e. a $\mathbb{Z}$-module. Following [7], we say that a system S of homogeneous $\mathbb{Z}$-linear equations with a set $X = \{x_i : i \in I\}$ of variables is *nontrivially solvable* in a group H if there exists a mapping $f : X \to H \setminus \{0\}$ such that, whenever $\sum_{j\in J} a_j x_j = 0$ is an equation from S (where J is a finite subset of I and $a_j \in \mathbb{Z}$ for each $j \in J$), then $\sum_{j\in J} a_j f(x_j) = 0$ holds in H.

This notion of nontriviality is a little bit unusual. If we assume instead that the mapping f goes to H and it is not constantly zero on all $x \in X$ that appear in the system S, we say that the system S is *weakly nontrivially solvable* in H. More natural as it might be, this weaker notion has got one significant disadvantage: unlike with nontrivial solvability, if a system S is weakly nontrivially solvable and T is a nonempty subsystem of S, then T need not be weakly nontrivially solvable. Notice also that an empty system S is (weakly) nontrivially solvable by definition.

Motivated by the work [7], our aim is to characterize the class $\mathcal{S}$ (or $\mathcal{WS}$) of all infinite cardinals κ such that any system S of homogeneous $\mathbb{Z}$-linear equations is nontrivially (or weakly nontrivially, respectively) solvable in $\mathbb{Z}$ provided that each subsystem $T \subseteq S$ of cardinality less than κ is nontrivially (weakly nontrivially, respectively) solvable in $\mathbb{Z}$. In [7, Section 2.2], the authors present several well-known examples of countable S which show in Zermelo–Fraenkel set theory (ZF)

DOI 10.14712/1213-7243.2020.017

that $\aleph_0 \notin \mathcal{S} \cup \mathcal{WS}$. They also discuss various interesting related questions in ZF: among other things, they provide a model of ZF without choice where $\aleph_1 \notin \mathcal{S}$ while they note that the result is not known in Zermelo–Fraenkel set theory with axiom of choice (ZFC).

In this short note, we use κ-free groups with trivial dual to show that ZFC actually proves $\aleph_\alpha \notin \mathcal{S}$ for each $\alpha < \omega_1 \cdot \omega$. Moreover, it is consistent with ZFC that $\mathcal{S} = \mathcal{WS} = \emptyset$ (see the discussion below Corollary 2.5 for both results). On the other hand, we are able to prove that $\kappa \in \mathcal{WS} \cap \mathcal{S}$ whenever there exists a regular $\mathcal{L}_{\omega_1\omega}$-compact cardinal less than or equal to κ, see Corollary 2.2 and Theorem 3.2.

For an unexplained terminology, we recommend, for instance, the very well-written extensive book [4].

2. The case of $\mathcal{S}$

Recall that, given an infinite cardinal κ, a filter $\mathcal{F}$ on a set I is called *κ-complete* if $\mathcal{F}$ is closed under intersections of systems of cardinality less than κ. In particular, every filter is trivially $\aleph_0$-complete.

Given an uncountable cardinal ν, we say that a cardinal κ is *$\mathcal{L}_{\nu\omega}$-compact* if every κ-complete filter on any set I can be extended to a ν-complete ultrafilter. Observe that a cardinal μ is $\mathcal{L}_{\nu\omega}$-compact whenever there exists an $\mathcal{L}_{\nu\omega}$-compact cardinal λ such that $\lambda \leq \mu$. This is obviously a large cardinal notion since the existence of an $\mathcal{L}_{\nu\omega}$-compact cardinal implies the existence of a measurable cardinal.

Alternatively, one can define the notion of $\mathcal{L}_{\nu\omega}$-compact cardinal by means of infinitary $\mathcal{L}_{\nu\omega}$ logic. We will not follow this approach, however the fact that there exists such a connection becomes rather apparent in the following proposition where the language L can be allowed to be of the infinitary type $\mathcal{L}_{\nu\omega}$. Although the proof of Proposition 2.1 is rather standard, see for instance the if part of [8, Proposition 4.1], we present it here for the reader's convenience.

Proposition 2.1. *Let λ be a regular $\mathcal{L}_{\nu\omega}$-compact cardinal, L a first-order language and $\mathcal{Z}$ an L-structure with the domain Z such that $|Z| < \nu$. Then a system S consisting of first-order L-formulas in variables from a set X is realized in $\mathcal{Z}$ provided that each of its subsystems T of cardinality less than λ is realized in $\mathcal{Z}$.*

Proof: First, let E denote the set Z^X of all mappings from X to Z. By the assumption for each $T \in [S]^{<\lambda}$ there exists $e \in E$ such that $\mathcal{Z} \models \varphi[e]$ for each $\varphi \in T$. Let $\mathcal{F}$ be the filter on E generated by the sets $E_T = \{e \in E:$

$\mathcal{Z} \models \varphi[e]$ for all $\varphi \in T\}$. Since λ is regular, we see that $\mathcal{F}$ is a λ-complete filter. Let $\mathcal{G}$ denote an extension of $\mathcal{F}$ to a ν-complete ultrafilter.

For each $(x, z) \in X \times Z$, put $E_{x,z} = \{e \in E\colon e(x) = z\}$ and define $f \in Z^X$ by the assignment $f(x) = z \Leftrightarrow E_{x,z} \in \mathcal{G}$. This is possible since the ultrafilter $\mathcal{G}$ picks for each fixed $x \in X$ exactly one element from the disjoint partition $E = \bigcup_{z\in Z} E_{x,z}$; recall that $|Z| < \nu$.

Now let $\varphi \in S$ be arbitrary and $x_1, \dots, x_n$ be variables freely occurring in φ. Then $\emptyset \neq E_{\{\varphi\}} \cap \bigcap_{i=1}^n E_{x_i, f(x_i)} \in \mathcal{G}$, and so $f \in E_{\{\varphi\}}$. We conclude that S is realized in $\mathcal{Z}$ using the evaluation f. □

Corollary 2.2. *Let κ be a cardinal and $\lambda \leq \kappa$ a regular $\mathcal{L}_{\omega_1\omega}$-compact cardinal. Then every system S of homogeneous $\mathbb{Z}$-linear equations in variables from a set X is nontrivially solvable in $\mathbb{Z}$ whenever each of its subsystems of cardinality less than κ is nontrivially solvable in $\mathbb{Z}$. In other words $\kappa \in \mathcal{S}$.*

Proof: In the system S replace each equation ψ in variables $x_1, \dots, x_n \in X$ by the formula $\psi \,\&\, \bigwedge_{i=1}^n x_i \neq 0$ and use Proposition 2.1. □

Before we turn our attention to the negative part, we need one preparatory lemma which holds in the general context of R-modules over an infinite commutative noetherian domain. Recall that an R-module M is *noetherian* provided that it does not contain an infinite strictly increasing chain of submodules. A commutative ring R is noetherian if R is noetherian as a module over itself.

For a module $M \in$ Mod-R and an ordinal number σ, an increasing chain $\mathcal{M} = (M_\alpha\colon \alpha \leq \sigma)$ of submodules of M is called a *filtration of* M if $M_0 = 0$, $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ whenever $\beta \leq \sigma$ is a limit ordinal, and $M_\sigma = M$.

Lemma 2.3. *Let R be an infinite commutative noetherian domain, M a free R-module of rank $\mu \geq \aleph_0$, and $\mathcal{M} = (M_\alpha\colon \alpha \leq \sigma)$ be a filtration of M where for all $\alpha < \sigma$, $M_{\alpha+1} = M_\alpha + \langle a_\alpha \rangle$ with $a_\alpha \in M \setminus M_\alpha$. For each $\alpha < \sigma$, let $z_\alpha \in R$ be arbitrary.*

Then there is a homomorphism $\psi\colon M \to R$ such that $\psi(a_\alpha) \neq z_\alpha$ for all $\alpha < \sigma$.

Proof: First, assume that $\mu = \aleph_0$. Let $\{g_n\colon n < \omega\}$ be a set of free generators of M. For each $\alpha < \sigma$, we express a_α as $\sum_{n\in I_\alpha} b_{n\alpha} g_n$, where I_α is a finite subset of ω and $b_{n\alpha} \in R \setminus \{0\}$ for every $n \in I_\alpha$.

Using the fact that a free R-module of finite rank is noetherian, we infer that for each $n < \omega$ the set $A_n = \{\alpha < \sigma\colon I_\alpha \subseteq \{0, 1, \dots, n\}\}$ is finite. Note that $\sigma = \bigcup_{n<\omega} A_n$. On the free generators of M, we recursively construct a homomorphism $\psi\colon M \to R$ as follows:

Let $\psi(g_0)$ be arbitrary such that for each $\alpha \in A_0$, $b_{0\alpha}\psi(g_0) \neq z_\alpha$. There is always an applicable choice by the hypothesis on R. Assume that $n > 0$, $\psi(g_{n-1})$ is defined, and $\psi(a_\alpha) \neq z_\alpha$ for each $\alpha \in A_{n-1}$.

We define $\psi(g_n)$ arbitrarily in such a way that for each $\alpha \in A_n \setminus A_{n-1}$ we have

$$b_{n\alpha}\psi(g_n) \neq z_\alpha - \sum_{k \in I_\alpha \setminus \{n\}} b_{k\alpha}\psi(g_k).$$

This is possible, since $A_n \setminus A_{n-1}$ is finite, $b_{n\alpha} \neq 0$ for each α from this set, and R is an infinite domain. It immediately follows that $\psi(a_\alpha) \neq z_\alpha$ for each $\alpha \in A_n$.

Now, let μ be an uncountable cardinal. Again, let $\{g_\beta \colon \beta < \mu\}$ be a set of free generators of M, and put $G_B = \langle g_\beta \colon \beta \in B\rangle$ for all $B \subseteq \mu$.

We use ideas from [6, Section 7.1]. First, we set $A_\alpha = \langle a_\alpha \rangle \leq M$. We say that a subset S of the ordinal σ is '*closed*' if every $\alpha \in S$ satisfies

$$M_\alpha \cap A_\alpha \subseteq \sum_{\beta \in S, \beta < \alpha} A_\beta.$$

Notice that any ordinal $\alpha \leq \sigma$ is a 'closed' subset of σ. For a 'closed' subset S, we define $M(S) = \sum_{\alpha \in S} A_\alpha$. The results from [6, Section 7.1] give us the following:

(1) For a system $(S_i \colon i \in I)$ of 'closed' subsets, $\bigcap_{i \in I} S_i$ and $\bigcup_{i \in I} S_i$ is 'closed' as well.
(2) For S, S' 'closed' subsets of σ, we have $S \subseteq S' \iff M(S) \subseteq M(S')$.
(3) Let S be a 'closed' subset of σ and X be a countable subset of M. Then there is a 'closed' subset S' such that $M(S) \cup X \subseteq M(S')$ and $|S' \setminus S| < \aleph_1$.

Using the properties listed above, we are going to construct a filtration $\mathcal{N} = (M(S_\alpha) \colon \alpha \leq \mu)$ of M such that for each $\alpha < \mu$: a) S_α is 'closed'; b) $S_{\alpha+1} \setminus S_\alpha$ is countable; and c) there exists $B_\alpha \subseteq \mu$ such that $G_{B_\alpha} = M(S_\alpha)$ and $\alpha \subseteq B_\alpha$.

We proceed by the transfinite recursion, starting with $S_0 = B_0 = \emptyset$. Let S_α and B_α be defined and $\alpha < \mu$. Then $|S_\alpha| + |B_\alpha| < \mu$ (using b) and c)). Let $B^0 \supseteq B_\alpha \cup \{\alpha\}$ be any subset of μ with $|B^0 \setminus B_\alpha| = \aleph_0$. By (3), we find $S^0 \supseteq S_\alpha$ such that $M(S^0) \supseteq G_{B^0}$ and $|S^0 \setminus S_\alpha| < \aleph_1$. Assuming B^n, S^n are defined for $n < \omega$, we can find $B^{n+1} \supseteq B^n$ with $|B^{n+1} \setminus B^n| < \aleph_1$ such that $G_{B^{n+1}} \supseteq M(S^n)$, and $S^{n+1} \supseteq S^n$ with $|S^{n+1} \setminus S^n| < \aleph_1$ such that $M(S^{n+1}) \supseteq G_{B^{n+1}}$. Put $S_{\alpha+1} = \bigcup_{n<\omega} S^n$ and $B_{\alpha+1} = \bigcup_{n<\omega} B^n$. This completes the isolated step. In limit steps, we simply take unions. Since $M(S_\mu) = M$, we have $S_\mu = \sigma$ by (2).

Now, for each $\alpha < \mu$ we have the countable sets $C_\alpha = B_{\alpha+1} \setminus B_\alpha$ and $T_\alpha = S_{\alpha+1} \setminus S_\alpha$, and the canonical projection $\pi_\alpha \colon M(S_{\alpha+1}) \to G_{C_\alpha}$. Let τ be the ordinal type of $(T_\alpha, <)$, and fix an order-preserving bijection $i \colon \tau \to T_\alpha$.

Since $S_\alpha \cup (S_{\alpha+1} \cap \beta)$ is 'closed' for any $\beta \leq \sigma$ by (1), the part (2) yields that the chain $(N_\beta \colon \beta \leq \tau)$ of modules defined as $N_\beta = M\big(S_\alpha \cup (S_{\alpha+1} \cap i(\beta))\big)$ for $\beta < \tau$, and $N_\tau = M(S_{\alpha+1})$ is strictly increasing. Notice that $N_0 = M(S_\alpha)$.

If we put $\overline{N}_\beta = \pi_\alpha[N_\beta]$ for all $\beta \leq \tau$, it follows that the strictly increasing chain $(\overline{N}_\beta \colon \beta \leq \tau)$ is a filtration of the free module G_{C_α} of countable rank. Moreover, for each $\beta < \tau$, we have $\overline{N}_{\beta+1} = \overline{N}_\beta + \langle \pi_\alpha(a_{i(\beta)}) \rangle$.

Finally, we recursively define the homomorphism $\psi \colon M \to R$. Let $\alpha < \mu$ and assume that $\psi \restriction G_{B_\alpha}$ is constructed with the property $\psi(a_\gamma) \neq z_\gamma$ for all $\gamma \in S_\alpha$. By the already proven part for $\mu = \aleph_0$, we can define $\psi \restriction G_{C_\alpha}$ in such a way that $\psi(\pi_\alpha(a_\gamma)) \neq z_\gamma - \psi(a_\gamma - \pi_\alpha(a_\gamma))$ for all $\gamma \in T_\alpha$; observe that the right-hand side of the inequality is already defined since $a_\gamma - \pi_\alpha(a_\gamma) \in G_{B_\alpha}$. We immediately get $\psi(a_\gamma) \neq z_\gamma$ for all $\gamma \in S_{\alpha+1}$. $\square$

Remark. Inspecting the proof more closely, we see that, instead of avoiding just one element z_α, we could have actually avoided a finite set $Z_\alpha \subset R$.

For the negative part, we start with an uncountable cardinal κ and a κ-free group G with the trivial dual property, i.e. with the property $G^* := \mathrm{Hom}(G, \mathbb{Z}) = 0$; here, κ-free means that any less than κ-generated subgroup of G is free. We will discuss the existence of such groups, as well as the question whether G can be taken with $|G| = \kappa$, later on. Firstly, we show how the existence of such G implies that $\kappa \notin \mathcal{S}$.

Let us denote by λ the cardinality of G and express G as a quotient F/K where F is a free group of rank λ. Notice that $\lambda \geq \kappa$. Let $\pi \colon F \to F/K$ denote the canonical projection and let $\{e_\alpha \colon \alpha < \lambda\}$ be a set of free generators of the group F. For each $A \subseteq \lambda$, let F_A denote the subgroup of F generated by $\{e_\alpha \colon \alpha \in A\}$. We can without loss of generality assume that

$$\mathrm{Im}(\pi \restriction F_\beta) \subsetneq \mathrm{Im}(\pi \restriction F_{\beta+1}) \qquad \text{for each ordinal } \beta < \lambda. \tag{$*$}$$

The group K is also free of rank λ. If it had a smaller rank, G would have possessed a free direct summand—a contradiction with $G^* = 0$. Let $\{k_\beta \colon \beta < \lambda\}$ denote a set of (free) generators of the group K. Consider the uncountable set

$$S = \left\{ \sum_{\alpha \in J_\beta} a_{\alpha\beta} x_\alpha = 0 \colon \beta < \lambda, J_\beta \in [\lambda]^{<\omega}, (\forall \alpha \in J_\beta)\,(a_{\alpha\beta} \in \mathbb{Z}) \sum_{\alpha \in J_\beta} a_{\alpha\beta} e_\alpha = k_\beta \right\}$$

of homogeneous $\mathbb{Z}$-linear equations with the set $\{x_\alpha \colon \alpha < \lambda\}$ of variables. We will show that this is the desired counterexample.

First of all, S does not have even a weakly nontrivial solution in $\mathbb{Z}$. Indeed, any such solution would define a nonzero homomorphism ψ from F to $\mathbb{Z}$ which is zero on K. Hence ψ would provide for a nonzero homomorphism from G to $\mathbb{Z}$, a contradiction.

On the other hand, we can show

Proposition 2.4. *Any system $T \subseteq S$ of cardinality less than κ is nontrivially solvable in $\mathbb{Z}$.*

Proof: Let $A \in [\lambda]^{<\kappa}$ be an infinite set such that whenever x_α appears in an equation from T then $\alpha \in A$. Put $M = \mathrm{Im}(\pi \restriction F_A)$.

Since G is κ-free, M is a free group (of infinite rank). Let σ denote the ordinal type of A and fix an order-preserving bijection $i\colon \sigma \to A$. For each $\alpha \leq \sigma$, set $M_\alpha = \langle \pi(e_{i(\beta)})\colon \beta < \alpha \rangle$. Then $(M_\alpha\colon \alpha \leq \sigma)$ is a filtration of M such that $M_{\alpha+1} = M_\alpha + \langle \pi(e_{i(\alpha)}) \rangle$ where $\pi(e_{i(\alpha)}) \notin M_\alpha$ for all $\alpha < \sigma$ (using $(*)$).

Applying Lemma 2.3 with $R = \mathbb{Z}$ and $z_\gamma = 0$ for all $\gamma < \sigma$, we obtain a homomorphism $\psi\colon M \to \mathbb{Z}$ such that $\psi(\pi(e_\alpha)) \neq 0$ for all $\alpha \in A$. The assignment $x_\alpha \mapsto \psi(\pi(e_\alpha))$, $\alpha \in A$, is the desired nontrivial solution of the system T in $\mathbb{Z}$. $\square$

Corollary 2.5. *Let κ be an uncountable cardinal. If there exists a κ-free group G with $G^* = 0$, then $\kappa \notin \mathcal{S} \cup \mathcal{WS}$.*

The problem of existence of κ-free groups with trivial dual turns out to be rather delicate. Under the assumption V = L (even a much weaker one), there are κ-free groups with trivial dual for any uncountable cardinal κ. Moreover, if κ is regular and not weakly compact, then the groups can be constructed of cardinality κ, see [3]. If κ is singular or weakly compact, then κ-free implies κ^+-free. For more information on the topic, we refer to [4, Chapter VII]. Anyway, we have $\mathcal{S} = \mathcal{WS} = \emptyset$ under $V = L$ by Corollary 2.5.

In [5], R. Göbel and S. Shelah show in ZFC that $\aleph_n$-free groups with cardinality $\beth_n$ and trivial dual exist for all $0 < n < \omega$. This is further generalized in [9] [1], where S. Shelah proves in ZFC the existence of κ-free groups with trivial dual for any uncountable $\kappa < \aleph_{\omega_1 \cdot \omega}$. On the other hand, he also shows (modulo the existence of a supercompact cardinal) that it is relatively consistent with ZFC that there is no $\aleph_{\omega_1 \cdot \omega}$-free group with trivial dual.

By Corollary 2.5, we thus know in ZFC that $\kappa \notin \mathcal{S}$ for $\kappa < \aleph_{\omega_1 \cdot \omega}$. However, we do not know what happens for larger cardinals κ since the existence of a κ-free group with trivial dual is just a sufficient condition for $\kappa \notin \mathcal{S}$. We have only the upper bound given by Corollary 2.2. It might still be possible that $\mathcal{S} = \mathcal{WS}$ where Theorem 3.2 contains a decent description of the latter class.

[1] Very heavy in content.

3. The case of $\mathcal{WS}$

For the weaker notion of nontrivial solvability, we have the following general result. Recall that $\operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$ denotes the class of all groups A such that $\operatorname{Hom}(A,\mathbb{Z}) = 0$.

Proposition 3.1. *Let κ be an uncountable cardinal. The following conditions are equivalent:*

(1) *There exists a regular cardinal $\lambda \leq \kappa$ which is $\mathcal{L}_{\omega_1\omega}$-compact.*
(2) *There is a regular cardinal $\lambda \leq \kappa$ such that each group $A \in \operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$ is the sum of its subgroups of cardinality less than λ which are contained in* $\operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$.
(3) *For any nonempty system S of homogeneous $\mathbb{Z}$-linear equations such that S has no weakly nontrivial solution in $\mathbb{Z}$, and any $C \in [S]^{<\kappa}$, there exists $T \in [S]^{<\kappa}$ such that $C \subseteq T$ and T has no weakly nontrivial solution in $\mathbb{Z}$.*

Proof: The equivalence of (1) and (2) follows directly from [1, Corollary 5.4]. Let us show that (2) is equivalent to (3). To this end, we are going to use the following two-way translation.

Given any system $S = \{k_j = 0 \colon j \in J\}$ of homogeneous $\mathbb{Z}$-linear equations with the set X of variables, we can build a group $A = F/K$ where F is freely generated by the elements of the set X and K is generated by the set $\{k_j \colon j \in J\}$. Then $\operatorname{Hom}(A,\mathbb{Z}) = 0$ if and only if S has no weakly nontrivial solution in $\mathbb{Z}$. On the other hand, for a given group A and its presentation F/K where F is freely generated by a set X, the same equivalence holds for the system $S = \{k_j = 0 \colon j \in J\}$ of homogeneous $\mathbb{Z}$-linear equations where $\{k_j \colon j \in J\}$ is a fixed set of generators of K expressed as $\mathbb{Z}$-linear combinations of elements from the set X.

Proving (2) $\Longrightarrow$ (3), we start with a system S and a set $C \in [J]^{<\kappa}$. Consider the group A constructed for S as in the previous paragraph, and let Y_0 denote the set of all the elements from X appearing in equations $k_j = 0$, $j \in C$.

Let $\mu \geq \lambda$ be a regular uncountable cardinal such that $|C| < \mu \leq \kappa$. Since $\operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$ is closed under direct sums and quotients, and μ is regular, there exists, by (2), $G_0 \in \operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$ such that G_0 is a subgroup of A, $|G_0| < \mu$ and $Y_0 + K := \{y + K \colon y \in Y_0\} \subseteq G_0$. Now, take any $Y_1 \in [X]^{<\mu}$, $Y_0 \subseteq Y_1$ such that:

(a) Group G_0 is contained in the subgroup of A generated by $Y_1 + K$.
(b) There exists $C_0 \in [J]^{<\mu}$ such that $\langle Y_0\rangle \cap K$ is contained in the subgroup of K generated by $\{k_j \colon j \in C_0\}$, and Y_1 contains all the elements from X appearing in equations $k_j = 0$, $j \in C_0$.

For this Y_1, we obtain, using (2), a subgroup G_1 of A with $|G_1| < \mu$, and so on.

After ω steps, we have the group $G = \sum_{n<\omega} G_n \in \operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$ generated by $Y + K$ where $Y = \bigcup_{n<\omega} Y_n \in [X]^{<\mu}$. By the construction, we have also $G = \langle y + K : y \in Y\rangle \cong \langle Y\rangle/\langle k_j : j \in \bigcup_{n<\omega} C_n\rangle$. Finally, we put $T = \{k_j = 0 : j \in \bigcup_{n<\omega} C_n\}$.

Now, let us prove the implication $\neg(1) \implies \neg(3)$. First, assume that κ is not $\mathcal{L}_{\omega_1\omega}$-compact. Following [1, Theorem 5.3] and its proof, we start with $A = \mathbb{Z}^I/\mathcal{F}$ where $\mathcal{F}$ is a κ-complete filter on I which cannot be extended to an ω_1-complete ultrafilter. From the latter part, it follows that $\operatorname{Hom}(A,\mathbb{Z}) = 0$. The κ-completeness of $\mathcal{F}$, on the other hand, assures that any subgroup of A of cardinality less than κ can be embedded into $\mathbb{Z}^I$.

Consider a system S of homogeneous $\mathbb{Z}$-linear equations associated to the group A presented as F/K where F is freely generated by a set X. We can without loss of generality assume that no $x \in X$ is contained in K. Let $C \in [J]^{<\kappa}$ be nonempty. We shall show that the system $\{k_j = 0 : j \in C\}$ has weakly nontrivial solution in $\mathbb{Z}$.

As in the proof of the other implication, we can possibly enlarge C to some $D \subseteq J$ such that $|D| \leq |C| + \aleph_0$ and $\langle y + K : y \in Y\rangle \cong \langle Y\rangle/\langle k_j : j \in D\rangle$, where Y denotes the set of all the elements from X appearing in equations $k_j = 0$, $j \in D$. Let us denote the latter group by H and fix an embedding $i : H \to \mathbb{Z}^I$ (which exists since $|H| < \kappa$).

Let $y \in Y$ be any element appearing in (one of the) equations $k_j = 0$, $j \in C$. Since $i(y + K) \neq 0$ there is a projection $\pi : \mathbb{Z}^I \to \mathbb{Z}$ such that $\pi i(y + K) \neq 0$. The assignment $x \mapsto \pi i(x + K)$ defines the desired weakly nontrivial solution of the system $\{k_j = 0 : j \in C\}$ in $\mathbb{Z}$.

It remains to tackle the possibility that κ is the least $\mathcal{L}_{\omega_1\omega}$-compact cardinal and κ is singular. We know by [2] that $\gamma = cf(\kappa)$ is greater than or equal to the first measurable cardinal in this case. Let $(\kappa_\alpha : \alpha < \gamma)$ be an increasing sequence of cardinals less than κ converging to κ.

Consider the group $A = \bigoplus_{\alpha<\gamma} A_\alpha$ where for each $\alpha < \gamma$, $A_\alpha \in \operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$ is not a sum of its subgroups of cardinality less than κ_α which belong to $\operatorname{Ker}\operatorname{Hom}(-,\mathbb{Z})$. Assume, for the sake of contradiction, that (3) holds for the system S of homogeneous $\mathbb{Z}$-linear equations associated to the group A (more precisely, to its presentation F/K).

By the definition of A, there exists for each $\alpha < \gamma$, an element $a_\alpha \in A$ such that a_α is not contained in any subgroup H of A of cardinality less than κ_α with the property $\operatorname{Hom}(H,\mathbb{Z}) = 0$.

We know that there is $C_0 \in [J]^{<\kappa}$ and $Y_0 \subseteq X$ consisting of the elements from X appearing in the equations $k_j = 0$, $j \in C_0$ such that $\{a_\alpha : \alpha < \gamma\} \subseteq \langle y + K : y \in Y_0\rangle \cong \langle Y_0\rangle/\langle k_j : j \in C_0\rangle$.

For this C_0, we obtain a corresponding $T_0 \in [J]^{<\kappa}$ using (3). We continue by finding $C_1 \in [J]^{<\kappa}$ and $Y_1 \in [X]^{<\kappa}$ such that $T_0 \subseteq C_1$, $Y_0 \subseteq Y_1$ and $\langle y + K \colon y \in Y_1 \rangle \cong \langle Y_1 \rangle / \langle k_j \colon j \in C_1 \rangle$, and so forth.

Put $T = \bigcup_{n<\omega} T_n = \bigcup_{n<\omega} C_n$ and $Y = \bigcup_{n<\omega} Y_n$. The system $\{k_j = 0 \colon j \in T\}$ has cardinality less than κ (since γ is uncountable) and it has no weakly nontrivial solution in $\mathbb{Z}$. Whence the subgroup $H = \langle y + K \colon y \in Y \rangle \cong \langle Y \rangle / \langle k_j \colon j \in T \rangle$ of A belongs to $\operatorname{Ker}\operatorname{Hom}(-, \mathbb{Z})$. However, this is impossible since $a_\alpha \in H$ for $\alpha < \gamma$ satisfying $|H| < \kappa_\alpha$. □

In the proof above, we have actually showed a little bit more. In fact, we have the following

Theorem 3.2. *Let κ be a cardinal, and assume that κ is not at the same time singular and the least $\mathcal{L}_{\omega_1\omega}$-compact cardinal. The following conditions are equivalent:*

1. *Cardinal κ is $\mathcal{L}_{\omega_1\omega}$-compact.*
2. *Every system S of homogeneous $\mathbb{Z}$-linear equations is weakly nontrivially solvable in $\mathbb{Z}$ provided that each of its subsystems of cardinality less than κ is weakly nontrivially solvable. In other words, $\kappa \in \mathcal{WS}$.*

Proof: The implication '(1) $\Longrightarrow$ (2)' follows immediately from '(1) $\Longrightarrow$ (3)' in Proposition 3.1. The other implication then follows from the first part of the proof of '$\neg$(1) $\Longrightarrow$ $\neg$(3)' in Proposition 3.1. □

As shown in [1], relative to the existence of a supercompact cardinal, there are models of ZFC where the smallest $\mathcal{L}_{\omega_1\omega}$-compact cardinal κ is singular. In this only case, we cannot resolve the question whether $\kappa \in \mathcal{WS}$ although we conjecture that this is not the case, which would readily imply that at least $\mathcal{WS} \subseteq \mathcal{S}$ always holds.

Apart from the subtlety above, a possible direction for further research is to investigate further what more can be proved in ZFC about the class $\mathcal{S}$.

Acknowledgement. I would like to thank Petr Glivický for a fruitful discussion about compactness problems in set theory, and an anonymous referee whose appropriate suggestions help to improve the quality of the paper.

References

[1] Bagaria J., Magidor M., *Group radicals and strongly compact cardinals*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 1857–1877.

[2] Bagaria J., Magidor M., *On ω_1-strongly compact cardinals*, J. Symb. Log. **79** (2014), no. 1, 266–278.

[3] Dugas M., Göbel R., *Every cotorsion-free ring is an endomorphism ring*, Proc. London Math. Soc. (3) **45** (1982), no. 2, 319–336.
[4] Eklof P. C., Mekler A. H., *Almost Free Modules*, Set-theoretic methods, North-Holland Mathematical Library, 65, North-Holland Publishing, Amsterdam, 2002.
[5] Göbel R., Shelah S., $\aleph_n$*-free modules with trivial duals*, Results Math. **54** (2009), no. 1–2, 53–64.
[6] Göbel R., Trlifaj J., *Approximations and Endomorphism Algebras of Modules, Volume 1., Approximations*, De Gruyter Expositions in Mathematics, 41, Walter de Gruyter GmbH & Co. KG, Berlin, 2012.
[7] Herrlich H., Tachtsis E., *On the solvability of systems of linear equations over the ring* $\mathbb{Z}$ *of integers*, Comment. Math. Univ. Carolin. **58** (2017), no. 2, 241–260.
[8] Kanamori A., *The Higher Infinite: Large Cardinals in Set Theory from Their Beginnings*, Springer Monographs in Mathematics, Springer, Berlin, 2003.
[9] Shelah S., *Quite free complicated abelian group, PCF and black boxes*, available at ArXiv: 1404.2775v2 [math.LO] (2019), 49 pages.

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(*Received* January 2, 2019*, revised* March 14, 2019)