# On left $\varphi$-biflat Banach algebras

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*Abstract.* We study the notion of left $\varphi$-biflatness for Segal algebras and semigroup algebras. We show that the Segal algebra $S(G)$ is left $\varphi$-biflat if and only if $G$ is amenable. Also we characterize left $\varphi$-biflatness of semigroup algebra $l^1(S)$ in terms of biflatness, when $S$ is a Clifford semigroup.

*Keywords:* left $\varphi$-biflat; Segal algebra; semigroup algebra; locally compact group

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## 1. Introduction and preliminaries

A Banach algebra $A$ is called amenable, if there exists an element $M \in (A \otimes_p A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_A^{**}(M)a = a$ for each $a \in A$. It is well-known that an amenable Banach algebra has a bounded approximate identity. For the history of amenability, see [12].

In homological theory, the notion of biflatness is an amenability-like property. In fact a Banach algebra $A$ is biflat if there exists a Banach $A$-bimodule $\varrho$ from $A$ into $(A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \varrho(a) = a$ for each $a \in A$. It is well-known that a Banach algebra $A$ with a bounded approximate identity is biflat if and only if $A$ is amenable.

E. Kanuith et al. in [9] defined a version of amenability with respect to a nonzero multiplicative functional $\varphi$. A Banach algebra $A$ is called left $\varphi$-amenable if there exists an element $m \in A^{**}$ such that $am = \varphi(a)m$ and $\widetilde{\varphi}(m) = 1$ for every $a \in A$. Note that the Segal algebra $S(G)$ is left $\varphi$-amenable if and only if $G$ is amenable, for further information see [1], [8] and [7].

Motivated by these considerations, M. Essmaili et al. in [2] defined a biflat-like property related to a multiplicative linear functional, called the condition $W$ (here called $\varphi$-biflatness).

**Definition 1.1** ([2])**.** Let $A$ be a Banach algebra and $\varphi \in \Delta(A)$. The Banach algebra $A$ is called left $\varphi$-biflat (right $\varphi$-biflat or is said to satisfy the condition $W$), if there exists a bounded linear map $\varrho\colon A \to (A \otimes_p A)^{**}$ such that

$$\varrho(ab) = \varphi(b)\varrho(a) = a \cdot \varrho(b) \qquad (\varrho(ab) = \varphi(a)\varrho(b) = \varrho(a) \cdot b)$$

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and

$$\widetilde{\varphi}\circ\pi_A^{**}\circ\varrho(a)=\varphi(a)$$

for each $a, b\in A$, respectively.

They showed that a symmetric Segal algebra $S(G)$ (on a locally compact group $G$) is right $\varphi$-biflat if and only if $G$ is amenable [2, Theorem 3.4]. As a consequence of this result in [2, Corollary 3.5] authors characterized the right $\varphi$-biflatness of Lebesgue-Fourier algebra $\mathcal{LA}(G)$, Weiner algebra $M_1$ and Feichtinger's Segal algebra $S_0(G)$ over a unimodular locally compact group.

In this paper, we extend [2, Theorem 3.4] for any Segal algebra (in left $\varphi$-biflat case). In fact we show that the Segal algebra $S(G)$ is left $\varphi$-biflat if and only if $G$ is amenable. Using this tool we characterize left $\varphi$-biflatness of the Lebesgue-Fourier algebra $\mathcal{LA}(G)$. Also we characterize left $\varphi$-biflatness of second dual of Segal algebra $S(G)^{**}$ in the term of amenability $G$. We study left $\varphi$-biflatness of some semigroup algebras.

We recall some standard notations and definitions that we shall need in this paper. Let $A$ be a Banach algebra. If $X$ is a Banach $A$-bimodule, then $X^*$ is also a Banach $A$-bimodule via the following actions

$$(a\cdot f)(x)=f(x\cdot a),\qquad (f\cdot a)(x)=f(a\cdot x),\qquad a\in A,\ x\in X,\ f\in X^*.$$

Throughout, the character space of $A$ is denoted by $\Delta(A)$, that is, all nonzero multiplicative linear functionals on $A$. Let $\varphi\in\Delta(A)$. Then $\varphi$ has a unique extension $\widetilde{\varphi}\in\Delta(A^{**})$ which is defined by $\widetilde{\varphi}(F)=F(\varphi)$ for every $F\in A^{**}$.

Let $A$ be a Banach algebra. The projective tensor product $A\otimes_p A$ is a Banach $A$-bimodule via the following actions

$$a\cdot(b\otimes c)ab\otimes c,\qquad (b\otimes c)\cdot a=b\otimes ca,\qquad a,b,c\in A.$$

The product morphism $\pi_A\colon A\otimes_p A\to A$ is given by $\pi_A(a\otimes b)=ab$ for every $a,b\in A$. Let $X$ and $Y$ be Banach $A$-bimodules. The map $T\colon X\to Y$ is called $A$-bimodule morphism, if

$$T(a\cdot x)=a\cdot T(x),\qquad T(x\cdot a)=T(x)\cdot a,\qquad a\in A,\ x\in X.$$

## 2. Left $\varphi$-biflatness

In this section we give two criteria which show the relation of left $\varphi$-biflatness and left $\varphi$-amenability.

**Lemma 2.1.** *Suppose that $A$ is a left $\varphi$-biflat Banach algebra with $\overline{A\ker\varphi}^{\|\cdot\|}=\ker\varphi$. Then $A$ is left $\varphi$-amenable.*

Proof: Let $A$ be left $\varphi$-biflat. Then there exists a bounded linear map $\varrho\colon A\to (A\otimes_p A)^{**}$ such that $\varrho(ab)=a\cdot\varrho(b)=\varphi(b)\varrho(a)$ and $\widetilde{\varphi}\circ\pi_A^{**}\circ\varrho(a)=\varphi(a)$ for all $a\in A$. We finish the proof in three steps:

*Step 1*: There exists a bounded left $A$-module morphism $\xi\colon A\to \big(A\otimes_p \frac{A}{\ker\varphi}\big)^{**}$ which $\xi(l)=0$ for each $l\in\ker\varphi$. To see this, we denote $\mathrm{id}_A\colon A\to A$ for the identity map. Also we denote $q\colon A\to\frac{A}{\ker\varphi}$ for the quotient map. Put

$$\xi:=(\mathrm{id}_A\otimes q)^{**}\circ\varrho\colon A\to\Big(A\otimes_p\frac{A}{\ker\varphi}\Big)^{**},$$

where $\mathrm{id}_A\otimes q(a\otimes b)=\mathrm{id}_A(a)\otimes q(b)$ for every $a,b\in A$. Clearly $\mathrm{id}_A\otimes q\colon A\otimes_p A\to A\otimes_p\frac{A}{\ker\varphi}$ is a bounded left $A$-module morphism, it follows that $(\mathrm{id}_A\otimes q)^{**}$ is also a bounded left $A$-module morphism. So $\xi\colon A\to\big(A\otimes_p\frac{A}{\ker\varphi}\big)^{**}$ is a bounded left $A$-module morphism. Let $l$ be an arbitrary element of $\ker\varphi$. Since $\overline{A\ker\varphi}^{\|\cdot\|}=\ker\varphi$, there exist two sequences $(a_n)$ in $A$ and $(l_n)$ in $\ker\varphi$ such that $a_nl_n\xrightarrow{\|\cdot\|}l$. Then

$$\xi(l)=(\mathrm{id}_A\otimes q)^{**}\circ\varrho(l)=\lim_n(\mathrm{id}_A\otimes q)^{**}\circ\varrho(a_nl_n)=\lim_n\varphi(l_n)(\mathrm{id}_A\otimes q)^{**}\circ\varrho(a_n)=0,$$

the last equality holds because $(l_n)$ is in $\ker\varphi$.

*Step 2*: There exists a bounded left $A$-module morphism $\eta\colon\frac{A}{\ker\varphi}\to A^{**}$ such that $\widetilde{\varphi}\circ\eta(a+\ker\varphi)=\varphi(a)$ for each $a\in A$. To see this, in Step 1 we showed that $\xi(\ker\varphi)=\{0\}$. It induces a map $\overline{\xi}\colon\frac{A}{\ker\varphi}\to\big(A\otimes_p\frac{A}{\ker\varphi}\big)^{**}$ which is defined by $\overline{\xi}(a+\ker\varphi)=\xi(a)$ for each $a\in A$. Define

$$\theta:=(\mathrm{id}_A\otimes\overline{\varphi})^{**}\circ\overline{\xi}\colon\frac{A}{\ker\varphi}\to\Big(A\otimes_p\frac{A}{\ker\varphi}\Big)^{**},$$

where $\overline{\varphi}$ is a character on $\frac{A}{\ker\varphi}$ given by $\overline{\varphi}(a+\ker\varphi)=\varphi(a)$ for each $a\in A$. Clearly $\theta$ is a bounded left $A$-module morphism. On the other hand we know that $\frac{A}{\ker\varphi}\cong\mathbb{C}$ and $A\otimes_p\frac{A}{\ker\varphi}\cong A$. Thus the composition of $\widetilde{\varphi}$ and $\theta$ can be defined. Since

$$\widetilde{\varphi}\circ(\mathrm{id}_A\otimes\overline{\varphi})^{**}=(\varphi\otimes\overline{\varphi})^{**},\qquad(\varphi\otimes\overline{\varphi})^{**}\circ\xi(a)=\widetilde{\varphi}\circ\pi_A^{**}\circ\varrho(a),\qquad a\in A,$$

we have

$$\begin{aligned}\widetilde{\varphi}\circ\theta(a+\ker\varphi)&=\widetilde{\varphi}\circ(\mathrm{id}_A\otimes\overline{\varphi})^{**}\circ\overline{\xi}(a+\ker\varphi)=(\varphi\otimes\overline{\varphi})^{**}\circ\xi(a)\\&=\widetilde{\varphi}\circ\pi_A^{**}\circ\varrho(a)=\varphi(a)\end{aligned}$$

for each $a\in A$.

*Step 3*: We prove that $A$ is left $\varphi$-amenable. To see that, choose an element $a_0$ in $A$ such that $\varphi(a_0)=1$. Put $m=\theta(a_0+\ker\varphi)\in A^{**}$. Since $aa_0-\varphi(a)a_0\in$

$\ker\varphi$, we have $aa_0 + \ker\varphi = \varphi(a)a_0 + \ker\varphi$. Consider

$$\begin{aligned} am &= a\theta(a_0 + \ker\varphi) = \theta(aa_0 + \ker\varphi) = \theta(\varphi(a)a_0 + \ker\varphi) \\ &= \varphi(a)\theta(a_0 + \ker\varphi) = \varphi(a)m \end{aligned}$$

and

$$\widetilde{\varphi}(m) = \widetilde{\varphi}\circ\theta(a_0 + \ker\varphi) = \varphi(a_0) = 1$$

for every $a \in A$. It implies that $A$ is left $\varphi$-amenable. □

**Theorem 2.2.** *Let $A$ be a Banach algebra with a left approximate identity and $\varphi \in \Delta(A)$. Then $A^{**}$ is left $\widetilde{\varphi}$-biflat if and only if $A$ is left $\varphi$-biflat.*

Proof: Suppose that $A^{**}$ is left $\widetilde{\varphi}$-biflat. Then there exists a bounded linear map $\varrho\colon A^{**} \to (A^{**}\otimes_p A^{**})^{**}$ such that $\widetilde{\widetilde{\varphi}}\circ\pi^{**}_{A^{**}}\circ\varrho(a) = \widetilde{\varphi}(a)$ for all $a \in A^{**}$. On the other hand, there exists a bounded linear map $\psi\colon A^{**}\otimes_p A^{**} \to (A\otimes_p A)^{**}$ such that for $a, b \in A$ and $m \in A^{**}\otimes_p A^{**}$, the following holds:

(i) $\psi(a\otimes b) = a\otimes b$;
(ii) $\psi(m)\cdot a = \psi(m\cdot a)$, $a\cdot\psi(m) = \psi(a\cdot m)$;
(iii) $\pi^{**}_A(\psi(m)) = \pi_{A^{**}}(m)$,

see [4, Lemma 1.7]. Clearly

$$\psi^{**}\circ\varrho|_A\colon A \to (A\otimes_p A)^{**}$$

is a bounded linear map for which

$$\psi^{**}\circ\varrho|_A(ab) = \varphi(b)\psi^{**}\circ\varrho|_A(a) = a\cdot\psi^{**}\circ\varrho|_A(b)$$

and

$$\widetilde{\widetilde{\varphi}}\circ\pi^{****}_A\circ\varrho(a) = \widetilde{\varphi}(a), \qquad a, b \in A.$$

By a similar argument as in the previous lemma (Step 1), we can find a bounded left $A$-module morphism $\xi\colon A \to \left(A\otimes_p \frac{A}{\ker\varphi}\right)^{****}$ such that $\xi(\ker\varphi) = \{0\}$. Now following the same course as in the previous lemma (Step 2) we can find a bounded linear map $\theta\colon \frac{A}{\ker\varphi} \to A^{****}$ such that $\widetilde{\widetilde{\varphi}}\circ\theta(a + \ker\varphi) = \varphi(a)$ for each $a \in A$. Choose $a_0$ in $A$ which $\varphi(a_0) = 1$. Set $m = \theta(a_0 + \ker\varphi)$. It is easy to see that

$$am = \varphi(a)m, \qquad \widetilde{\widetilde{\varphi}}(m) = 1, \qquad a \in A.$$

Applying Goldestine's theorem, we can find a bounded net $(m_\alpha)$ in $A^{**}$ such that $am_\alpha - \varphi(a)m_\alpha \xrightarrow{w^*} 0$ and $\widetilde{\varphi}(m_\alpha) \to 1$ for each $a \in A$. On the other hand $(m_\alpha)$ is a bounded net, therefore $(m_\alpha)$ has a $w^*$-limit point, say $M$. It is easy to see that $aM = \varphi(a)M$ and $\widetilde{\varphi}(M) = 1$ for each $a \in A$. Define $\eta\colon A \to (A\otimes_p A)^{**}$ by

$\eta(a) = \varphi(a)M \otimes M$ for each $a \in A$. It is easy to see that $\eta$ is a bounded linear map such that

$$\eta(ab) = a \cdot \eta(b) = \varphi(b)\eta(a), \qquad \widetilde{\varphi} \circ \pi_A^{**} \circ \eta(a) = \varphi(a), \qquad a, b \in A.$$

It follows that $A$ is left $\varphi$-biflat.

Conversely, suppose that $A$ is left $\varphi$-biflat. Since $A$ has a left approximate identity, we have $\overline{A \ker \varphi}^{\|\cdot\|} = \ker \varphi$. By the previous lemma $A$ is left $\varphi$-amenable. Applying [9, Proposition 3.4] $A^{**}$ is left $\widetilde{\varphi}$-amenable. Thus there exists an element $m \in A^{****}$ such that $am = \varphi(a)m$ and $\widetilde{\widetilde{\varphi}}(m) = 1$ for each $a \in A^{**}$. Define $\gamma\colon A \to (A^{**} \otimes_p A^{**})^{**}$ by $\gamma(a) = \varphi(a)m \otimes m$ for each $a \in A$. It is easy to see that $\gamma$ is a bounded linear map such that

$$\gamma(ab) = a \cdot \gamma(b) = \varphi(b)\gamma(a), \qquad \widetilde{\varphi} \circ \pi_{A^{**}}^{**} \circ \gamma(a) = \varphi(a), \qquad a, b \in A.$$

It follows that $A^{**}$ is left $\widetilde{\varphi}$-biflat. □

## 3. Segal and semigroup algebras

A linear subspace $S(G)$ of $L^1(G)$ is said to be a Segal algebra on $G$ if it satisfies the following conditions:

(i) subspace $S(G)$ is dense in $L^1(G)$;
(ii) subspace $S(G)$ with a norm $\|\cdot\|_{S(G)}$ is a Banach space and $\|f\|_{L^1(G)} \leq \|f\|_{S(G)}$ for every $f \in S(G)$;
(iii) for $f \in S(G)$ and $y \in G$, we have $L_y(f) \in S(G)$ and the map $y \mapsto L_y(f)$ from $G$ into $S(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$;
(iv) $\|L_y(f)\|_{S(G)} = \|f\|_{S(G)}$ for every $f \in S(G)$ and $y \in G$.

For various examples of Segal algebras, we refer the reader to [11].

It is well-known that $S(G)$ always has a left approximate identity. For a Segal algebra $S(G)$ it has been shown that

$$\Delta(S(G)) = \{\varphi_{|_{S(G)}} : \varphi \in \Delta(L^1(G))\},$$

see [1, Lemma 2.2].

**Theorem 3.1.** *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(i) *subspace $S(G)^{**}$ is left $\widetilde{\varphi}$-biflat;*
(ii) *subspace $S(G)$ is left $\varphi$-biflat;*
(iii) *group $G$ is amenable.*

Proof: (i) $\Rightarrow$ (ii) Let $S(G)^{**}$ be left $\widetilde{\varphi}$-biflat. Since $S(G)$ has a left approximate identity, by Theorem 2.2, $S(G)$ is left $\varphi$-biflat.

(ii) $\Rightarrow$ (iii) Suppose that $S(G)$ is left $\varphi$-biflat. Since $S(G)$ has a left approximate identity, $\overline{S(G)\ker\varphi}^{\|\cdot\|} = S(G)$. Applying Lemma 2.1, it follows that $S(G)$ is left $\varphi$-amenable. Now by [1, Corollary 3.4] $G$ is amenable.

(iii) $\Rightarrow$ (i) Let $G$ be amenable. By [1, Corollary 3.4] $S(G)$ is left $\varphi$-amenable. Thus $S(G)$ is left $\varphi$-biflat. Using Theorem 2.2, $S(G)^{**}$ is left $\widetilde{\varphi}$-biflat. □

Let $G$ be a locally compact group. Define $\mathcal{LA}(G) = L^1(G) \cap A(G)$, where $A(G)$ is the Fourier algebra over $G$. For $f \in \mathcal{LA}(G)$ put

$$|||f||| = \|f\|_{L^1(G)} + \|f\|_{A(G)},$$

with this norm and the convolution product $\mathcal{LA}(G)$ becomes a Banach algebra called Lebesgue–Fourier algebra. In fact $\mathcal{LA}(G)$ is a Segal algebra in $L^1(G)$, see [3]. Following corollary is an easy consequence of the previous theorem:

**Corollary 3.2.** *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(i) *algebra $\mathcal{LA}(G)^{**}$ is left $\widetilde{\varphi}$-biflat;*
(ii) *algebra $\mathcal{LA}(G)$ is left $\varphi$-biflat;*
(iii) *group $G$ is amenable.*

Let $G$ be a locally compact group and let $\widehat{G}$ be its dual group, which consists of all nonzero continuous homomorphism $\zeta\colon G \to \mathbb{T}$. It is well-known that $\Delta(L^1(G)) = \{\varphi_\zeta \colon \zeta \in \widehat{G}\}$, where $\varphi_\zeta(f) = \int_G \overline{\zeta(x)} f(x)\,\mathrm{d}x$ and $\mathrm{d}x$ is a left Haar measure on $G$, for more details see [5, Theorem 23.7].

Using the previous corollary, we can easily show the following result.

**Corollary 3.3.** *Let $G$ be a locally compact group. Then the following statements are equivalent:*

(i) *algebra $L^1(G)^{**}$ is left $\widetilde{\varphi}$-biflat;*
(ii) *algebra $L^1(G)$ is left $\varphi$-biflat;*
(iii) *group $G$ is amenable.*

A discrete semigroup $S$ is called inverse semigroup if for each $s \in S$ there exists an element $s^* \in S$ such that $ss^*s = s^*$ and $s^*ss^* = s$. There is a partial order on each inverse semigroup $S$, that is,

$$s \le t \Leftrightarrow s = ss^*t, \qquad s,t \in S.$$

Let $(S, \le)$ be an inverse semigroup. For each $s \in S$, set $(x] = \{y \in S : y \le x\}$. Semigroup $S$ is called uniformly locally finite if $\sup\{|(x]| \colon x \in S\} < \infty$.

Suppose that $S$ is an inverse semigroup and $e \in E(S)$, where $E(S)$ is the set of all idempotents of $S$. Then $G_e = \{s \in S \colon ss^* = s^*s = e\}$ is a maximal subgroup of $S$ with respect to $e$. An inverse semigroup $S$ is called Clifford semigroup if for each $s \in S$ there exists $s^* \in S$ such that $ss^* = s^*s$, for more details see [6].

**Proposition 3.4.** *Let $S = \bigcup_{e\in E(S)} G_e$ be a Clifford semigroup such that $E(S)$ is uniformly locally finite. Then the followings are equivalent:*

(i) *Algebra $l^1(S)^{**}$ is left $\widetilde{\varphi}$-biflat for each $\varphi \in \Delta(l^1(S))$.*
(ii) *Algebra $l^1(S)$ is left $\varphi$-biflat for each $\varphi \in \Delta(l^1(S))$.*
(iii) *Each $G_e$ is an amenable group.*
(iv) *Algebra $l^1(S)$ is biflat.*

Proof: (i) $\Rightarrow$ (ii) Suppose that $l^1(S)^{**}$ is left $\varphi$-biflat for all $\varphi \in \Delta(l^1(S))$. By [10, Theorem 2.16], $l^1(S) \cong l^1 - \bigoplus_{e\in E(S)} l^1(G_e)$. Since each $l^1(G_e)$ has an identity, $l^1(S) \cong l^1 - \bigoplus_{e\in E(S)} l^1(G_e)$ has an approximate identity. Applying Theorem 2.2 gives that $l^1(S)$ is left $\varphi$-biflat.

(ii) $\Rightarrow$ (iii) Suppose that $l^1(S)$ is left $\varphi$-biflat for each $\varphi \in \Delta(l^1(S))$. Since $l^1(S) \cong l^1 - \bigoplus_{e\in E(S)} l^1(G_e)$ has an approximate identity, Lemma 2.1 implies that $l^1(S)$ is left $\varphi$-amenable for each $\varphi \in \Delta(l^1(S))$. We know that each $l^1(G_e)$ is a closed ideal in $l^1(S)$, so every nonzero multiplicative linear functional $\varphi \in \Delta(l^1(G_e))$ can be extended to $l^1(S)$. Thus by [9, Lemma 3.1] left $\varphi$-amenability of $l^1(S)$ implies that each $l^1(G_e)$ is left $\varphi$-amenable. Using [1, Corollary 3.4] each $G_e$ is amenable.

(iv) $\Rightarrow$ (i) It is clear by [10, Theorem 3.7]. $\square$

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