Fixed point approximation under Mann iteration beyond Ishikawa

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Abstract. Consider the Mann iteration $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n$ for a nonexpansive mapping $T\colon K \to K$ defined on some subset K of the normed space X. We present an innovative proof of the Ishikawa almost fixed point principle for nonexpansive mapping that reveals deeper aspects of the behavior of the process. This fact allows us, among other results, to derive convergence of the process under the assumption of existence of an accumulation point of $\{x_n\}$.

Keywords: Mann iteration; fixed point; nonexpansive mapping

Classification: 47H10

1. Introduction

In 1953 W. R. Mann in [9] introduced an iteration process of the form

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n \tag{1}$$

for scalars $\alpha_n \in [0,1]$. Since then, this process has been extensively studied by a number of authors. Among them, we find W. R. Mann [9], W. G. Dotson [5], F. E. Browder and W. V. Petryshyn [3], Z. Opial [10], M. Edelstein [6], M. A. Krasnosel'skii [8], H. Schaefer [14], C. Outlaw and C. W. Groetsch [11], and more recently [1], [2], [12], [13]. However, one of the most significant contribution on the topic was done by S. Ishikawa [7] in 1976 for nonexpansive mappings, which stimulated immensely the study of this process for various other types of operators.

The main purpose of this paper is three-fold. First, we present a quite different proof (see Theorem 1) of Ishikawa's work. In this case, we are able to extend the result, including a refined and unknown behavior of the Mann iteration process.

Second, we connect the approximation of the fixed points of a nonexpansive mapping to the behavior of all the Mann iterations (see Theorem 2). Others [2] have derived similar results, but not to the generality presented in this paper.

Third, in the light of a failed attempt of proving convergence of the process in [4], we succeed in proving convergence of the process (see Theorem 3).

DOI 10.14712/1213-7243.2020.031

Throughout the paper we assume that X is a normed space, and K is a subset of X. In addition, some of the proofs appeal to various Mann iterations that only differ in the initial point, since the scalars $\{\alpha_n\}$ will remain exactly the same in a given proof. We also adopt for a typical Mann iteration $\{x_n\}$ with initial point x_0, and scalars $\{\alpha_n\}$, as in (1), the notation

$$\eta(x_0) = \lim_{n\to\infty} \|x_n - Tx_n\|. \tag{2}$$

We make some standard assumptions such as the infinite sum of the α_k is infinite, as well as, the sequence $\{\alpha_k\}$ is bounded away from 1. In these two cases, we show examples (see Section 5) that reflect the necessity of these conditions to prove that $\eta(x_0) = 0$.

2. Preliminaries

Lemma 1. *Let K be a subset of a normed space X and let $T\colon K \to X$ be a nonexpansive mapping. If the iteration $\{x_n\}$ is defined by (1), then $\|x_{n+1} - Tx_{n+1}\| \leq \|x_n - Tx_n\|$ for each n.*

Lemma 2. *If $\alpha \in (0,1)$ and $\{\alpha_k\}_{k=1}^{m>0}$ lives in $(0,\alpha]$, then*

$$P_m(\alpha_1,\cdots,\alpha_m) \equiv \frac{1}{1-\alpha_1}\frac{1}{1-\alpha_2}\cdots\frac{1}{1-\alpha_m} \leq \Big(\frac{1}{1-\alpha}\Big)^n \frac{1}{1-\delta}$$

where

$$n = \Big\lfloor \frac{\sigma}{\alpha} \Big\rfloor, \qquad \delta = \sigma - n\alpha, \qquad \sigma = \sum_{k=1}^{m} \alpha_k.$$

Proof: Define $\beta(x) = (1-x)^{-1}$. The inequality is obviously true for $m = 1$. Suppose $m = 2$, and let $\sigma \in (0, 2\alpha)$. Maximizing P_2 subject to the constraints $a_k \in (0,\alpha]$, $a_1 + a_2 = \sigma$, equates to maximizing $p_2(a) = \beta(a)\beta(\sigma - a)$, $\max(0, \sigma-\alpha) \leq a \leq \min(\sigma,\alpha)$.

Since $p_2'(a) = \beta(a)\beta(\sigma-a)(\beta(a) - \beta(\sigma-a))$, it follows that $p_2'(\sigma/2) = 0$. We conclude that p_2 must achieve its maximum at either endpoint, so

$$P_2(\alpha_1,\alpha_2) \leq p_2(\min(\sigma,\alpha)) = \left\{ \begin{array}{ll} \dfrac{1}{1-\sigma}, & \sigma \leq \alpha \\ \dfrac{1}{1-\alpha}\dfrac{1}{1-\sigma+\alpha}, & \sigma > \alpha \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} \dfrac{1}{1-\delta}, & \sigma \leq \alpha \\ \dfrac{1}{1-\alpha}\dfrac{1}{1-\delta}, & \sigma > \alpha \end{array} \right\} = \Big(\frac{1}{1-\alpha}\Big)^n \frac{1}{1-\delta}.$$

Suppose inequality holds for some $m \geq 2$, then

$$P_{m+1}(\alpha_1, \cdots, \alpha_m, \alpha_{m+1}) \leq \Big(\frac{1}{1-\alpha}\Big)^{n_m} \frac{1}{1-\delta_m} \frac{1}{1-\alpha_{m+1}},$$

where

$$\sigma_m = \sum_{k=1}^{m} \alpha_k, \qquad n_m = \Big\lfloor \frac{\sigma_m}{\alpha} \Big\rfloor, \qquad \delta_m = \sigma_m - n_m \alpha.$$

If $\sigma_m + \alpha_{m+1} < \alpha$, then

$$n_{m+1} = \Big\lfloor \frac{\sigma_m + \alpha_{m+1}}{\alpha} \Big\rfloor = \Big\lfloor \frac{\sigma_m}{\alpha} \Big\rfloor = n_m, \qquad \delta_{m+1} = \delta_m + \alpha_{m+1}.$$

If $\sigma_m + \alpha_{m+1} \geq \alpha$, then

$$n_{m+1} = \Big\lfloor \frac{\sigma_m + \alpha_{m+1}}{\alpha} \Big\rfloor = \Big\lfloor \frac{\sigma_m}{\alpha} \Big\rfloor + 1 = n_m + 1, \qquad \delta_{m+1} = \delta_m + \alpha_{m+1} - \alpha.$$

These facts, combined with the case $m = 2$, allow us to obtain the following

$$\begin{aligned} P_{m+1}(\alpha_1, \cdots, &\alpha_m, \alpha_{m+1}) \\ &\leq \Big(\frac{1}{1-\alpha}\Big)^{n_m} \left\{ \begin{array}{ll} \dfrac{1}{1-\delta_m-\alpha_{m+1}}, & \sigma_m + \alpha_{m+1} < \alpha \\ \dfrac{1}{1-\alpha} \dfrac{1}{1-\delta_m-\alpha_{m+1}+\alpha}, & \sigma_m + \alpha_{m+1} \geq \alpha \end{array} \right\} \\ &= \Big(\frac{1}{1-\alpha}\Big)^{n_{m+1}} \frac{1}{1-\delta_{m+1}}. \end{aligned}$$

The principle of mathematical induction provides the *coup de grâce.* □

Lemma 3. *If* $\alpha \in (0,1)$, $\eta > 0$, *and* $\varepsilon \geq 0$, *then*

$$\begin{aligned} S &= P_m \eta - (\eta + \varepsilon)(\alpha_0 + \alpha_1\beta_1 + \alpha_2\beta_1\beta_2 + \cdots + \alpha_m P_m) \\ &\geq (1-\alpha)\eta - \varepsilon\Big(\frac{1}{1-\alpha}\Big)^{1+\sigma/\alpha} \end{aligned}$$

for each sequence $\{\alpha_k\}_{k=0}^{m>0}$ *in* $(0, \alpha]$ *where*

$$\sigma = \sum_{k=1}^{m} \alpha_k, \qquad \beta_k = \frac{1}{1-\alpha_k}, \qquad P_k = \prod_{j=1}^{k} \beta_j \ \text{ for each } k.$$

Proof: Note that

$$\begin{aligned} \widehat{S} &= 1 + \alpha_1\beta_1 + \alpha_2\beta_1\beta_2 + \cdots + \alpha_m P_m = \beta_1 + \alpha_2\beta_1\beta_2 + \cdots + \alpha_m P_m \\ &= \beta_1\beta_2 + \cdots + \alpha_m P_m = \cdots = P_m, \end{aligned}$$

so

$$S = P_m\eta - (\eta+\varepsilon)(\alpha_0 - 1 + P_m) = (1-\alpha_0)(\eta+\varepsilon) - \varepsilon P_m \geq (1-\alpha)\eta - \varepsilon P_m.$$

For a fixed σ, the product P_m achieves a maximum when we set as many $\alpha_k = \alpha$ as possible (this follows from single variable optimization for $m = 2$, and extends to $m > 2$ via induction, see Lemma 2), the cardinality of which we denote with

$$n = \left\lfloor \frac{\sigma}{\alpha} \right\rfloor \leq \frac{\sigma}{\alpha}.$$

So

$$P_m \leq \Big(\frac{1}{1-\alpha}\Big)^{n+1} \leq \Big(\frac{1}{1-\alpha}\Big)^{1+\sigma/\alpha}$$

independently of m. □

3. Ishikawa

In this section, we present the inequality, see Lemma 4, crucial in proving our two main theorems, Theorems 2 and 3. We also use this inequality to provide a simpler proof (done in contrapositive form by Theorem 1) of Ishikawa's famous Lemma 2, see [7]. The following notation will be used from now on.

$$\sigma(m,n) = \sum_{k=m}^{n} \alpha_k, \qquad \alpha_k \in (0,a] \text{ for some } a<1, \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha_k = \infty.$$

Lemma 4. *Let K be a subset of the normed space X, and $T\colon K \to X$. Define a Mann iteration $\{x_n\}$ on K, $\eta(x_0) > 0$, and $\|Tx_{n+1} - Tx_n\| \leq \|x_{n+1} - x_n\|$ for each n, then*

$$\|x_n - x_{m-1}\| \geq \bigg((1-a)\eta(x_0) - \varepsilon_m \Big(\frac{1}{1-a}\Big)^{1+\sigma(m,n)/a}\bigg)\sigma(m-1,n-1)$$

for each $0 < m < n$, where $\varepsilon_n = \|x_n - Tx_n\| - \eta(x_0)$.

Proof: Let $\varphi \in X^*$ such that $\|\varphi\| = 1$. Define

$$\Delta_k = x_k - x_{k-1}, \qquad \delta_k = \|\Delta_k\|, \qquad \eta_k = \|x_k - Tx_k\|,$$

and set

$$\begin{aligned}\Gamma_k &= \frac{1}{\alpha_{k-1}}\Delta_k - \frac{1-\alpha_{k-2}}{\alpha_{k-2}}\Delta_{k-1} \\ &= Tx_{k-1} - x_{k-1} - (1-\alpha_{k-2})(Tx_{k-2} - x_{k-2}) = Tx_{k-1} - Tx_{k-2}\end{aligned}$$

for each k. The sequential nonexpansive nature of T implies that

$$\delta_{k-1} = \|\Delta_{k-1}\| \geq \|Tx_{k-1} - Tx_{k-2}\| = \|\Gamma_k\| \geq \operatorname{Re}\langle \Gamma_k, \varphi\rangle,$$

or

$$\operatorname{Re}\langle \Delta_{k-1}, \varphi\rangle \geq \frac{\alpha_{k-2}}{1-\alpha_{k-2}}\Big(\frac{1}{\alpha_{k-1}}\operatorname{Re}\langle \Delta_k, \varphi\rangle - \delta_{k-1}\Big).$$

To reduce the notational burden, set

$$\gamma_k = \operatorname{Re}\langle \Delta_k, \varphi\rangle, \qquad \beta_k = \frac{1}{1-\alpha_k} > 1,$$

then

$$\gamma_k \geq \alpha_{k-1}\beta_{k-1}\Big(\frac{\gamma_{k+1}}{\alpha_k} - \delta_k\Big) = \alpha_{k-1}\beta_{k-1}\Big(\frac{\gamma_{k+1}}{\alpha_k} - \alpha_{k-1}\eta_{k-1}\Big).$$

Lemma 1 implies that $\{\eta_k\}$ does not increase so

$$\begin{aligned}\gamma_k &\geq \alpha_{k-1}\Big(\beta_k\Big(\frac{\gamma_{k+2}}{\alpha_{k+1}} - \alpha_k\eta_k\Big) - \alpha_{k-1}\eta_{k-1}\Big)\\ &\geq \alpha_{k-1}\Big(\beta_k\frac{\gamma_{k+2}}{\alpha_{k+1}} - \eta_{k-1}(\alpha_{k-1}+\alpha_k\beta_k)\Big) \qquad (3)\\ &\geq \alpha_{k-1}\Big(\beta_k\beta_{k+1}\frac{\gamma_{k+3}}{\alpha_{k+2}} - \eta_{k-1}(\alpha_{k-1}+\alpha_k\beta_k+\alpha_{k+1}\beta_k\beta_{k+1})\Big)\cdots\end{aligned}$$

Define

$$P_{k,q} = \beta_k\beta_{k+1}\cdots\beta_{k+q}, \qquad S_{k,q} = \alpha_{k-1} + \alpha_k\beta_k + \cdots + \alpha_{k+q}P_{k,q}.$$

Then (3) becomes

$$\gamma_k \geq \alpha_{k-1}\Big(P_{k,q}\frac{\gamma_{k+q+2}}{\alpha_{k+q+1}} - \eta_{k-1}S_{k,q}\Big) \qquad \text{for any } q \geq 0. \tag{4}$$

Let $0 < m < n$, choose $j \in J\Delta_{n+2}$, where $J\colon X \to 2^{X^*}$ represents the normalized duality mapping. Set $\varphi = j/\|j\|$, then $\gamma_{n+2} = \delta_{n+2}$. For $m \leq k \leq n$ equation (4) along with Lemma 3 imply that

$$\begin{aligned}\gamma_k &\geq \alpha_{k-1}\Big(P_{k,q}\frac{\gamma_{n+2}}{\alpha_{n+1}} - \eta_k S_{k,q}\Big) = \alpha_{k-1}\Big(P_{k,q}\eta_{n+1} - \eta_k S_{k,q}\Big)\\ &\geq \alpha_{k-1}\Big(P_{k,q}\eta(x_0) - (\eta(x_0)+\varepsilon_m)S_{k,q}\Big) \qquad (5)\\ &\geq \alpha_{k-1}\Big((1-a)\eta(x_0) - \varepsilon_m\Big(\frac{1}{1-a}\Big)^{1+\sigma(k,n)/a}\Big)\end{aligned}$$

where $q = n-k$. Thus,

$$\begin{aligned}\|x_n - x_{m-1}\| = \operatorname{Re}\langle x_n - x_{m-1}, \varphi\rangle &= \sum_{k=m}^{n}\operatorname{Re}\langle \Delta_k, \varphi\rangle = \sum_{k=m}^{n}\gamma_k\\ &\geq \Big((1-a)\eta(x_0) - \varepsilon_m\Big(\frac{1}{1-a}\Big)^{1+\sigma(m,n)/a}\Big)\sum_{k=m}^{n}\alpha_{k-1}\end{aligned}$$

for any $0 < m < n$. □

Now we are ready to state and prove Ishikawa's Lemma 2, see [7], in contrapositive form.

Theorem 1. *Let K be a convex subset of the normed space X and let $T\colon K \to X$ be a nonexpansive mapping with the iteration $\{x_n\}$ defined in K. If $\eta(x_0) > 0$, then $\{x_n\}$ is unbounded. Moreover, if $\eta(x_0)$ is attained, then $\|x_n\| \to \infty$ as $n \to \infty$.*

Proof: Suppose η is attained. This means there exists $n_0 \in \mathbb{N}$ such that $\|x_n - Tx_n\| = \eta$ for all $n \geq n_0$. Then $\varepsilon_m = \|x_m - Tx_m\| - \eta(x_0) = 0$ for all $m > n_0$. Lemma 4 implies that

$$\|x_n - x_{m-1}\| \geq (1-a)\eta\sigma(m-1, n-1) \to \infty$$

as $n \to \infty$. Suppose η is not attained. Fix $r > 1$ and choose $\varepsilon > 0$ small enough so that

$$(1-a)\eta - \varepsilon\Big(\frac{1}{1-a}\Big)^{1+r/a} \geq \frac{1-a}{2}\eta.$$

Select $m > 1$ large enough such that $\varepsilon_m < \varepsilon$. Lemma 4 implies that

$$\begin{aligned}\|x_n - x_{m-1}\| &\geq \Big((1-a)\eta - \varepsilon_m\Big(\frac{1}{1-a}\Big)^{1+\sigma(m,n)/a}\Big)\sigma(m-1, n-1)\\ &\geq \Big((1-a)\eta - \varepsilon\Big(\frac{1}{1-a}\Big)^{1+r/a}\Big)\sigma(m-1, n-1)\\ &\geq \frac{1-a}{2}\eta\sigma(m-1, n-1)\end{aligned}$$

for $n > m$ where $\sigma(m,n) \leq r$. We can certainly choose n such that $\sigma(m,n) > r-1$, which implies that

$$\|x_n - x_{m-1}\| \geq \frac{1-a}{2}\eta(r-1).$$

Since we place no restrictions on $r > 1$, we can find two elements in the sequence $\{x_n\}$ that lie arbitrarily far apart, making the sequence unbounded. $\square$

We derive as a direct consequence of Theorem 1, the well-known result of S. Ishikawa, see Lemma 2 of [7].

Corollary 1 (S. Ishikawa [7])**.** *Let K be a subset of the normed space X and $T\colon K \to X$ be a nonexpansive mapping. If the iteration $\{x_n\}$ is defined and bounded, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

4. Main results

Theorems 2 and 3 represent the main results of this paper.

J. Borwein, S. Reich, and I. Shafrir came close to proving Theorem 2, see Corollary 9 in their paper [2], but fell short because they utilize the Banach contraction theorem in [2, Lemma 6], which they require for the proof of Corollary 9 in [2]. Thus, their result requires completeness of the underlying space whereas ours does not.

C. E. Chidume in [4, Theorem 4] claims to have proved Theorem 3, but a close inspection of his proof reveals several fundamental and uncorrectable errors. J. Borwein, S. Reich, and I. Shafrir have a version, see [2, Corollary 10], of Theorem 3, but, again, they require completeness of the space.

To obtain Theorems 2 and 3, we start with three lemmas, the first of which appears as Theorem 3 in [2] in Borwein, Reich, and Shafrir's paper.

Lemma 5 ([2, Theorem 3])**.** *Let K be a convex subset of the normed space X and let $T\colon K \to K$ be a nonexpansive mapping. For any $\{x_n\}$ on K,*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \inf_{x\in K} \|x - Tx\|.$$

As we observe in Theorem 1 and Corollary 1, boundedness of the iteration appears to be important. The next result establishes an equivalent condition for a bounded iteration.

Lemma 6 (Bounded)**.** *Let K be a convex subset of X, and $T\colon K \to K$ be a nonexpansive mapping. If there exists a bounded sequence $\{z_i\}$ in K such that $\|z_i - Tz_i\| \to 0$, then $\{x_n\}$ is bounded.*

Proof: Let $\varrho = \sup_i \|z_i - x_0\|$, then $\varrho < \infty$. For a particular i,

$$\|x_{n+1} - z_i\| \le \|x_n - z_i\| + \alpha_n \|Tz_i - z_i\|,$$

hence,

$$\|x_{n+1} - z_i\| \le \|x_0 - z_i\| + \|Tz_i - z_i\| \sum_{k=0}^{n} \alpha_k \le \varrho + \|Tz_i - z_i\| \sum_{k=0}^{n} \alpha_k.$$

Consequently,

$$\|x_{n+1} - x_0\| \le \|x_{n+1} - z_i\| + \|z_i - x_0\| \le 2\varrho + \|Tz_i - z_i\| \sum_{k=0}^{n} \alpha_k \to 2\varrho$$

as $i \to \infty$. Therefore, $\{x_n\}$ lives in $B(x_0; 2\varrho)$. □

Lemma 7 (Divergence)**.** *Let K be a convex subset of the normed space X, $T\colon K \to K$ be a nonexpansive mapping, $\{z_i\}$ be a bounded sequence in K, and $\eta(z_i)$ come from the Mann iteration created using z_i as a starting point. If*

$$\|z_i - Tz_i\| - \eta(z_i) \to 0, \qquad \eta = \liminf_{i\to\infty} \eta(z_i) > 0$$

as $i \to \infty$, then $\|x_n\| \to \infty$ as $n \to \infty$ for any initial point $x_0 \in K$.

Proof: Set $\eta = \frac{1}{2}\liminf_{i\to\infty} \eta(z_i)$, then $\eta > 0$ and without loss of generality we assume that each $\eta(z_i) > 0$. Suppose $\{x_n\}$ has a bounded subsequence $\{x_{n_k}\}$, then there exists $\varrho > 0$ such that $z_i, x_{n_k} \in B(x_0; \varrho)$ for each i, k. Choose k such that

$$\sigma(0, n_k - 1) \geq \frac{3\varrho}{m_1}, \qquad m_1 = \frac{1-a}{2}\,\eta.$$

Set $r = \sigma(1, n_k) + 1$, and choose $\varepsilon > 0$ small enough so that

$$(1-a)\eta - \varepsilon\Big(\frac{1}{1-a}\Big)^{1+r/a} > m_1.$$

Choose i large enough such that $\|z_i - Tz_i\| - \eta(z_i) \leq \varepsilon$ and set $\zeta_0 = z_i$, then

$$\varepsilon_n = \|\zeta_n - T\zeta_n\| - \eta(z_i) \leq \|z_i - Tz_i\| - \eta(z_i) \leq \varepsilon$$

for each n. Lemma 4 implies that

$$\begin{aligned}\|\zeta_n - \zeta_0\| &\geq \Big((1-a)\eta(z_i) - \varepsilon_1\Big(\frac{1}{1-a}\Big)^{1+\sigma(1,n)/a}\Big)\sigma(0, n-1)\\ &\geq \Big((1-a)\eta - \varepsilon\Big(\frac{1}{1-a}\Big)^{1+r/a}\Big)\sigma(0,n-1) m_1 \sigma(0, n-1)\end{aligned}$$

for n where $\sigma(1,n) \leq r$. Since $\sigma(1, n_k) < r$,

$$\|\zeta_{n_k} - \zeta_0\| > m_1\sigma(0, n_k - 1) \geq 3\varrho,$$

hence,

$$\|\zeta_{n_k} - x_0\| \geq \|\zeta_{n_k} - \zeta_0\| - \|\zeta_0 - x_0\| > 3\varrho - \varrho = 2\varrho.$$

But

$$\|\zeta_{n_k} - x_0\| \leq \|\zeta_{n_k} - x_{n_k}\| + \|x_{n_k} - x_0\| < \|\zeta_0 - x_0\| + \varrho \leq 2\varrho.$$

Therefore, we have reached a contradiction. $\square$

Theorem 2 (AFP). *Let K be a convex subset of the normed space X and let $T\colon K \to K$ be a nonexpansive mapping.*

(1) *If T does not have an almost fixed point, then $\|x_n\| \to \infty$.*
(2) *If $\{x_n\}$ has a bounded subsequence, then T has an almost fixed point and the entire sequence is bounded, hence, $\|Tx_n - x_n\| \to 0$.*

Proof: Set

$$\eta = \inf_{x\in K} \|Tx - x\|,$$

then Lemma 5 ([2, Theorem 3]) implies that $\|Tx_n - x_n\| \to \eta$. Let $\{z_i = x_{n_i}\}$ represent a subsequence of $\{x_n\}$, then

$$\|Tz_i - z_i\| \to \eta.$$

Let $\eta(z_i)$ come from the Mann iteration created using z_i as a starting point, then

$$\|Tz_i - z_i\| - \eta(z_i) \leq \|Tz_i - z_i\| - \eta \to 0. \tag{6}$$

Suppose T does not have an almost fixed point, $\eta > 0$. If $\{x_n\}$ has a bounded subsequence, then equation (6) implies we are under the purview of Lemma 7, which implies that $\|x_n\| \to \infty$.

Assume now that $\{x_n\}$ does have a bounded subsequence, then Part 1 implies that $\eta = 0$. Since $\{z_i\}$ is a subsequence of $\{x_n\}$, then $\|z_i - Tz_i\| \to \eta(x_0)$ and due to Lemma 5, $\eta = \eta(x_0) = 0$. Hence, by Lemma 6, the sequence $\{x_n\}$ is bounded and $\|Tx_n - x_n\| \to 0$ as $n \to \infty$. □

We derive the following corollary, which appears to be an extension of a well-known lemma of S. Ishikawa.

Corollary 2. *Let K be a subset of the normed space X and let $T\colon K \to X$ be a nonexpansive mapping. Let $\{x_n\}$ be a Mann iteration defined on K containing a bounded subsequence, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Theorem 3 (Accumulation). *Let K be a convex subset of the normed space X and let $T\colon K \to K$ be a nonexpansive mapping. If p is an accumulation point of $\{x_n\}$, then p is a fixed point of T and $x_n \to p$.*

Proof: Theorem 2 implies that $\|Tx_n - x_n\| \to 0$. Since $\{x_n\}$ accumulates at p, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$. So

$$\|Tp - p\| \leq \|p - x_{n_k}\| + \|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\| \to 0$$

as $k \to \infty$, hence, p is a fixed point of T. Since

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tx_n - Tp\| \leq \|x_n - p\|,$$

$x_n \to p$. □

5. Remarks

The following examples show the necessity of the two most standard assumptions on the $\{\alpha_n\}$ concerning the Mann iteration process.

Example 1. Necessity of bounding $\{\alpha_n\}$ away from 1 in Theorem 1. Let $X = l^\infty$ and define $T\colon X \to X$ as the right shift operator, then T is nonexpansive. Set $x_1 = (1, 0, 0, \cdots)$, $\alpha_n = \mathrm{e}^{-1/n^2}$, then

$$\|x_{n+1} - x_n\| \geq \prod_{k=1}^{n} \mathrm{e}^{-1/n^2} \equiv p_n.$$

But

$$\ln p_n > -\frac{\pi^2}{6} \quad \text{for each} \quad n, \ \text{and hence} \quad \|x_n - Tx_n\| \not\to 0.$$

Example 2. Necessity of requiring that

$$\sum_{n=1}^{\infty} \alpha_n = \infty$$

in Theorem 1. Let $\alpha_n = 1/n^2$, $X = \mathbb{R}$, and define $T\colon X \to X$ by $Tx = x + a$ for some $a \neq 0$, then T is nonexpansive and

$$M = \sum_{n=0}^{\infty} \alpha_n < \infty.$$

Thus,

$$|x_{k+1} - x_0| \leq |Tx_0 - x_0| \sum_{n=0}^{\infty} \alpha_n = M|x_0 - Tx_0|$$

for each k. Therefore, $\{x_n\}$ is a bounded sequence, but

$$|x_k - Tx_k| = |a| > 0,$$

$|x_k - Tx_k|$ does not converge to 0.

The fundamental result of Ishikawa ensures *the almost fixed point property of* T under the assumption that the Mann iteration process is bounded. However, the convergence of this process to a fixed point of T is not guaranteed.

Example 3. The following example shows that assuming T has a fixed point is not enough for the Mann iteration to converge to a fixed point. Let $X = l^{\infty}$ and define $T\colon X \to X$ as the right shift operator. Then T is nonexpansive, and 0 is the unique fixed point of T. Set

$$x_1 = (1, 0, 0, \cdots), \qquad \alpha_n = \frac{1}{n},$$

then $\{x_n\}$ is bounded since T has a fixed point. However, the iteration orbits around 0, but never gets closer than about 0.15 to 0.

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(*Received* April 5, 2019, *revised* March 27, 2020)