Translation surfaces of finite type in Sol_3

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Abstract. In the homogeneous space Sol_3, a translation surface is parametrized by $r(s,t) = \gamma_1(s) * \gamma_2(t)$, where γ_1 and γ_2 are curves contained in coordinate planes.

In this article, we study translation invariant surfaces in Sol_3, which has finite type immersion.

Keywords: Laplacian operator; homogeneous space; invariant surface; surfaces of coordinate finite type

Classification: 53C30, 53B25

Introduction

A Euclidean submanifold is said to be of finite Chen-type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian, see [3]. B.-Y. Chen posed the problem of classifying the finite type surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$. Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space.

Let M^2 be a 2-dimensional surface of the Euclidean 3-space $\mathbb{E}^3$. If we denote by r, H and Δ the position vector field, the mean curvature vector field and the Laplace operator of M^2, respectively, then it is well-known, see [3], that

$$\Delta r = -2H. \tag{1}$$

A well-known result due to T. Takahashi in [9] states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$ for a real constant λ. Equation (1) shows that M^2 is a minimal surface of $\mathbb{E}^3$ if and only if its coordinate functions are harmonic. In [2], M. Bekkar and B. Senoussi studied the translation surfaces in the 3-dimensional Euclidean and Lorentz–Minkowski space under the condition $\Delta^{III} r_i = \mu_i r_i$, $\mu_i \in \mathbb{R}$, where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III.

DOI 10.14712/1213-7243.2020.018

In [11], D. W. Yoon studied translation surfaces in Sol_3 satisfying the condition

$$\Delta x = Ax + B,$$

where $A \in \mathrm{Mat}(3,\mathbb{R})$ is a 3×3 real matrix and $B \in \mathbb{R}^3$. In [1], H. Al-Zoubi, S. Stamatakis, W. Al-Mashaleh and M. Awadallah studied the translation surfaces of coordinate finite type.

The main purpose of this paper is to complete classification of translation surfaces in Sol_3 in terms of the position vector field and the Laplacian operator

$$\Delta r_i = \lambda_i r_i, \qquad \lambda_i \in \mathbb{R},\ i = 1,2,3, \tag{2}$$

where (r_1, r_2, r_3) are the components of r and $\Delta r = (\Delta r_1, \Delta r_2, \Delta r_3)$.

As a result, we are to complete [11] classification of translation surfaces in Sol_3 satisfying the condition (2).

1. Preliminaries

The space Sol_3 is the space $\mathbb{R}^3$ equipped with the metric

$$\mathrm{d}s^2 = (\mathrm{e}^z \mathrm{d}x)^2 + (\mathrm{e}^{-z}\mathrm{d}y)^2 + (\mathrm{d}z)^2,$$

where (x, y, z) are usual coordinates of $\mathbb{R}^3$, see for instance [8].

The space Sol_3 is a Lie group with the multiplication

$$(x, y, z) * (x', y', z') = (x + \mathrm{e}^{-z}x', y + \mathrm{e}^z y', z + z'),$$

where '$*$' denotes the group operation of Sol_3. A left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ in Sol_3 is given by

$$E_1 = \mathrm{e}^{-z}\frac{\partial}{\partial x}, \qquad E_2 = \mathrm{e}^{z}\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}.$$

Proposition 1.1 ([10]). *The Levi–Civita connection $\widetilde{\nabla}$ of* Sol_3 *with respect to this frame is*

$$\begin{aligned}
\begin{pmatrix} \widetilde{\nabla}_{E_1}E_1 \\ \widetilde{\nabla}_{E_1}E_2 \\ \widetilde{\nabla}_{E_1}E_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \\
\begin{pmatrix} \widetilde{\nabla}_{E_2}E_1 \\ \widetilde{\nabla}_{E_2}E_2 \\ \widetilde{\nabla}_{E_2}E_3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}
\end{aligned} \tag{3}$$

$$\begin{pmatrix} \widetilde{\nabla}_{E_3}E_1 \\ \widetilde{\nabla}_{E_3}E_2 \\ \widetilde{\nabla}_{E_3}E_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

The immersion (M^2, r) is said to be of finite Chen-type k, if the position vector r of M^2 can be written as a finite sum of nonconstant eigenvectors of the Laplacian Δ, that is, if

$$r = y_0 + \sum_{i=1}^{k} y_i,$$

where $\Delta y_i = \lambda_i y_i$, $i = 1, 2, \dots, k$, y_0 is a fixed vector and $\lambda_1, \lambda_2, \dots, \lambda_k$ are eigenvalues of Δ. In particular, if all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ are mutually distinct, then M^2 is said to be of finite type k. However, if $\lambda_i = 0$ for some $i = 1, 2, \dots, k$, then M^2 is said to be of finite null type k. Otherwise M^2 is said to be of infinite type.

For the matrix $G = (g_{ij})$ consisting of the components of the induced metric on M^2, we denote by $G^{-1} = (g^{ij})$ (or $D = \det(g_{ij})$) the inverse matrix (the determinant, respectively) of the matrix (g_{ij}). The Laplacian Δ on M^2 is, in turn, given by [11]

$$\Delta = \frac{-1}{\sqrt{D}} \sum_{ij} \frac{\partial}{\partial x^i}\Big(\sqrt{D} g^{ij} \frac{\partial}{\partial x^j}\Big). \tag{4}$$

2. Translation surfaces in Sol_3

A surface M^2 in the Euclidean 3-space $\mathbb{E}^3$ is called minimal when locally each point on the surface has a neighborhood which is the surface of least area with respect to its boundary. In 1775, J. B. Meusnier showed that the condition of minimality of a surface in $\mathbb{E}^3$ is equivalent with the vanishing of its mean curvature function, $H = 0$.

In 1835, H. F. Scherk studied translation surfaces in $\mathbb{E}^3$ and proved that, besides the planes, the only minimal translation surfaces are given by

$$z(x, y) = \frac{1}{a} \log|\cos(ax)| - \frac{1}{a} \log|\cos(ay)|,$$

where a is a nonzero constant. The minimal translation surfaces were generalized to minimal translation hypersurfaces by F. Dillen, L. Verstraelen and G. Zafindratafa in [4].

R. López and M. I. Munteanu constructed translation surfaces in Sol_3 and investigated properties of minimal ones in [7]. In [6], the authors defined and classified two types of constant angle surfaces in the homogeneous 3-manifold Sol_3.

In [5] J. Inoguchi, R. López and M. I. Munteanu defined translation surfaces in the 3-dimensional Heisenberg group H_3 in terms of a pair of two planar curves lying in orthogonal planes. D. W. Yoon, C. W. Lee and M. K. Karacan in [12] considered translation surfaces in H_3 generated as product of two planar curves lying in planes, which are not orthogonal, and the authors classified such minimal translation surfaces.

In the space Sol_3, a translation surface is parameterized by $r(s,t) = \gamma_1(s) * \gamma_2(t)$, where γ_1 and γ_2 are curves contained in coordinate planes and '$*$' denotes the group operation of Sol_3.

Definition 2.1 ([7])**.** A translation surface $M(\gamma_1, \gamma_2)$ in Sol_3 is a surface parametrized by $r(s,t) = \gamma_1(s) * \gamma_2(t)$, where $\gamma_1 \colon I \subset \mathbb{R} \to \mathrm{Sol}_3$, $\gamma_2 \colon J \subset \mathbb{R} \to \mathrm{Sol}_3$ are curves in two coordinate planes of $\mathbb{R}^3$.

We distinguish six types of translation surfaces in Sol_3.

2.1 Translation surfaces of type *I* and type *IV*. Let the curves γ_1 and γ_2 be given by $\gamma_1(s) = (s, f(s), 0)$ and $\gamma_2(t) = (t, 0, g(t))$. We have two translation surfaces $M(\gamma_1, \gamma_2)$ and $M(\gamma_2, \gamma_1)$ parametrized by, respectively,

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (s + t, f(s), g(t))$$

and

$$r(s,t) = \gamma_2(t) * \gamma_1(s) = (s\mathrm{e}^{-g(t)} + t, f(s)\mathrm{e}^{g(t)}, g(t)),$$

which are called the translation surfaces of type *I* and *IV*.

2.2 Translation surfaces of type *II* and type *V*. Let the curves γ_1 and γ_2 be given by $\gamma_1(s) = (s, f(s), 0)$ and $\gamma_2(t) = (0, t, g(t))$. We have two translation surfaces $M(\gamma_1, \gamma_2)$ and $M(\gamma_2, \gamma_1)$ parametrized by, respectively,

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (s, t + f(s), g(t))$$

and

$$r(s,t) = \gamma_2(t) * \gamma_1(s) = (s\mathrm{e}^{-g(t)}, t + f(s)\mathrm{e}^{g(t)}, g(t)),$$

which are called the translation surfaces of type *II* and *V*.

2.3 Translation surfaces of type *III* and type *VI*. Let the curves γ_1 and γ_2 be given by $\gamma_1(s) = (s, 0, f(s))$ and $\gamma_2(t) = (0, t, g(t))$. We have two translation surfaces $M(\gamma_1, \gamma_2)$ and $M(\gamma_2, \gamma_1)$ parametrized by, respectively,

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (s, t\mathrm{e}^{f(s)}, f(s) + g(t))$$

and

$$r(s,t) = \gamma_2(t) * \gamma_1(s) = (s\mathrm{e}^{-g(t)}, t, f(s) + g(t)),$$

which are called the translation surfaces of type *III* and *VI*.

3. Translation surfaces in Sol_3 satisfying $\Delta r_i = \lambda_i r_i$

3.1 Translation surfaces of type *II*. Let $M(\gamma_1, \gamma_2)$ be a translation surface of type *II* in Sol_3. Then, $M(\gamma_1, \gamma_2)$ is parametrized by

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (s, t + f(s), g(t)). \tag{5}$$

We have the natural frame $\{\frac{\partial r}{\partial s}, \frac{\partial r}{\partial t}\}$ given by

$$\begin{aligned} \frac{\partial r}{\partial s} &= r_s = \frac{\partial}{\partial x} + f' \frac{\partial}{\partial y} = \mathrm{e}^g E_1 + f' \mathrm{e}^{-g} E_2, \\ \frac{\partial r}{\partial t} &= r_t = \frac{\partial}{\partial y} + g' \frac{\partial}{\partial z} = \mathrm{e}^{-g} E_2 + g' E_3. \end{aligned} \tag{6}$$

Let $\mathbf{N}$ be a unit normal vector of $M(\gamma_1, \gamma_2)$. Then it is defined by

$$\mathbf{N} = \frac{r_s \times r_t}{\|r_s \times r_t\|}$$

and hence we get

$$\mathbf{N} = \Big(\frac{f'g'\mathrm{e}^{-g}}{W}\Big) E_1 - \Big(\frac{g'\mathrm{e}^{g}}{W}\Big) E_2 + \Big(\frac{1}{W}\Big) E_3,$$

where $W = \sqrt{\|r_s \times r_t\|} = \sqrt{g'^2\mathrm{e}^{2g} + f'^2 g'^2 \mathrm{e}^{-2g} + 1}$.

The first fundamental form I of $M(\gamma_1, \gamma_2)$ is defined by

$$I = E\mathrm{d}s^2 + 2F\mathrm{d}s\mathrm{d}t + G\mathrm{d}t^2,$$

where

$$E = \langle r_s, r_s \rangle = \mathrm{e}^{2g} + f'^2\mathrm{e}^{-2g}, \quad F = \langle r_s, r_t \rangle = f'\mathrm{e}^{-2g}, \quad G = \langle r_t, r_t \rangle = \mathrm{e}^{-2g} + g'^2,$$

and $\langle\ ,\ \rangle$ denotes the standard scalar product in $\mathbb{E}^3$.

To compute the second fundamental form of $M(\gamma_1, \gamma_2)$, we have to calculate the following:

$$\begin{aligned} r_{ss} &= \widetilde{\nabla}_{r_s} r_s = f''\mathrm{e}^{-g} E_2 + (f'^2 - 1)\mathrm{e}^{2g} E_3, \\ r_{st} &= \widetilde{\nabla}_{r_s} r_t = \widetilde{\nabla}_{r_t} r_s = g'\mathrm{e}^{g} E_1 - f'g'\mathrm{e}^{-g} E_2 + f'\mathrm{e}^{-2g} E_3, \\ r_{tt} &= \widetilde{\nabla}_{r_t} r_t = -2g'\mathrm{e}^{-g} E_2 + (\mathrm{e}^{-2g} + g'') E_3, \end{aligned} \tag{7}$$

which imply the coefficients of the second fundamental form of $M(\gamma_1, \gamma_2)$ are given by

$$L = \langle \widetilde{\nabla}_{r_s} r_s, \mathbf{N} \rangle = \frac{-1}{W}(f''g' - f'^2\mathrm{e}^{-2g} + \mathrm{e}^{2g}),$$

$$M = \langle \widetilde{\nabla}_{r_s} r_t, \mathbf{N} \rangle = \frac{1}{W}(2f'g'^2 + f'\mathrm{e}^{-2g}),$$
$$N = \langle \widetilde{\nabla}_{r_s} r_s, \mathbf{N} \rangle = \frac{1}{W}(g'' + 2g'^2 + \mathrm{e}^{-2g}).$$

A surface is minimal if its mean curvature, computed by the formula

$$H = \frac{EN - 2FM + GL}{2W^3},$$

vanishes identically. Thus, the mean curvature of $M(\gamma_1, \gamma_2)$ is given by

$$H = \frac{-f''g'^3 - (f''g' + f'^2g'^2 - f'^2g'')\mathrm{e}^{-2g} + (g'^2 + g'')\mathrm{e}^{2g}}{2W^3}. \tag{8}$$

Then $M(\gamma_1, \gamma_2)$ is a minimal surface if and only if

$$-f''g'^3 - (f''g' + f'^2g'^2 - f'^2g'')\mathrm{e}^{-2g} + (g'^2 + g'')\mathrm{e}^{2g} = 0.$$

By (4), the Laplacian operator Δ of r can be expressed as

$$\begin{aligned} \Delta r &= \frac{-1}{W}\Big[\frac{\partial}{\partial s}\Big(\frac{Gr_s - Fr_t}{W}\Big) + \frac{\partial}{\partial t}\Big(\frac{Er_t - Fr_s}{W}\Big)\Big] \\ &= -\frac{1}{2W^4}\big(2W^2(G\widetilde{\nabla}_{r_s} r_s - 2F\widetilde{\nabla}_{r_s} r_t + E\widetilde{\nabla}_{r_t} r_t) + \mathcal{H}_1 r_s + \mathcal{H}_2 r_t\big), \end{aligned} \tag{9}$$

where

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{e}^{-2g}(4f'g' - 2f'f''g'^4) + \mathrm{e}^{-4g}(2f'^3g'^3 - 2f'f''g'^2 + 2g'g''f'^3) \\ &\quad + 6f'g'^3 + 2f'g'g'', \\ \mathcal{H}_2 &= 4g'\mathrm{e}^{2g} + \mathrm{e}^{-2g}(-2f'' - 4f'^2g') + \mathrm{e}^{-4g}(-2f'^4g'^3 - 2g'g''f'^4) \\ &\quad + \mathrm{e}^{4g}(-2g'g'' + 2g'^3) - 2f''g'^2 - 4g'g''f'^2. \end{aligned}$$

The substituting of (6) and (7) into (9) gives

$$\begin{aligned} \Delta r &= -\frac{1}{2W^4}\big(2W^2(Gr_{ss} - 2Fr_{st} + Er_{tt}) + \mathcal{H}_1 r_s + \mathcal{H}_2 r_t\big) \\ &= -\frac{1}{2W^4}\big(2W^2(Gr_{ss} - 2Fr_{st} + Er_{tt}) \\ &\qquad + \mathcal{H}_1(\mathrm{e}^g E_1 + f'\mathrm{e}^{-g}E_2) + \mathcal{H}_2(\mathrm{e}^{-g}E_2 + g'E_3)\big) \\ &= \frac{-2H}{W}(f'g'\mathrm{e}^{-g}E_1 - g'e^g E_2 + E_3) \\ &= -2H\mathbf{N}. \end{aligned} \tag{10}$$

$M(\gamma_1, \gamma_2)$ is a minimal surfaces in Sol_3 if and only if its coordinate functions are harmonic.

Equations (2) and (4) imply

$$\frac{2Hf'g'}{W} = -\lambda_1 s\mathrm{e}^{2g}, \tag{11}$$

$$\frac{2Hg'}{W} = \lambda_2(f+t)\mathrm{e}^{-2g}, \tag{12}$$

$$\frac{2H}{W} = -\lambda_3 g. \tag{13}$$

Case 1. Let $\lambda_3 = 0$. (13) implies that the mean curvature H is identically zero. Thus, the surface $M(\gamma_1, \gamma_2)$ is minimal.

Case 2. Let $\lambda_3 \neq 0$.

2-1) Let $\lambda_1 = 0$ and $\lambda_2 = 0$. In this case the system (11), (12) and (13) is reduced equivalently to

$$\begin{aligned} Hf'g' &= 0, \\ Hg' &= 0, \\ \frac{2H}{W} &= -\lambda_3 g. \end{aligned}$$

2-1-1) If $H = 0$, then $\lambda_3 = 0$, a contradiction.

2-1-2) If $g' = 0$, then $H = 0$. So we get a contradiction.

2-2) Let $\lambda_1 = 0$ and $\lambda_2 \neq 0$. In this case the system (11), (12) and (13) is reduced equivalently to

$$\left(\frac{2H}{W}\right)f'g' = 0, \tag{14}$$

$$\frac{2Hg'}{W} = \lambda_2(f+t)\mathrm{e}^{-2g}, \tag{15}$$

$$\frac{2H}{W} = -\lambda_3 g. \tag{16}$$

2-2-1) If $H = 0$, then $\lambda_3 = 0$ and $\lambda_2 = 0$, a contradiction.

2-2-2) If $g' = 0$, then $\lambda_2 = 0$. So we get a contradiction.

2-2-3) If $f' = 0$. Then $f(s) = \alpha$, $\alpha \in \mathbb{R}$.

Substituting (16) into (15), we get

$$-\lambda_3 gg' = \lambda_2(\alpha + t)\mathrm{e}^{-2g}.$$

A direct integration implies that there exist $\alpha_1, \alpha_2, \alpha_3$ such that

$$(2g-1)\mathrm{e}^{2g} = \alpha_1 t^2 + \alpha_2 t + \alpha_3.$$

2-3) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$. In this case the system (11), (12) and (13) is reduced equivalently to

$$\frac{2Hg'}{W} = 0. \tag{17}$$

(17) implies that the mean curvature H is identically zero. Then (13) gives $\lambda_3 = 0$, a contradiction.

2-4) If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Substituting (13) into (12), we get

$$\lambda_3 gg' = -\lambda_2 (f+t)\mathrm{e}^{-2g}. \tag{18}$$

Differentiating (18) with respect to s we get $\lambda_2 f' = 0$. If $f' = 0$, then $\lambda_1 = 0$. So we get a contradiction.

Therefore, we have the following:

Theorem 3.1. *Let $M(\gamma_1, \gamma_2)$ be a translation surface of type II in* Sol_3. *Then, $M(\gamma_1, \gamma_2)$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $i = 1, 2, 3$, $\lambda_i \in \mathbb{R}$, if and only if one of the following statement is true:*

1) *A surface $M(\gamma_1, \gamma_2)$ has zero mean curvature everywhere.*
2) *A surface $M(\gamma_1, \gamma_2)$ is parametrized as*

$$r(s,t) = (s, t+\alpha, g(t)),$$

where $(2g-1)\mathrm{e}^{2g} = \alpha_1 t^2 + \alpha_2 t + \alpha_3$; $\alpha, \alpha_i \in \mathbb{R}$.

3.2 Translation surfaces of type V. Let $M(\gamma_1, \gamma_2)$ be a translation surface of type V in Sol_3. Then, $M(\gamma_1, \gamma_2)$ is parametrized by

$$r(s,t) = \gamma_2(t)\gamma_1(s) * \gamma_1(s) = (s\mathrm{e}^{-g(t)}, t + f(s)\mathrm{e}^{g(t)}, g(t)). \tag{19}$$

By differentiating (19) with respect to s and to t we deduce the following:

$$r_s = E_1 + f'E_2, \qquad r_t = (-sg')E_1 + (\mathrm{e}^{-g} + fg')E_2 + g'E_3. \tag{20}$$

The coefficients of first fundamental form of $M(\gamma_1, \gamma_2)$ are

$$\begin{aligned} E &= 1 + f'^2, \\ F &= -sg' + f'(fg' + \mathrm{e}^{-g}), \\ G &= g'^2(s^2+1) + (fg' + \mathrm{e}^{-g})^2. \end{aligned} \tag{21}$$

The unit normal vector field $\mathbf{N}$ of $M(\gamma_1, \gamma_2)$ is given by

$$\mathbf{N} = \Big(\frac{f'g'}{W}\Big)E_1 - \Big(\frac{g'}{W}\Big)E_2 + \Big(\frac{fg' + sf'g' + \mathrm{e}^{-g}}{W}\Big)E_3,$$

where $W = \sqrt{g'^2(1+f'^2) + (fg' + sf'g' + \mathrm{e}^{-g})^2}$. From (20) and (3), we have

$$\begin{aligned} \widetilde{\nabla}_{r_s} r_s &= f'' E_2 + (f'^2 - 1)E_3, \\ \widetilde{\nabla}_{r_s} r_t &= \widetilde{\nabla}_{r_t} r_s = (ff'g' + sg' + f'\mathrm{e}^{-g})E_3, \\ \widetilde{\nabla}_{r_t} r_t &= -s(g'' + g'^2)E_1 + (-2g'\mathrm{e}^{-g} + f(g'' - g'^2))E_2 \\ &\quad + (g'' - s^2 g'^2 + (fg' + \mathrm{e}^{-g})^2)E_3. \end{aligned} \tag{22}$$

The coefficients of the second fundamental form are given by

$$\begin{aligned} WL &= -g'f'' + (f'^2 - 1)(fg' + sf'g' + \mathrm{e}^{-g}), \\ WM &= (fg' + sf'g' + \mathrm{e}^{-g})(ff'g' + sg' + f'\mathrm{e}^{-g}), \\ WN &= (fg' + sf'g' + \mathrm{e}^{-g})(g'' - s^2 g'^2 + (fg' + \mathrm{e}^{-g})^2) \\ &\quad - g'(-2g'\mathrm{e}^{-g} + f(g'' - g'^2)) - sf'g'(g'' + g'^2). \end{aligned}$$

The mean curvature H of $M(\gamma_1, \gamma_2)$ is given by

$$H = \frac{\mathcal{H}}{2W^3}, \tag{23}$$

where

$$\begin{aligned} \mathcal{H} = &-f''g'(g'^2(1 + s^2 + f^2) + 2fg'\mathrm{e}^{-g} + \mathrm{e}^{-2g}) + g''\mathrm{e}^{-g}(f'^2 + 1) \\ &+ g'^2\mathrm{e}^{-g}(1 + 3f'^2 + 2s^2 f'^2) + 2f'g'^3(ff' - s). \end{aligned}$$

The substituting of (21) and (19) into (4) gives

$$\Delta r = -2H\mathbf{N}. \tag{24}$$

Then, from (24) and (2), we get

$$\begin{aligned} -\Big(\frac{2H}{W}\Big) f'g' &= \lambda_1 s, \\ \Big(\frac{2H}{W}\Big) g' &= \lambda_2 (f + t\mathrm{e}^{-g}), \\ -\frac{2H}{W}(fg' + sf'g' + \mathrm{e}^{-g}) &= \lambda_3 g. \end{aligned}$$

Therefore, the problem of classifying the translation surfaces $M(\gamma_1, \gamma_2)$ satisfying (2) is reduced to the integration of this system of ordinary differential equations.

Then, by using similar method as for the translation surface of type II, we have the following result:

Theorem 3.2. *Let $M(\gamma_1, \gamma_2)$ be a translation surface of type V in* Sol_3*. Then, $M(\gamma_1, \gamma_2)$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $i = 1, 2, 3$, $\lambda_i \in \mathbb{R}$, if and only if one of the following statement is true:*

1) *A surface $M(\gamma_1, \gamma_2)$ has zero mean curvature everywhere.*
3) *A surface $M(\gamma_1, \gamma_2)$ is parametrized as*

$$r(s,t) = (s\mathrm{e}^{-g(t)}, t + a\mathrm{e}^{g(t)}, g(t)),$$

where $(\delta_1 g + \delta_2)\mathrm{e}^{2g} = \delta_3 t^2 + \delta_4 t\mathrm{e}^{g}$; $a, \delta_i \in \mathbb{R}$, $1 \leq i \leq 4$.

3.3 Translation surfaces of type *III*. Let $M(\gamma_1, \gamma_2)$ be a translation surface of type *III* in Sol_3. Then, $M(\gamma_1, \gamma_2)$ is parametrized by

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (s, t\mathrm{e}^{f(s)}, f(s) + g(t)). \tag{25}$$

The first derivatives are

$$\begin{aligned} \frac{\partial r}{\partial s} &= r_s = \mathrm{e}^{g+f} E_1 + t f' \mathrm{e}^{-g} E_2 + f' E_3, \\ \frac{\partial r}{\partial t} &= r_t = \mathrm{e}^{-g} E_2 + g' E_3. \end{aligned} \tag{26}$$

From this, the unit normal vector field $\mathbf{N}$ of $M(\gamma_1, \gamma_2)$ is given by

$$\mathbf{N} = -\Big(\frac{f'(1 - tg')\mathrm{e}^{-g}}{W}\Big) E_1 - \Big(\frac{g'\mathrm{e}^{f+g}}{W}\Big) E_2 + \Big(\frac{\mathrm{e}^{f}}{W}\Big) E_3,$$

where $W = \sqrt{g'^2 \mathrm{e}^{2(f+g)} + f'^2(1 - tg')^2 \mathrm{e}^{-2g} + \mathrm{e}^{2f}}$.

The coefficients of the first fundamental form are:

$$\begin{aligned} E &= \langle r_s, r_s \rangle = \mathrm{e}^{2(f+g)} + f'^2(1 + t^2 \mathrm{e}^{-2g}), \\ F &= \langle r_s, r_t \rangle = f'(g' + t\mathrm{e}^{-2g}), \\ G &= \langle r_t, r_t \rangle = \mathrm{e}^{-2g} + g'^2. \end{aligned} \tag{27}$$

The covariant derivatives are:

$$\begin{aligned} r_{ss} &= \widetilde{\nabla}_{r_s} r_s = (2f'\mathrm{e}^{f+g}) E_1 + t(f'' - f'^2)\mathrm{e}^{-g} E_2 \\ &\qquad\qquad + (f'' + t^2 f'^2 \mathrm{e}^{-2g} - \mathrm{e}^{2(f+g)}) E_3, \\ r_{st} &= \widetilde{\nabla}_{r_s} r_t = \widetilde{\nabla}_{r_t} r_s = g'\mathrm{e}^{f+g} E_1 - t f' g' \mathrm{e}^{-g} E_2 + t f' \mathrm{e}^{-2g} E_3, \\ r_{tt} &= \widetilde{\nabla}_{r_t} r_t = -2g'\mathrm{e}^{-g} E_2 + (\mathrm{e}^{-2g} + g'') E_3. \end{aligned} \tag{28}$$

The coefficients of the second fundamental form are given by

$$\begin{aligned} WL &= \mathrm{e}^f\left[(f''-2f'^2)(1-tg')+tg'f'^2+t^2f'^2\mathrm{e}^{-2g}-\mathrm{e}^{2(f+g)}\right],\\ WM &= \mathrm{e}^f[-f'g'(1-tg')+tf'(\mathrm{e}^{-2g}+g'^2)],\\ WN &= \mathrm{e}^f[g''+2g'^2+\mathrm{e}^{-2g}]. \end{aligned}$$

The mean curvature H of $M(\gamma_1,\gamma_2)$ is given by

$$H=\frac{\mathcal{H}}{2W^3}, \tag{29}$$

where

$$\begin{aligned} \mathcal{H} = \mathrm{e}^f\big[&(g'^2+g'')\mathrm{e}^{2(f+g)}+(f''(1-tg')-f'^2(1-tg')^2+f'^2t(tg''+g'))\mathrm{e}^{-2g}\\ &+f''g'^2(1-tg')+f'^2g'^2(1-tg')+f'^2(g''+g'^2)\big]. \end{aligned}$$

By (4), the Laplacian operator Δ of r can be expressed as

$$\Delta r=-\frac{1}{2W^4}(2W^2(Gr_{ss}-2Fr_{st}+Er_{tt})+\mathcal{H}_1r_s+\mathcal{H}_2r_t), \tag{30}$$

where

$$\begin{aligned} \mathcal{H}_1 &= \mathrm{e}^{-2g}(4f'g'-2f'f''g'^4)+\mathrm{e}^{-4g}(2f'^3g'^3-2f'f''g'^2+2g'g''f'^3)\\ &\quad+6f'g'^3+2f'g'g'',\\ \mathcal{H}_2 &= 4g'\mathrm{e}^{2g}+\mathrm{e}^{-2g}(-2f''-4f'^2g')+\mathrm{e}^{-4g}(-2f'^4g'^3-2g'g''f'^4)\\ &\quad+\mathrm{e}^{4g}(-2g'g''+2g'^3)-2f''g'^2-4g'g''f'^2. \end{aligned}$$

The substituting of (26) and (28) into (30) gives

$$\begin{aligned} \Delta r =&\ -\frac{1}{2W^4}[2W^2(Gr_{ss}-2Fr_{st}+Er_{tt})+\mathcal{H}_1r_s+\mathcal{H}_2r_t]\\ =&\ -\frac{1}{2W^4}\big[2W^2(G\widetilde{\nabla}_{r_s}r_s-2F\widetilde{\nabla}_{r_s}r_t+E\widetilde{\nabla}_{r_t}r_t)\\ &\quad+\mathcal{H}_1(\mathrm{e}^{g+f}E_1+tf'\mathrm{e}^{-g}E_2+f'E_3)+\mathcal{H}_2(\mathrm{e}^{-g}E_2+g'E_3)\big]\\ =&\ -\frac{\mathcal{H}}{W^4}(-(f'(1-tg')\mathrm{e}^{-g})E_1-(g'\mathrm{e}^{f+g})E_2+(\mathrm{e}^f)E_3)\\ =&\ -2H\mathbf{N}. \end{aligned} \tag{31}$$

Then, from (31) and (2), we get

$$\left(\frac{2H}{W}\right)f'(1-tg') = \lambda_1 s\mathrm{e}^{f+2g}, \tag{32}$$

$$\left(\frac{2H}{W}\right)g' = \lambda_2 t\mathrm{e}^{-f-2g}, \tag{33}$$

$$\frac{2H}{W} = -\lambda_3(f+g)\mathrm{e}^{-f}. \tag{34}$$

Therefore, the problem of classifying the affine translation surfaces $M(\gamma_1, \gamma_2)$ satisfying (2) is reduced to the integration of this system of ordinary differential equations. Next we study it according to the constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1. Let $\lambda_3 = 0$. (34) implies that the mean curvature H is identically zero. Thus, the surface $M(\gamma_1, \gamma_2)$ is minimal.

Case 2. Let $\lambda_3 \neq 0$.

2-1) Let $\lambda_1 = 0$ and $\lambda_2 = 0$. In this case the system (32), (33) and (34) is reduced equivalently to

$$\begin{aligned} Hf'(1-tg') &= 0, \\ Hg' &= 0, \\ \frac{2H}{W} &= -\lambda_3(f+g)\mathrm{e}^{-f}. \end{aligned}$$

2-1-1) If $H = 0$, then $f = a$ and $g = -a$, $a \in \mathbb{R}$. Then $H = 0$.

2-1-2) If $g' = 0$, then $H = 0$. Thus, the surface $M(\gamma_1, \gamma_2)$ is minimal.

2-2) Let $\lambda_1 = 0$ and $\lambda_2 \neq 0$. In this case the system (32), (33) and (34) is reduced equivalently to

$$\left(\frac{2H}{W}\right)f'(1-tg') = 0, \tag{35}$$

$$\left(\frac{2H}{W}\right)g' = \lambda_2 t\mathrm{e}^{-f-2g}, \tag{36}$$

$$\frac{2H}{W} = -\lambda_3(f+g)\mathrm{e}^{-f}. \tag{37}$$

2-2-1) If $H = 0$, then $\lambda_2 = 0$. So we get a contradiction.

2-2-2) If $1 - tg' = 0$, then $H = 0$. So we get a contradiction.

2-2-3) If $f' = 0$. Then $f(s) = b$, $b \in \mathbb{R}$.

Substituting (37) into (36), we get

$$-\lambda_3(b+g)g' = \lambda_2 t\mathrm{e}^{-2g}.$$

A direct integration implies that there exist β_1, β_2 such that

$$(2b + 2g - 1)\mathrm{e}^{2g} = \beta_1 t^2 + \beta_2.$$

2-3) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$. From (33), we have $g' = 0$. Then $g(t) = c$, $c \in \mathbb{R}$. Substituting (34) into (32), we get

$$-\lambda_3(c+f)f' = \lambda_1 s\mathrm{e}^{2(f+c)}.$$

A direct integration implies that there exist γ_1, γ_2 such that

$$(2c+2f+1)\mathrm{e}^{-2f} = \gamma_1 s^2 + \gamma_2.$$

2-4) If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. Substituting (34) into (33), we get

$$-\lambda_3(f+g)g' = \lambda_2 t\mathrm{e}^{-2g}. \tag{38}$$

Differentiating (38) with respect to s we get $\lambda_3 f'g' = 0$.
2-4-1) If $f' = 0$, then $\lambda_1 = 0$, a contradiction.
2-4-2) If $g' = 0$, then $\lambda_2 = 0$, a contradiction.
Therefore, we have the following

Theorem 3.3. *Let $M(\gamma_1, \gamma_2)$ be a translation surface of type III in* Sol_3. *Then, $M(\gamma_1, \gamma_2)$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $i = 1, 2, 3$, $\lambda_i \in \mathbb{R}$, if and only if one of the following statement is true:*

1) *A surface $M(\gamma_1, \gamma_2)$ has zero mean curvature everywhere.*
2) *A surface $M(\gamma_1, \gamma_2)$ is parametrized as*

$$r(s,t) = (s, t\mathrm{e}^b, b + g(t)),$$

where $(2b + 2g - 1)\mathrm{e}^{2g} = \beta_1 t^2 + \beta_2$; $\beta_1, \beta_2 \in \mathbb{R}$.

3) *A surface $M(\gamma_1, \gamma_2)$ is parametrized as*

$$r(s,t) = (s, t\mathrm{e}^{f(s)}, f(s) + c),$$

where $(2c + 2f + 1)\mathrm{e}^{-2f} = \gamma_1 s^2 + \gamma_2$; $\gamma_1, \gamma_2 \in \mathbb{R}$.

3.4 Translation surfaces of type *VI*. Let $M(\gamma_1, \gamma_2)$ be a translation surface of type *VI* in Sol_3. Then, $M(\gamma_1, \gamma_2)$ is parametrized by

$$r(s,t) = \gamma_2(s) * \gamma_1(t) = (s\mathrm{e}^{-g(t)}, t, f(s) + g(t)). \tag{39}$$

The first derivatives are

$$\begin{aligned} r_s &= \mathrm{e}^f E_1 + f'E_3, \\ r_t &= (-g's\mathrm{e}^f)E_1 + \mathrm{e}^{-(f+g)}E_2 + g'E_3. \end{aligned} \tag{40}$$

The unit normal vector $\mathbf{N}$ of the surface is defined by

$$\mathbf{N} = \Big(\frac{-f'\mathrm{e}^{-(f+g)}}{W}\Big)E_1 - \Big(\frac{g'(1+sf')\mathrm{e}^f}{W}\Big)E_2 + \Big(\frac{\mathrm{e}^{-g}}{W}\Big)E_3,$$

where $W=\sqrt{f'^2\mathrm{e}^{-2(f+g)}+g'^2(1+sf')^2\mathrm{e}^{2f}+\mathrm{e}^{-2g}}$. The coefficients of the first fundamental form are:

$$\begin{aligned} E&=\mathrm{e}^{2f}+f'^2,\\ F&=g'(f'-s\mathrm{e}^{2f}),\\ G&=\mathrm{e}^{-2(f+g)}+g'^2(1+s^2\mathrm{e}^{2f}). \end{aligned}\tag{41}$$

The covariant derivatives are:

$$\begin{aligned} r_{ss}&=\widetilde{\nabla}_{r_s}r_s=(2f'\mathrm{e}^f)E_1+(f''-\mathrm{e}^{2f})E_3,\\ r_{st}&=\widetilde{\nabla}_{r_t}r_s=-(sg'f'\mathrm{e}^f)E_1-(f'\mathrm{e}^{-(f+g)})E_2+(sg'\mathrm{e}^{2f})E_3,\\ r_{tt}&=-s\mathrm{e}^f(g''+g'^2)E_1-(2g'\mathrm{e}^{-(f+g)})E_2+(\mathrm{e}^{-2(f+g)}+g''-g'^2s^2\mathrm{e}^{2f})E_3. \end{aligned}\tag{42}$$

The coefficients of the second fundamental form are given by

$$\begin{aligned} WL&=\mathrm{e}^{-g}(f''-2f'^2-\mathrm{e}^{2f}),\\ WM&=\mathrm{e}^{-g}(f'g'(1+sf')+sg'(\mathrm{e}^{2f}+f'^2)),\\ WN&=\mathrm{e}^{-g}((g''+2g'^2)(1+sf')+sf'g'^2+\mathrm{e}^{-2(f+g)}-g'^2s^2\mathrm{e}^{2f}). \end{aligned}$$

The mean curvature H of $M(\gamma_1,\gamma_2)$ is given by

$$H=\frac{\mathcal{H}}{2W^3},\tag{43}$$

where

$$\begin{aligned} \mathcal{H}=\;&g''\mathrm{e}^{-g}(1+sf')(\mathrm{e}^{2f}+f'^2)+f''\mathrm{e}^{-g}(g'^2(1+s^2\mathrm{e}^{2f})+\mathrm{e}^{-2(f+g)})\\ &+\mathrm{e}^{-g}(1+sf')g'^2(\mathrm{e}^{2f}+sf'-f'^2)-f'\mathrm{e}^{-g}(f'g'^2+f'\mathrm{e}^{-2f}-sg'^2\mathrm{e}^{2f}). \end{aligned}$$

The substituting of (40), (41) and (42) into (4) gives

$$\Delta r=-2H\mathbf{N}.\tag{44}$$

Then, from (44) and (2), we get

$$\Big(\frac{2H}{W}\Big)f'=\lambda_1 s\mathrm{e}^{2f+g},\tag{45}$$

$$\Big(\frac{2H}{W}\Big)g'(1+sf')=\lambda_2 t\mathrm{e}^{-2f-g},\tag{46}$$

$$-\frac{2H}{W}=\lambda_3(f+g)\mathrm{e}^{g}.\tag{47}$$

Therefore, the problem of classifying the translation surfaces $M(\gamma_1,\gamma_2)$ satisfying (2) is reduced to the integration of this system of ordinary differential equations. Next we study it according to the constants $\lambda_1,\lambda_2,\lambda_3$.

Then, by using similar method as for the translation surface of type *III*, we have the following result:

Theorem 3.4. *Let $M(\gamma_1, \gamma_2)$ be a translation surface of type VI in* Sol_3. *Then, $M(\gamma_1, \gamma_2)$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $i = 1, 2, 3$, $\lambda_i \in \mathbb{R}$, if and only if one of the following statement is true:*

1) *A surface $M(\gamma_1, \gamma_2)$ has zero mean curvature everywhere.*
2) *A surface $M(\gamma_1, \gamma_2)$ is parametrized as*

$$r(s,t) = (s\mathrm{e}^{-a}, t, f(s) + a),$$

where $(2a + 2f + 1)\mathrm{e}^{-2f} = \beta_1 s^2 + \beta_2$; $a, \beta_1, \beta_2 \in \mathbb{R}$.

3) *A surface $M(\gamma_1, \gamma_2)$ is parametrized as*

$$r(s,t) = (s\mathrm{e}^{-g(t)}, t, c + g(t)),$$

where $(2c + 2g - 1)\mathrm{e}^{2g} = \gamma_1 t^2 + \gamma_2$; $c, \gamma_1, \gamma_2 \in \mathbb{R}$.

3.5 Translation surfaces of type *IV*. Let $M(\gamma_1, \gamma_2)$ be a translation surface of type *IV* in Sol_3. Then, $M(\gamma_1, \gamma_2)$ is parametrized by

$$r(s,t) = \gamma_2(s) * \gamma_1(t) = (s\mathrm{e}^{-g(t)} + t, f(s)\mathrm{e}^{g(t)}, g(t)). \tag{48}$$

The first derivatives are

$$\begin{aligned} \frac{\partial r}{\partial s} &= r_s = E_1 + f' E_2, \\ \frac{\partial r}{\partial t} &= r_t = (\mathrm{e}^g - sg')E_1 + g' f E_2 + g' E_3. \end{aligned} \tag{49}$$

From this, the unit normal vector field $\mathbf{N}$ of $M(\gamma_1, \gamma_2)$ is given by

$$\mathbf{N} = -\Big(\frac{f'g'}{W}\Big)E_1 - \Big(\frac{g'}{W}\Big)E_2 + \Big(\frac{fg' - f'(\mathrm{e}^g - sg')}{W}\Big)E_3,$$

where

$$W = \sqrt{(1 + f'^2 + f^2)g'^2 + f'^2(\mathrm{e}^g - sg')^2 - 2ff'g'(\mathrm{e}^g - sg')}. \tag{50}$$

The coefficients of the first fundamental form are:

$$E = 1 + f'^2, \qquad F = \mathrm{e}^g - g'(s - ff'), \qquad G = (\mathrm{e}^g - sg')^2 + g'^2(1 + f^2). \tag{51}$$

The covariant derivatives are:

$$\begin{aligned} \widetilde{\nabla}_{r_s} r_s &= f'' E_2 + (f'^2 - 1)E_3, \\ \widetilde{\nabla}_{r_t} r_s &= (ff'g' + sg' - e^g)E_3, \end{aligned} \tag{52}$$

$$\widetilde{\nabla}_{r_t} r_t = (2g'\mathrm{e}^g - s(g'' + g'^2))E_1 + f(g'' - g'^2)E_2 + (g'' - (\mathrm{e}^g - sg')^2 + f^2 g'^2)E_3.$$

The coefficients of the second fundamental form are given by

$$\begin{aligned} WL &= (1 - f'^2)(f'(\mathrm{e}^g - sg') - fg') - g'f'', \\ WM &= (ff'g' - (\mathrm{e}^g - sg'))(fg' - f'(\mathrm{e}^g - sg')), \\ WN &= (fg' - f'(\mathrm{e}^g - sg'))(g'' + f^2 g'^2 - (\mathrm{e}^g - sg')^2) \\ &\quad - f'g'(s(g'' + g'^2) - 2g'\mathrm{e}^g) - fg'(g'' - g'^2). \end{aligned}$$

The mean curvature H of $M(\gamma_1, \gamma_2)$ is given by

$$H = \frac{\mathcal{H}}{2W^3}, \tag{53}$$

where

$$\begin{aligned} \mathcal{H} &= (1 + f'^2)((fg' - f'(\mathrm{e}^g - sg'))(g'' + f^2 g'^2 - (\mathrm{e}^g - sg')^2) \\ &\quad - f'g'(s(g'' + g'^2) - 2g'\mathrm{e}^g) - fg'(g'' - g'^2)) \\ &\quad + ((\mathrm{e}^g - sg')^2 + g'^2(1 + f^2))((1 - f'^2)(f'(\mathrm{e}^g - sg') - fg') - g'f'') \\ &\quad - 2(\mathrm{e}^g - g'(s - ff'))((ff'g' - (\mathrm{e}^g - sg'))(fg' - f'(\mathrm{e}^g - sg'))). \end{aligned}$$

The substituting of (49), (51) and (52) into (4) gives

$$\Delta r = -2H\mathbf{N}. \tag{54}$$

Then, from (54) and (2), we get

$$\left(\frac{-2H}{W}\right) f'g' = \lambda_1(s + t\mathrm{e}^g), \tag{55}$$

$$\left(\frac{2H}{W}\right) g' = \lambda_2 f, \tag{56}$$

$$\left(\frac{-2H}{W}\right)(fg' - f'\mathrm{e}^g + sf'g') = \lambda_3 g. \tag{57}$$

Therefore, the problem of classifying the affine translation surfaces $M(\gamma_1, \gamma_2)$ satisfying (2) is reduced to the integration of this system of ordinary differential equations. Next we study it according to the constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1. Let $\lambda_2 = 0$. Then, the equation (56) gives rise to $g'H = 0$. If $g' = 0$, then $H = 0$, which means that the surfaces are minimal.

Case 2. Let $\lambda_2 \neq 0$.

i) If $f = 0$, then $H = 0$.

ii) If $f \neq 0$, in this case we have four possibilities:

a) Let $\lambda_1 = 0$ and $\lambda_3 = 0$. (55) gives rise to $f'g'H = 0$. Then (53) implies $H = 0$. Then $\lambda_2 = 0$, a contradiction.

b) Let $\lambda_1 = 0$ and $\lambda_3 \neq 0$. In this case the system (55), (56) and (57) is reduced equivalently to

$$\left(\frac{-2H}{W}\right) f'g' = 0, \tag{58}$$

$$\left(\frac{2H}{W}\right) g' = \lambda_2 f, \tag{59}$$

$$\frac{-2H}{W}(fg' - f'\mathrm{e}^g + sf'g') = \lambda_3 g. \tag{60}$$

i) If $f = 0$, then (60) gives $g = 0$. Then (50) gives $W = 0$, a contradiction.

ii) If $f \neq 0$, from (58), we have $f'g'H = 0$. So, we get $H = 0$, it is a contradiction.

c) If $\lambda_1 \neq 0$ and $\lambda_3 = 0$. In this case the system (55), (56) and (57) is reduced equivalently to

$$\left(\frac{2H}{W}\right) f'g' = \lambda_1(s + t\mathrm{e}^g), \tag{61}$$

$$\left(\frac{2H}{W}\right) g' = \lambda_2 f, \tag{62}$$

$$\left(\frac{-2H}{W}\right)(fg' - f'\mathrm{e}^g + sf'g') = 0. \tag{63}$$

i) If $f = 0$, then (61) gives $\lambda_1 = 0$, a contradiction.

ii) If $f \neq 0$, from (63), we have $(fg' - f'\mathrm{e}^g + sf'g')H = 0$. We discuss by cases:

(1) The case $H = 0$. Then (62) implies $\lambda_2 f = 0$, a contradiction.

(2) The case when

$$fg' - f'\mathrm{e}^g + sf'g' = 0. \tag{64}$$

(2-1) If $g' = 0$, then $H = 0$, a contradiction.

(2-2) If $f' = 0$, then $\lambda_1 = 0$, a contradiction.

(2-3) If $f'g' \neq 0$, combining equations (61) and (62), we have

$$-\lambda_2 ff' - \lambda_1 s = \lambda_1 t\mathrm{e}^g. \tag{65}$$

We have an identity of two functions, one depending only on t and the other one depending only on s. Hence we deduce the existence of a real number $k \in \mathbb{R}\setminus\{0\}$ such that

$$\lambda_2 ff' + \lambda_1 s = -k = -\lambda_1 t\mathrm{e}^g. \tag{66}$$

This differential equation admits the solutions

$$\lambda_2 f^2 + \lambda_1 s^2 + 2ks + a = 0, \qquad a \in \mathbb{R}, \tag{67}$$

$$g(t) = \ln \frac{k}{\lambda_1 t}. \tag{68}$$

From (64) and (68), there exists a constant $c \in \mathbb{R} \setminus \{0\}$ such that

$$f(s) = \frac{c}{k+s}. \tag{69}$$

By combining (69) and (67), we have

$$\lambda_1 s^2 (k+s)^2 + 2ks(k+s)^2 + a(k+s)^2 + \lambda_2 c^2 = 0.$$

This is a polynomial in s. Then $\lambda_1 = 0$, a contradiction.

d) If $\lambda_1 \neq 0$ and $\lambda_3 \neq 0$, from (68), we have $g'(t) = -1/t$. We put this value of $g'(t)$ into (53) and we obtain

$$\frac{2H}{W} = \frac{\psi(s)}{\varphi^2(s)} t, \tag{70}$$

where

$$\begin{aligned} \psi(s) &= 2f'(\delta + s) - 2ff'^2 + f''(1 + (\delta+s)^2) + f(ff'' - 2f'^2),\\ \varphi(s) &= 1 + f'^2(1 + (\delta+s)^2) + 2ff'(\delta+s) + f^2,\\ \delta &= \frac{k}{\lambda_1}. \end{aligned}$$

By combining (60) and (70), we have

$$-\frac{\psi(s)}{\varphi^2(s)}(f + \delta f' + sf') = \lambda_3 g. \tag{71}$$

Differentiating (71) with respect to t, we have $\lambda_3 = 0$, it is a contradiction.

Theorem 3.5. *Let $M(\gamma_1, \gamma_2)$ be a translation surface of type IV in* Sol_3*. Then, $M(\gamma_1, \gamma_2)$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $i = 1, 2, 3$, $\lambda_i \in \mathbb{R}$, if and only if $M(\gamma_1, \gamma_2)$ has zero mean curvature.*

3.6 Translation surfaces of type I. Let $M(\gamma_1, \gamma_2)$ be a translation surface of type I in Sol_3. Then, $M(\gamma_1, \gamma_2)$ is parametrized by

$$r(s,t) = \gamma_1(s) * \gamma_2(t) = (s + t, f(s), g(t)). \tag{72}$$

The coefficients of the first fundamental form of M^2 are given by

$$E = \mathrm{e}^{2g} + f'^2 \mathrm{e}^{-2g}, \qquad F = \mathrm{e}^{2g}, \qquad G = \mathrm{e}^{2g} + g'^2.$$

The unit normal vector is given by

$$\mathbf{N} = \Big(\frac{f'g'\mathrm{e}^{-g}}{W}\Big)E_1 - \Big(\frac{g'\mathrm{e}^{g}}{W}\Big)E_2 - \Big(\frac{f'}{W}\Big)E_3,$$

where $W = \sqrt{g'^2\mathrm{e}^{2g} + f'^2 + f'^2g'^2\mathrm{e}^{-2g}}$.

The mean curvature H of $M(\gamma_1,\gamma_2)$ is given by

$$H = \frac{\mathcal{H}}{2W^3},$$

where $\mathcal{H} = -f''g'^3 - (f''g' + f'g'^2 + f'g'')\mathrm{e}^{2g} + f'^3(g'^2 - g'')\mathrm{e}^{-2g} - f''g'^3$.

Then, by using similar method as for the translation surface of type *IV*, we have the following result:

Theorem 3.6. *Let $M(\gamma_1,\gamma_2)$ be a translation surface of type I in* Sol_3*. Then, $M(\gamma_1,\gamma_2)$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $i = 1,2,3$, $\lambda_i \in \mathbb{R}$, if and only if $M(\gamma_1,\gamma_2)$ has zero mean curvature.*

Acknowledgment. The authors would like to express their sincere gratitude to the referee for the valuable suggestions which improve the paper.

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(*Received* December 4, 2018, *revised* March 27, 2019)